Non-Classical Constructions in Tensor Categories and Conformal Field Theory (22frg002)

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1 Overview of the Field

At the turn of the 21st century a common language was solidifying a bond between many distinct branches of mathematics. Definitions and results in the study of conformal field theory, topological quantum field theory, vertex operator algebras, subfactors of von Neumann algebras, Hopf algebras, and others, could now be understood under the uniform language of *tensor categories* [3]. With this common framework, novel results and examples in the study of tensor categories have implications for each of these diverse branches of research.

Major families of examples of tensor categories, most commonly *fusion categories*, arose from the representation theory of Hopf algebras and quantum groups at roots of unity, the latter related to the Wess-Zumino-Witten conformal field theories. The structure and properties of these families of examples have been rich enough to sustain 30+ years of research, and some came to believe that these examples generated all others with a certain finite list of allowed constructions. But this worldview is challenged by an increasing list of "exotic" categories, the first of which came from the *Haagerup subfactor*. Using constructions from von Neumann algebras to explicitly realize these examples, there is now a long list of *Haagerup-Izumi*, *generalized near-group*, and *quadratic* fusion categories which have no known connection to classical representation-theoretic examples by existing constructions.

2 Recent Developments

Patterns begin to emerge when one passes to the Drinfeld center of these exotic fusion categories, providing a highly-structured *modular tensor category*. Modular tensor categories have the benefit of being studied via their *modular data*: an associated representation ρ of the modular group $\mathrm{SL}(2,\mathbb{Z})$. In particular, $\mathrm{SL}(2,\mathbb{Z})$ has generators

$$\mathfrak{s}=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathfrak{t}=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{1}$$

and there exists a basis such that $\rho(\mathfrak{s})$ is a symmetric unitary matrix and $\rho(\mathfrak{t})$ is a diagonal matrix of roots of unity, satisfying copious structural identities such as the Verlinde formula which describes the fusion rules of the category. Evans and Gannon first noticed that the modular data of the Drinfeld centers of many exotic fusion categories could be realized as amalgamations of modular data of classical examples. The simplest amalgamation of modular data is a tensor product of matrices, realized on the categorical level by the *Deligne product*. The amalgamations of Evans and Gannon [5, 4], and more recently Grossman and Izumi [6], are more subtle, resembling more a sum than a product. For this reason they have been called "smashed sums" in the past. Since this initial observation, infinite families of modular data have been found using these amalgamation constructions and have been checked for feasibility/consistency using stringent measurements such as Frobenius-Schur indicators. Using cumbersome von Neumann algebraic methods we have categorical realizations for well over 100 of these, and it is very likely that there exist modular tensor categories realizing all of this numerical data. All this suggests that we are missing a completely new class of constructions of fusion categories or modular tensor categories – a class missed by physics, vertex operator algebras, and others.

3 Scientific outcome

One key aspect of proving that modular data comes from a modular tensor category is that the kernel of the corresponding modular group representation is a congruence subgroup whose level is the conductor of the category. Since all irreducible representations of $\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})$ for $n \in \mathbb{Z}_{\geq 2}$ are known, one should be able to identify precisely which irreducible representation their modular data arises from. This technique has been used effectively to produce classifications of fusion rules for modular tensor categories of rank up to, and including 6.

The irreducible representations corresponding to the hyopothetical amalgamation constructions are currently unknown, but much progress was made during this workshop toward this end. Recent results of S.-H. Ng et. al. imply that there are essentially canonical bases for the direct sum of two nondegenerate irreducible symmetric $\mathrm{SL}(2, \mathbb{Z})$ representations whose $\mathfrak{t}$-eigenvalues intersect at a single value. This is precisely the case for the original amalgamation constructions described by Evans & Gannon. The smallest example of this construction is the direct sum of the trivial representation with $\mathfrak{t}$-eigenvalue 1, and either of the two 3-dimensional irreducible representations of $\mathrm{SL}(2, \mathbb{Z}/5\mathbb{Z})$ with $\mathfrak{t}$-eigenvalues $\{1, \zeta_5, \zeta_5^{-1}\}$ or $\{1, \zeta_5^2, \zeta_5^{-2}\}$ with ζ_5 a primitive fifth root of unity. The modular data of the direct sum of the corresponding irreducible $\mathrm{SL}(2, \mathbb{Z})$ representations is that of the double of the Fibonacci modular tensor category, the smallest near-group fusion category. This has raised the important question whether the apparent existence of amalgamations of modular tensor categories is really an observation about how distinct irreducible $\mathrm{SL}(2, \mathbb{Z})$ representations can intertwine to realize new modular data. But is equally likely that the implication would run in the other direction.

Another discovery made at this workshop is that many of the known Witt group relations [1] involving a mix of pointed and quadratic dimension modular tensor categories can be realized by the above amalgamation construction. The smallest nontrivial example is to take the modular data for a pointed modular tensor category $\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)$ of rank 3 for an appropriate choice of quadratic form q, and amalgamate with the modular data of either a pointed modular tensor category of rank 7 or 15, the only two options which will produce consistent modular data by the Verlinde formula. These cases lie outside the above results on modular group representations since the corresponding irreducible representations in this case are degenerate. Surprisingly, both constructions correspond to very nontrivial Witt group relations

$$[\mathcal{C}(\mathfrak{g}_2, 3)] = [\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)] \qquad \text{and} \qquad [\mathcal{C}(\mathfrak{g}_2, 1)][\mathcal{C}(\mathfrak{sl}_3, 2)] = [\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)] \tag{2}$$

which come from conformal embeddings of affine Lie algebras.

Even more surprising has been the discovery that one can expand amalgamations of modular tensor categories to possibly degenerately braided fusion categories and achieve similar results. One reason for this is that the "shape" of the s-matrices for the amalgamation force sensible fusion rules via the Verlinde formula independent from whether the input data was degenerate or not. One example is to amalgamate the degenerate braided fusion category of super vector spaces with a pointed fusion category $\mathcal{C}(\mathbb{Z}/6\mathbb{Z}, q)$ where q is a slightly degenerate quadratic form. There is no *a priori* reason to believe that the resulting slightly degenerate data would be sensical, yet the output s-matrix is

$$s = \frac{1}{\sqrt{D}} \begin{bmatrix} 1 & 1+\sqrt{3} & 2+\sqrt{3} & 2+\sqrt{3} & 1+\sqrt{3} & 1 \\ 1+\sqrt{3} & 1+\sqrt{3} & -(1+\sqrt{3}) & -(1+\sqrt{3}) & 1+\sqrt{3} & 1+\sqrt{3} \\ 2+\sqrt{3} & -(1+\sqrt{3}) & 1 & 1 & -(1+\sqrt{3}) & 2+\sqrt{3} \\ 2+\sqrt{3} & -(1+\sqrt{3}) & 1 & 1 & -(1+\sqrt{3}) & 2+\sqrt{3} \\ 1+\sqrt{3} & 1+\sqrt{3} & -(1+\sqrt{3}) & -(1+\sqrt{3}) & 1+\sqrt{3} & 1+\sqrt{3} \\ 1 & 1+\sqrt{3} & 2+\sqrt{3} & (2+\sqrt{3}) & 1+\sqrt{3} & 1 \end{bmatrix} \tag{3}$$

with $\sqrt{D}$ the positive square root of $12(2+\sqrt{3})$. The $\mathfrak{t}$-eigenvalues in this basis are $1, \zeta_3, 1, -1, -\zeta_3, -1$ where $\zeta_3 := \exp(2\pi i/3)$. This is precisely the (pre)-modular data corresponding to $\mathcal{C}(\mathfrak{sl}_2, 10)_{\mathrm{ad}}$, and together this hypothetically implies the known nontrivial (super) Witt group relation $[\mathcal{C}(\mathfrak{sl}_2, 10)_{\mathrm{ad}}] = [\mathrm{sVec}]$.

4 Open Questions and Future Directions

1. It is clear that there are "exotic" modular tensor categories which have the form of an amalgamation of pointed modular tensor categories, and have associated modular group representations which are the direct sum of two irreducible representations with almost disjoint $\mathfrak{t}$-spectra. Which direct sums of irreducible modular group representations with almost disjoint $\mathfrak{t}$-spectra can be realized by modular tensor categories? Which irreducible modular group representations can be realized by modular tensor categories? (this problem was approached to some extent by Eholzer in the 1990's [2].)

2. Which known relations in the pointed subgroup of the Witt group of nondegenerately braided fusion categories can be realized by amalgamation constructions? Are there unknown Witt group relations which are implied the existence of amalgamation constructions?

3. Can amalgamation constructions be generalized in a rigorous way to arbitrary braided fusion categories? When does "degenerate" modular data still satisfy the Verlinde formula?

4. The hypothetical amalgamation constructions seem to involve tensor autoequivalences of modular tensor categories φ of order 2 (involution) such that the fixed-point simple objects of φ generate a φ-invariant fusion subcategory. This is a very non-trivial condition. When the modular tensor category is pointed, this is true of any tensor autoequivalence. When does a tensor autoequivalence φ of a fusion (braided, modular) category have a φ-invariant subcategory? When does a fusion category have a fixed-point free (tensor) involution?

References

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