Modern Breakthroughs in Diophantine Problems (Online)

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1 Overview of the Field and Recent Developments

The subject of Diophantine equations is currently experiencing a rapid succession of breakthroughs. These include:

(i) The work of Rafael von Känel, Benjamin Matschke, Hector Pasten, and others, proving powerful results on classical Diophantine equations by associating solutions to points on modular or Shimura curves.

(ii) Recent successes in making the Chabauty-Kim method effective, explicit and practical, due to Balakrishnan, Dogra, Müller, and others.

(iii) Progress on Manin's conjecture and other quantitative questions by a new generation of analytic number theorists, including Browning, Loughran, Schindler, Tanimoto and many others.

(iv) The introduction of the notion of Campana points which interpolate between rational and integral points, and which give rise to a host of new Diophantine problems.

(iv) Applications of modularity over number fields to the asymptotic Fermat conjecture and other Diophantine problems due to Bennett, Dahmen, Freitas, Kraus, Sengun, Siksek and others.

Whilst these and other successes constitute dramatic progress on problems of tremendous historical importance, there has also been a divergence of methods and approaches, and the subject is undergoing a period of fragmentation. A primary objective of the workshop was to reverse this fragmentation by bringing together researchers belonging to disparate Diophantine traditions, and who would otherwise rarely interact.

2 Presentation Highlights

2.1 Benjamin Matschke: A general S-unit equation solver and tables of elliptic curves over number fields

Many Diophantine problems can be algorithmically reduced to solving unit and S-unit equations, including the determination of integral points on elliptic and hyperelliptic curves, the resolution of Thue and Thue–Mahler equations, and the enumeration elliptic and hyperelliptic curves of good reduction outside a given finite set of primes. Matschke presented work in progress on a new highly optimized solver for general and constraint S-unit equations over number fields. He previewed some impressive applications to computing tables of elliptic curves over number fields which involve improvements to the method of Koutsianas (Parshin, Shafarevich, Elkies). For example, Matschke has computed all elliptic curves with everywhere good reduction over all number fields K with absolute discriminants ≤ 20000.

2.2 Josha Box: Modularity of elliptic curves over totally real quartic fields not containing $\sqrt{5}$

The proof by Wiles, Breuil, Conrad, Diamond and Taylor that elliptic curves over the rationals are modular was one of the highlights of 20th century mathematics. More recently, modularity of elliptic curves over totally real fields of degree 2 and 3 has been proved by Freitas, Le Hung and Siksek, and Derickx, Najman and Siksek respectively. In fact, the strategy involves reducing the problem to the determination of low degree points on some collection of complicated modular curves. Box tackles the problem for totally real quadratic fields. Recent strong results of Thorne and Kalyanswami allow him to eliminate some of the modular curves, subject to the assumption that $\sqrt{5}$ is not contained in the field. For the remaining modular curves, Box used Chabauty's method and sieving to describe the quartic points. This allowed him to prove the following theorem.

Theorem 1 (Box)**.** *Let K be a totally real quartic fields not containing $\sqrt{5}$. Let E be an elliptic curve defined over K. Then E is modular. More precisely, there is a Hilbert eigenform $\mathfrak{f}$ over K with parallel weight 2 and rational Hecke eigenvalues such that $L(E, s) = L(\mathfrak{f}, s)$.*

2.3 Hector Pasten: A Chabauty–Coleman bound for surfaces in cubic threefolds

Let $C/\mathbb{Q}$ be a curve of genus $g \geq 2$, and write J for the Jacobian of C. Let $r = \operatorname{rank}(J(\mathbb{Q}))$ and suppose $r < g$. Let $p > 2g$ be a prime of good reduction for C. A famous theorem of Coleman assert that

$$\#C(\mathbb{Q}) \leq \#C(\mathbb{F}_p) + (2g - 2). \tag{1}$$

The method of Chabauty–Coleman can often be refined to determine the rational points $C(\mathbb{Q})$ provided the condition $r < g$ holds, and this is the most popular method for determining rational points on curves. There are extensions to Chabauty–Coleman higher dimension, which apply to symmetric powers of curves, or to Weil restrictions of curves defined over number fields, but these have yet to yield an analogue of Coleman's elegant bound (1).

Pasten sketched a proof of the following elegant theorem, which is the first instance of a Coleman-style bound in higher dimension.

Theorem 2 (Caro and Pasten)**.** *Let $A/\mathbb{Q}$ be an elliptic variety of dimension 3 such that $\operatorname{rank}(A(\mathbb{Q})) = 1$. Let $X/\mathbb{Q}$ be a smooth projective hyperbolic surface contained in A. Write $c_1^2(X) = (K_X, K_X)$ (this is the first Chern number of X). Let p be a prime $> 15c_1^2(X)^2$ of good reduction such that $X \otimes \overline{\mathbb{F}}_p$ does not contain elliptic curves. Then*

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + (p + 4\sqrt{p} + 8) \cdot c_1^2(X).$$

3 Open Problems

3.1 Adam Logan: Quicksand K3s

Define a K3 surface X to be 'quicksand' if there is no map of finite degree from X to a nonisomorphic K3 surface Y. (In characteristic p I exclude supersingular K3 surfaces on both sides. I do not require every finite-degree map from X to itself to be an isomorphism.) Obviously a K3 surface is not quicksand if it has an elliptic fibration with an isogeny of degree greater than 1, or if it has a genus 1 fibration without a section.

If a K3 does not have one of these types of fibration, should it be expected to be quicksand? In particular, what about:

- X of Picard number 1;
- X with Picard lattice $U + E_6 + E_8$, where U is the hyperbolic lattice generated by x, y with $x^2 = y^2 = 0$, $(x, y) = 1$?

Probably easy: prove that there are no examples with rank greater than 16.

3.2 Lajos Hajdu: Arithmetic Progressions of Powers

Problem 1: Is it true that the length of any non-constant arithmetic progression of perfect powers (possibly with different exponents) with initial term 1 or -1 is bounded by an absolute constant?
Problem 2: More generally, is it true that the length of any primitive non-constant arithmetic progression of perfect powers (possibly with different exponents) is bounded by an absolute constant? (An arithmetic progression $a + td$ $(t = 0, 1, 2, \dots)$ is primitive if $\gcd(a, d) = 1$.)
Remarks by Lajos Hajdu: Both problems are open. Note that the cases where the exponents of the perfect powers are the same, immediately follow from results of Darmon and Merel [1].

- If in Problem 1, the first term of the progression is a with $|a| \geq 2$, then the length of the progression can be bounded in terms of a. If $a = 0$, then the length of the progression cannot be bounded. For details, see [2].
- The question in Problem 2 was answered affirmatively in [3], assuming the abc conjecture.

3.3 Benjamin Matschke: Szpiro Ratio

How large can you make $\frac{\log|\Delta_E|}{\log \mathrm{rad}(N_E)}$ for elliptic curves $E/\mathbb{Q}$? Here,

- Δ_E is the minimal discriminant of E, and
- $\mathrm{rad}(N_E)$ is the radical of the conductor of E, that is, the product of all primes of bad reduction.

Remarks:

- The limsup over all $E/\mathbb{Q}$ might be 6.
- $E : y^2 = x^3 - 54540x + 9958896$ yields $21.2187\dots$.
- Any uniform upper bound would yield a strengthening of the best currently known bounds for the abc conjecture.

4 Outcome of the Meeting

We were initially overwhelmed by the idea of running an online workshop, but the BIRS staff were really supportive and guided us through the process. The combination of using Zoom for the lectures and Zulip for the discussions worked unexpectedly well, and most talks generated good feedback and interactions. With the cancellation of many workshops and local seminar series, there are fewer opportunities for young mathematicians to shine, and so we made a choice to have as many talks by younger participants as possible. Whilst the online format lacked many of the informal exchanges that are integral to a face-to-face workshop, it has allowed us to welcome a much larger number of participants. We are particularly pleased that many PhD students and postdocs were able to join the workshop, and also with the surprising geographical spread of the participants.

References

[1] H. Darmon and L. Merel, Winding quotients and some variants of Fermats Last Theorem, *J. Reine Angew. Math.* **490** (1997), 81100.

[2] L. Hajdu, Powerful arithmetic progressions, *Indag. Math.* **19** (2008), 547–561.

[3] L. Hajdu, Perfect powers in arithmetic progression. A note on the inhomogeneous case, *Acta Arith.* **113** (2004), 343–349.