Random Motions in Markov and Semi-Markov Environments and Their Applications

Anatoliy Swishchuk (University of Calgary, Calgary, AB, Canada),
Anatoliy Pogoruy (Zhytomyr Ivan Franko State University, Zhytomyr, Ukraine),
Ramon M. Rodriquez-Dagnino (Tecnologico de Monterrey, Monterrey, Mexico)

June 19, 2022-June 26, 2022

The theory of dynamical systems operating under the influence of random factors is one of the fields of modern mathematics that is under intensive study. The theory of stochastic processes is the basic mathematical tool to study these systems, and a good representative of them is the theory of random evolutions or random motion. Random evolution in a random environment means that the system depends on the state of the environment, and it occurs in many real systems in nature. Similarly, if the evolution of the system does not affect the random environment, but the environment is described by a random process, say a Markov renewal processes, then such systems are called stochastic. Some stochastic system change their states abruptly, that is, in every state the system is spending a random holding time and then immediately is transferred to another state. A better modelling strategy for such systems is the notion of semi-Markov (Markov) evolution, which is given by two processes: the switching Markov renewal process describing the random environment, and the switched process that describes the evolution of the system (see [20, 21, 22, 23]).

1 Overview of the Field

In the study of the asymptotic distribution of probability of reaching a "hard-to-reach domain" by semi-Markov processes, the theory of perturbation for linear operators is systematically used, as well as potential operators and generators theory. These theories were used for the asymptotic analysis (large deviations) of semi-Markov processes. A phase merging scheme or state lumping procedure for Markov chains and for the investigation of the probability distribution of reaching "hard-to-reach domain" in semi-Markov processes have been studied.. This state lumping scheme was developed and introduced in seminal book by Korolyuk V. and Turbin A. in 1993 (see [19]), and it is also called asymptotic average scheme [6]. Transport stochastic processes with reflecting boundaries is a good model for multiphase supplying system with feedback which are also studied in this book. For instance, the estimation of effectiveness of a supplying system with feedback is reduced to the calculations of the stationary distribution of a switched process that models the system. The superposition of processes is also used to model aggregates in combination with reservoirs.

S. Goldstein in 1951 (see [14]) and M. Kac in 1974 (see [17]) studied the movement of a particle on a line with a speed that changes its sign driven by the Poisson process. Subsequently, this process was called the telegraph process or the Goldstein- Kac process. Further developments of this theory have been presented in the works of R. Griego and R. Hersh in 1969 (see [15]), who gave a definition of stochastic evolutions in a general setting. R. Hersh wrote a nice survey paper in 1974 (see [16]) on the results and problems in random evolutions and their applications. Several models of random evolutions in Markov and semi-Markov media, which generalize the Goldstein-Kac telegraph process, are considered and distributions of such evolutions. Papers [25, 26, 27, 28] investigated fading evolutions, where the velocity of a particle tends to zero as the number of switches growths at infinite.

The papers [25, 26, 27, 28] also studied a generalization of the telegraph process to the case of Erlang interarrivals between successive switches of particle velocities. For such processes a differential equation for the pdf of the particles position on the line was obtained. In addition, a method for solving such equations by using monogenic functions was developed. A recursive expression for the conditional characteristic functions of random walk with Erlang switching considering a non-Markov switching process was also obtained. Papers [4, 25] generalized the result obtained by A. D. Kolesnik [18] (an integral equation for the characteristic function of multidimensional walk in the case of Poisson switching process) for the semi-Markov case, and investigated the case of Erlang distributed stay of the switching process in the states. The paper [28] extended these results to multidimensional random motion at random velocities.

2 Recent Developments and Open Problems

In the recent two-volume book [9, 10] both discrete systems for which the model are Markov and semi-Markov processes, and continuous systems which are simulated by random evolutions have been studied. More precisely, this two-volume book [9, 10] is devoted to the description of different homogeneous and inhomogeneous one- and multi-dimensional random motions in Markov and semi-Markov random environment and their applications, including financial ones. The latter application contains, e.g., modelling of financial markets with Markov and semi-Markov volatilities and pricing of covariance and correlation swaps. This book incorporated the approach based on martingales and also considers applications such as telegraph process in finance to model stock price and an analogue of Black-Scholes formula, a generalization of Black-76 formula in commodity markets by regarding Markov or semi-Markov modulated volatility in the forward pricing of energy products. We apply there the method of inhomogeneous random evolutions introduced in [24]. The set of particles with interaction, where each particle moves on a line according to a telegraph process (with Markov and semi-Markov switching) up to collision with another particle, are also studied in this book.

Recent paper [8] considered various transformations of classical telegraph process. We also gave three applications of transform telegraph process in finance: 1) application of classical telegraph process in the case of balance, 2) application of classical telegraph process in the case of dis-balance, and 3) application of asymmetric telegraph process in finance. For these three cases, we presented European call and put option prices. The novelty of the paper consists of new results for transformed classical telegraph process, new models for stock prices and new applications of these models to option pricing.

One of many important directions in the study and applications of telegraph process is collision of particles under different motion conditions, see [1, 2, 3]. And one of the open problems here was of how to asymptotically estimate two telegraph particle collisions and to find an application of this result. We solved this problem in [11], see Sec. 4.

Other problems associated with telegraph process are: telegraph random motion on an ellipse and telegraph Cox-based process. That's because they have important applications in random harmonic oscillators theory and in finance, respectively. We started to consider and to solve these problems during our workshop [12, 13]. See Sec. 4.

3 Presentation Highlights

During our workshop we made several presentations devoted to reviewing many papers on the topic including [29, 30, 31] and the following book [18].

We also discussed, reviewed and corrected our paper [11] that finally was sent to Mathematics journal and published during our stay at BIRS!

4 Scientific Progress Made

During our workshop we reviewed, discussed and successfully published the following paper "Asymptotic Estimation of Two Telegraph Particle Collisions and Spread Options Valuations", Mathematics, 2022, 10, 2201. See [11]. In this paper [11] we studied collisions of two telegraph particles on a line that are described by telegraph processes between collisions. We obtained an asymptotic estimation of the number of collisions under Kac's condition for the cases where the direction-switching processes have the same parameters and different parameters. We also considered the application of these results to evaluate Margrabe's spread option (see [7]) for two assets of spot prices modelled by two telegraph processes.

Regarding the problem associated with the telegraph process on an ellipse [12], we made a significant progress in solving this problem: we proposed the model, stated main results and highlighted main methodologies and approaches in proving those results. Application to random harmonic oscillator on an ellipse was considered as well.
Regarding the problem associated with the telegraph Cox-based (see [5]) process [13], we also made a significant progress in solving this problem by proposing the model, by creating main results with highlighted approaches and methodologies, and by proposing an application in finance, in particular, how to price option for a stock modelled via telegraph Cox-based process.

5 Outcome of the Meeting

The main outcomes of the workshop are two working papers we started to develop and write with prospective to publish them:

1) [12] -"Telegraph process on an ellipse": we introduced and described a new model for telegraph process on an ellipse, highlighted main approaches and application of this process in random harmonic oscillator theory. The novelty of the paper consists in a new model for telegraph process on an ellipse and in a new application: before the telegraph process was considered only on a circle [29];

2) [13]-"Telegraph Cox-based process": we introduced and described a new model for telegraph Cox-based process, highlighted main approached and application of this process in finance. The novelty of the paper consists of a new model for telegraph Cox-based process, namely, the governing process was replaced by Cox process comparing with previous models with Poisson process. Also, we consider a new application of this process in finance associated with frequency of treading described by Cox process.

6 Acknowledgement

All three members of the workshop (22rit234) would like to thank BIRS very much for their hospitality and friendly environment that resulted in a very productive week. Also, Anatoliy Swishchuk thanks NSERC for continuing support.

References

[1] Harris, T.E. Diffusion with “Collisions” between Particles. *J. Appl. Probab.* **1965**, *2*, 323–338.

[2] Spitzer, F. Uniform Motion with Elastic Collision of an Infinite Particle System. *J. Math. Mech.* **1969**, *18*, 973–989.

[3] Major, P.; Domokos, S. On the Effect of Collisions on the Motion of an Atom in R^1. *Ann. Probab.* **1980**, *8*, 1068–1078.

[4] Pogorui, A.A.; Rodríguez-Dagnino, R.M. One-dimensional semi-Markov evolutions with general Erlang sojourn times. *Random Oper. Stoch. Equ.* **2005**, *13*, 399–405.

[5] Cox D.R. *Renewal Theory*, Published by Methuen; John Wiley & Sons: London, UK; New York, NY, USA, 1962.

[6] Korolyuk, V.S.; Limnios, N. Average and diffusion approximation for evolutionary systems in an asymptotic split phase space. *Ann. Appl. Probab.* **2004**, *14*, 489–516.

[7] Margrabe, W. The value to exchange one asset for another. *J. Financ.* **1978**, *33*, 177–186.

[8] Pogorui, A.; Swishchuk, A.; Rodríguez-Dagnino, R.M. Transformations of Telegraph Processes and Their Financial Applications. *Risks* **2021**, *9*, 147. https://doi.org/10.3390/risks9080147

[9] Pogorui, A.; Swishchuk, A.; Rodríguez-Dagnino, R.M. *Random Motion in Markov and Semi-Markov Random Environment 1: Homogeneous and Inhomogeneous Random Motions*; ISTE Ltd. & Wiley: London, UK, 2021; Volume 1.

[10] Pogorui, A.; Swishchuk, A.; Rodríguez-Dagnino, R.M. *Random Motion in Markov and Semi-Markov Random Environment 2: High-Dimensional Random Motions and Financial Applications*; ISTE Ltd. & Wiley: London, UK, 2021; Volume 2.

[11] Pogorui, A.; Swishchuk, A.; Rodríguez-Dagnino, R.M. Asymptotic estimation of two telegraph particle collisions and spread options valuations. *Mathematics* 2022, 10, 2022. 14 pages.

[12] Swishchuk, A.; Pogorui, A.; Rodríguez-Dagnino, R.M. Telegraph Process on Ellipse. *Working paper*. Created during BIRS RiT workshop (22rit234), June 19-26, 2022.

[13] Swishchuk, A.; Pogorui, A.; Rodríguez-Dagnino, R.M. Telegraph Cox-based Process. *Working paper*. Created during BIRS RiT workshop (22rit234), June 19-26, 2022.

[14] Goldstein, S.: On diffusion by discontinuous movements and on the telegraph equation. *Q. J. Mech. Appl. Math*. 4, 129-156 (1951).

[15] Griego, R. and Hersh, R.: Random evolutions, Markov chains, and system of partial differential equations. *Proc. Nat. Acad. Sci. USA*, 62(2), 305-308 (1969).

[16] Hersh, R.: Random evolutions: A survey of results and problems. *Rocky Mountain J. Math.* 4, no. 3, 443–478 (1974).

[17] Kac, M.: A stochastic model related to the telegrapher's equation. *Rocky Mt. J. Math*. 4, 497-509 (1974).

[18] Kolesnik, A.D.; Ratanov, N. Telegraph Processes and Option Pricing; Springer: Heidelberg, Germany, 2013; Volume 204.

[19] Korolyuk, V. S. and Turbin, A.: *Mathematical foundations of the state lumping of large systems*. Kluwer AP, 1993.

[20] Korolyuk, V.S., Swishchuk, A.V. *Evolution of Systems in Random Media*. CRC Press, Boca Raton, U.S.A., 1995.

[21] Korolyuk V.S. and Swishchuk A.V.: *Semi-Markov Random Evolutions*. Kluwer Academic Publishers, Dordrecht, The Netherlands. 310 p. 1995.

[22] Swishchuk, A.: *Random Evolutions and their Applications*, Kluwer Academic Publishers, 1997.

[23] Swishchuk, A.V.: *Random Evolutions and Their Applications. New Trends*. Kluwer AP, Dordrecht, 2000.

[24] Swishchuk, A.: *Inhomogeneous Random Evolutions*. CRC Press, Boca Raton, USA, 2019.

[25] Pogorui, A. A., and Rodríguez-Dagnino, R. M. (2011). Isotropic random motion at finite speed with K-Erlang distributed direction alternations. *Journal of Statistical Physics*, 145(1), 102.

[26] Rodríguez-Said, R. D., Pogorui, A. A., and Rodríguez-Dagnino, R. M. (2008). Stationary Effectiveness of an Information Server with a Single Buffer and Bursty Demands of Two Different Customers. *Stochastic Models*, 24(sup1), 246-269.

[27] Pogorui, A. A., and Rodríguez-Dagnino, R. M. (2012). Random motion with uniformly distributed directions and random velocity. *Journal of Statistical Physics*, 147(6), 1216-1225.

[28] Pogorui, A. A., and Rodríguez-Dagnino, R. M. (2013). Random motion with gamma steps in higher dimensions. *Statistics and Probability Letters*, 83(7), 1638-1643.

[29] De Gregorio, A. and Iafrate, F. Telegraph random evolutions on a circle. *Stoch. Proc. Appl.*, 141 (2021), 79-108.

[30] Marchione, M. and Orsingher, E. Hitting distribution of a correlated planar Brownian motion in a disk. *Mathematics*, 2022, 10, 536. See also *arXive:2019.00144v3*, 20 Mar 2022.

[31] Cinque, F. and Orsingher, E. Random motions in R^3 with orthogonal directions. *arXive:2205v1*, 18 May 2022.