RESEARCH IN TEAMS: Functor Calculus, Cartesian Differential Categories, and Operads

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1 Overview of the Field

A basic goal of algebraic topology is to find algebraic invariants that classify topological spaces up to various notions of equivalence. Computing such invariants can be extremely difficult, yet can lead to spectacular outcomes. Some of the most acclaimed results in algebraic topology and K-theory in recent years, including Hill, Hopkins and Ravenel's solution to the Kervaire Invariant One problem and Voevodsky's use of motivic homotopy theory in resolving longstanding conjectures in K-theory, are founded on such computations.

The calculus of homotopy functors, first introduced by Goodwillie in the 1980's, is a framework that makes it possible to understand and gain information about algebraic invariants even when they cannot be computed explicitly. Within this framework, one replaces an algebraic invariant with a related functor of topological objects and then approximates this functor by a tower of "polynomial" functors analogous to the Taylor series for a real-valued function. These polynomial functor approximations have properties that often make them easier to analyze than the original invariants. This approach has led to significant breakthroughs in the understanding of periodic homotopy theory, algebraic K-theory, and embeddings of manifolds. (See [AM99, DGM13, Wei99].)

Goodwillie's original formulation of functor calculus has been generalized and applied to a variety of settings, all sharing the common theme of approximating functors (invariants) with easier to control "polynomial" functors. The resulting tower of approximations can be analyzed through study of the fibers, which can often be classified by "derivatives." Some versions of functor calculus include

- homotopy calculus for Quillen model categories and ∞-categories, generalizes Goodwillie's homotopy calculus [Ku07, Lur17],
- abelian calculus, developed by Johnson and McCarthy [JM04] and applied to algebraic invariants like Hochschild homology (by Kantorovicz and McCarthy, see [KM02]), and
- orthogonal and manifold calculus (see [Wei95, GW99]) , invented by Weiss and Goodwillie-Weiss and applied to questions in embedding and surgery theory.

One major thread of current activity in functor calculus is unifying these approaches categorically. For example, comparisons between various functor calculi have been studied by Bauer, Johnson & McCarthy [BJM15] and Barnes & Eldred [BE15]. Work of Johnson and Hess seeks a categorical context uniting manifold, homotopy, and abelian calculi that can also be used to generate new functor calculi. The WIT II project by Bauer, Johnson, Osborne, Riehl and Tebbe [BJO+18] and ongoing work of Bauer, Burke and Ching tie abelian and homotopy calculus with the notion of cartesian differential categories of Blute-Cockett-Seely [BCS09].

2 Recent Developments and Open Problems

Broadly speaking, we seek to further these comparisons by

1. finding analogues of theorems from homotopy calculus in abelian calculus,
2. identifying how such theorems are a result of the differential category structure in abelian calculus, and
3. generalizing this relationship to create and compare results in various versions of functor calculus.

In the long term, we expect the flow of information to yield a new framework for dealing with unreduced functors in homotopy calculus, a topic with few results and important applications.

Our first goal is to find an operad structure for derivatives in abelian calculus. Ching showed that the derivatives of the identity functor in homotopy calculus form an operad ([Chi05, Chi]). Synthesizing results from [BJO+18] and methods in [Yea19], we will show that the derivatives of certain functors (including the identity functor) in abelian calculus form a functor-operad, which recovers an operad upon evaluation at particular objects. As part of this process, we will show that the operad structures naturally arise as a consequence of a particular lax monoidal functor built using the differential category structure identified in [BJO+18].

The next goal will be to compare classifications of polynomial functors given by Arone-Ching in homotopy calculus [AC15] and Johnson-McCarthy in abelian calculus [JM03a, JM03b] and determine how these classifications are tied to differential categories. Ching's operad was instrumental in the classification of functors obtained by Arone and Ching [AC11], and we plan to use our operad in a similar way to obtain classifications in abelian calculus that can be compared to those observed by Johnson and McCarthy.

3 Scientific Progress Made

Prior to the BIRS RIT program, we had established that our desired operad structure for the derivatives in abelian calculus could be obtained by finding a bicategory homomorphism from $AbCat$, the bicategory of abelian categories (suitably defined) and another bicategory which we will call Faà$(AbCat)$. The latter category is a bicategorical version of the Faà category originally defined by Cockett and Seely [CS11], and may be of independent interest. During the BIRS RIT, we focused on constructing this bicategory homomorphism, ∇.

Our goal of extracting specific operad structures from this framework dictates what the source and target of this bicategory homomorphism should be while the definition of bicategory homomorphism requires that two technical conditions, the hexagon axiom and the unit axiom, are satisfied by the homomorphism. In particular, to successfully construct the homomorphism, we needed a very concrete version of a chain rule for abelian functor calculus – we needed to find a concrete natural weak equivalence

$$D_1F \odot D_1G \to D_1(F \odot G),$$

where D_1 denotes the degree 1 homogeneous approximation of a functor, and $\odot$ denotes the horizontal composition in the bicategory $AbCat$. Abstractly, such an equivalence is known to exist by work of Bauer, Johnson, Osborne, Riehl, and Tebbe [BJO+18], but verifying the hexagon and unit axioms entails building an explicit model for this homomorphism and showing that it is a natural weak equivalence. Part of the challenge in doing so arises from the manner in which horizontal composition in $AbCat$ is defined – the definition relies on the Dold-Kan correspondence, a well-known equivalence between categories of chain complexes and simplicial objects in abelian categories.

During the first half of our RIT program, we constructed a candidate for this chain rule map. We showed that it provided the desired natural weak equivalence between $D_1F \odot D_1G$ and $D_1(F \odot G)$, and we were able to prove that the hexagon axiom holds for that candidate. In the process, we proved several technical lemmas that should prove useful as we continue our work on this project.

During the second half of the week, we attempted to verify the unit axiom - but here we ran into trouble. This will be resolved in ongoing collaboration.

4 Outcome of the Meeting

The Research in Teams program provided us with the opportunity to focus on a highly technical aspect of our project. We made far more progress on the construction of the bicategory homomorphism in this one week than we had in many previous months of long-distance collaboration. We are very grateful to Banff International Research Station for making this possible. In addition to the results we obtained while in Banff, we now have several new tools and ideas that we can use in tackling the remaining steps in this problem. Once these steps have been completed, we will have a paper that

- proves a new chain rule for abelian functor calculus,
- provides a bicategorical version of Cockett and Seely's Faà category,
- demonstrates how operad structures in functor calculus arise from these categorical constructions, at least in the case of the abelian functor calculus.

These results will pave the way for future work in two significant directions. First, as outlined in Section 2, identification of the operad structures is the first step in a program to obtain a classification of polynomial functors in abelian functor calculus in a manner similar to that done for the calculus of homotopy functors by Arone and Ching [AC15], and compare that with the classification obtained by Johnson and McCarthy [JM03a, JM03b]. The second direction would explore the new Faà bicategory and the extent to which Cockett and Seely's characterization of cartesian differential categories in terms of the Faà comonad can be extended to a bicategorical setting, and the consequences of such an extension.

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