Future targets in the classification program for amenable C*-algebras

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1 Overview of the Field

C*-algebra theory is a branch of mathematics with connections to other areas such as topology, symbolic and measurable dynamics, harmonic analysis, number theory, differential geometry, set theory, and geometric group theory. A C*-algebra is an algebra of bounded operators on Hilbert space, that is closed under taking adjoints and in the operator norm topology. Operator algebras – C*-algebras and von Neumann algebras – arose in the mathematical rigorization of quantum mechanics in the 1920s and '30s.

Constructions of C*-algebras from other mathematical objects, such as groups or dynamical systems, appeared shortly after the original physics-inspired inception of C*-algebras. These constructions encode interesting information about the input objects: e.g., for groups the C*-algebra captures unitary representation theory; for dynamical systems it concerns the orbit-equivalence relation. Many other constructions have since arisen, producing C*-algebras associated to such objects as directed graphs, foliations, coarse metric spaces, integral domains, higher-rank graphs, and C*-dynamical systems. Understanding the structure of such C*-algebras, and how this structure reflects the input, has been a driving force behind C*-algebra research.

The classification program is a major research direction in C*-algebra theory, with the aim of showing that C*-algebras which agree on K-theoretic (and thereby computable) invariants are isomorphic. K-theory for C*-algebras is a generalization of topological K-theory; it is also necessary to include the simplex of tracial states and its pairing with the K_0-group in the K-theoretic invariants used for classification. The classification program systematically tackles the question of the structure of C*-algebras, since a given abstract algebra will then be isomorphic to a model with known structure – constructed to have the same invariant.

Classification of C*-algebras has its origins in the work of Dixmier and Glimm in the '60s; in the early '90s, Elliott brought it to prominence as a problem in its own right, and it has subsequently seen tremendous development. Much of the recent focus has been on simple, separable, amenable, unital, infinite dimensional C*-algebras; the nonunital case has also seen very recent developments. The following are the major turning points in the theory, which have set the foundations for exciting recent developments:

- The classification, by Bratteli and Elliott, of approximately finite dimensional (AF) C*-algebras [2, 6].
- The development by Brown–Douglas–Fillmore and Kasparov of KK-theory, a bivariant version of K-theory [3, 18].

- Elliott's conjecture of the classifiability of simple separable amenable unital C*-algebras [7].
- The Kirchberg–Phillips classification of purely infinite simple separable amenable unital C*-algebras which satisfy the Universal Coefficient Theorem (UCT) [29].
- The unexpected discovery of simple separable amenable unital C*-algebras which cannot be distinguished by traditional K-theoretic invariants, with obstructions related to high topological dimension [45, 46, 47, 36, 42, 38].
- The Toms–Winter conjecture, postulating, in response to the last point, a robust characterization of low topological dimension for simple separable amenable unital C*-algebras (see more in Recent Developments and Open Problems.) [44, 10]
- Tracial approximation, pioneered by Lin based on results of Elliott, Gong, and L. Li (see [23, 8]), and on Popa's local quantization ([31]).

2 Recent Developments

The recent developments in C*-algebra theory may be viewed in the context of three streams:

Classification. Classification, underpinned by the concept of tracial approximation, culminated recently in the following:

Theorem A (Elliott, Gong, Lin, Niu)**.** *[9] Let A, B be simple unital infinite dimensional separable amenable C*-algebras with finite nuclear dimension and for which all traces are quasidiagonal, which satisfy the Universal Coefficient Theorem for KK-theory, and which have a non-zero trace. Then $A \cong B$ if and only if* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

In the above theorem, the *Elliott invariant*, $\mathrm{Ell}(A)$ of a C*-algebra A is

$$\mathrm{Ell}(A) := (K_0(A), K_0(A)_+, [1_A]_{K_0(A)}, K_1(A), T(A), \rho_A : K_0(A) \times T(A) \to \mathbb{R}).$$

The hypotheses of finite nuclear dimension and the Universal Coefficient Theorem are discussed below. The case with no non-zero trace was dealt with by Kirchberg and Phillips (see [20, 29]). (In both cases, the result followed a significant amount of earlier work, such as [35, 24, 25, 50, 51].)

In fact, this result came about in two stages. In the first stage, Gong, Lin, and Niu proved classification of $\mathcal{Z}$-stable C*-algebras which satisfy certain tracial approximation hypothesis. Then, using a key result of Winter, it was proven that the hypotheses of the above theorem are sufficient to establish the tracial approximation hypothesis.

These ideas have been pushed further, yielding certain (but not yet complete) classification results in the nonunital setting. For instance, Elliott, Gong, Lin, and Niu have classified simple separable C*-algebras with finite nuclear dimension which are KK-contractible (UCT with zero K-groups) – the Elliott invariant in this (non-unital) case reduces to the cone of lower semicontinuous traces, together with the subset of tracial states

Structure. The Toms–Winter conjecture is an idea central to the structure theory of C*-algebras, which complements the classification results mentioned above. It states the following:

Conjecture B (The Toms–Winter conjecture)**.** *[44, 10] Let A be a simple separable unital infinite dimensional amenable C*-algebra. The following conditions are equivalent.*
(i) *A has finite nuclear dimension:* $\dim_{\mathrm{nuc}}(A) < \infty$.
(ii) *A is $\mathcal{Z}$-stable:* $A \cong A \otimes \mathcal{Z}$.
(iii) *A has strict comparison of positive elements.*

The three eclectic properties in this conjecture have varied pertinence in different settings, or constructions. Powerful structural consequences ensue for C*-algebras known to have all three of these properties.

A C*-algebra is said to have nuclear dimension at most n if it has the completely positive approximation property (the identity map approximately factorizes via uniformly bounded completely positive maps through finite dimensional algebras), in a particularly controlled manner which is inspired by Lebesgue's notion of covering dimension. This property was introduced by Winter and Zacharias, and is based on an earlier property called decomposition rank due to Kirchberg and Winter [22].

The property of $\mathcal{Z}$-stability says that the C*-algebra A is isomorphic to its tensor product with a certain special C*-algebra, namely the Jiang–Su algebra $\mathcal{Z}$. This algebra $\mathcal{Z}$ has the important property of being a strongly self-absorbing C*-algebra, meaning that $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$, and the isomorphism can be chosen to be approximately unitarily equivalent to the first-factor embedding $\mathcal{Z} \to \mathcal{Z} \otimes 1 \subset \mathcal{Z} \otimes \mathcal{Z}$, and it is not isomorphic to $\mathbb{C}$ [43]. Every other strongly self-absorbing C*-algebra is $\mathcal{Z}$-stable [48], and the Jiang–Su algebra is in fact characterized by this property. The Jiang–Su algebra may be built indirectly using UHF-algebras; this enables certain properties of UHF algebras to be generalized to the algebra $\mathcal{Z}$, a prominent example being [50].

Strict comparison of positive elements is a property for a C*-algebra that factors through a particular invariant, the Cuntz semigroup. The Cuntz semigroup is an ordered abelian semigroup which captures certain cohomological information about the C*-algebra. Strict comparison is equivalent to almost unperforation of the Cuntz semigroup, a property which captures a sense of good (or expected) behaviour in this invariant.

The conjecture is closely linked to the classification of C*-algebras. It arose in response to the counterexamples to Elliott's classification conjecture. The hypothesis of finite nuclear dimension appears in Theorem A, and is known to be a necessary hypothesis. In fact, $\mathcal{Z}$-stability is also used crucially in parts of the proof of Theorem A, whence the known implication (i) $\Rightarrow$ (ii) of the Toms–Winter conjecture is essential to this theorem.

In fact, this conjecture is very close to being established, owing to results that were announced at this workshop. (i) $\Rightarrow$ (ii) is due to Winter [49] and (ii) $\Rightarrow$ (iii) to Rørdam [37]. After being established in various special cases (including [26, 39, 1]), it was announced at this workshop that the implication (ii) $\Rightarrow$ (i) has been proven in full generality, by Castillejos, Evington, Tikuisis, White, and Winter. While the validity of (iii) $\Rightarrow$ (ii) has yet to be determined (there are partial results, such as [21]), a significant step forward was presented at the workshop by Thiel. He showed that C*-algebras with stable rank one and strict comparison of positive elements automatically have almost divisible Cuntz semigroups; when one additionally assumes locally finite nuclear dimension, $\mathcal{Z}$-stability follows by [49].

Quasidiagonality, of C*-algebras and of traces, is another key hypothesis in Theorem A. In 2015, it was proven by Tikuisis, White, and Winter that for amenable C*-algebras which satisfy the Universal Coefficient Theorem, all faithful traces are automatically quasidiagonal [41]. It remained mysterious what role the Universal Coefficient Theorem really played in this result: was it just a technical hypothesis, or was it truly necessary? Indeed, many questions around the Universal Coefficient Theorem remain mysterious (more on this below).

Putting together the exciting developments mentioned above yields the following result of a fairly definitive nature.

Theorem C. *Let A, B be a simple unital infinite dimensional separable amenable C*-algebras with the following properties:*

(i) either $\mathcal{Z}$-stability, or stable rank one plus locally finite nuclear dimension, and

(ii) satisfying the Universal Coefficient Theorem.

Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

A separable C*-algebra A satisfies the Universal Coefficient Theorem if a canonical sequence,

$$0 \to \mathrm{Ext}_1^{\mathbb{Z}}(K_*(A), K_*(B)) \to KK(A,B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0,$$

is exact, for every separable C*-algebra A. Many C*-algebras are known to satisfy the Universal Coefficient Theorem; one of the key results in this direction is due to Tu, which says that C*-algebras constructed from amenable groupoids automatically satisfy the Universal Coefficient Theorem. However, the following is a major open question.

Question D. *Does every separable amenable C*-algebra satisfy the Universal Coefficient Theorem?*

Dadarlat gave an excellent talk at the workshop discussing some of the subtleties of this problem.

Group actions and crossed products. As the classification of simple unital separable amenable C*-algebras reaches maturity, there is increasing interest in looking at group actions on C*-algebras and at crossed products, where the classification results may be put to use.

Given a group G and an action α of the group (by *-automorphisms) on a C*-algebra A, the crossed product, denoted $A \rtimes_\alpha G$, is a new C*-algebra which contains A and encodes the action. An important special case is when A is a commutative C*-algebra, necessarily of the form $C_0(X, \mathbb{C})$; in this case, the action corresponds to an action of G on X by homeomorphisms.

There have been a variety of approaches to showing classifiability of crossed products; however, while the standard classification hypotheses (separability, unitality, simplicity, amenability) are either automatic or well-understood for crossed products, we still have a lot to learn about how the hypotheses of $\mathcal{Z}$-stability/finite nuclear dimension and of the Universal Coefficient Theorem behave.

Some highlights among what is already known:

- If G is a finitely generated group with polynomial growth (equivalently, a virtually nilpotent group [14, 27]) and X is a finite dimensional locally compact Hausdorff space, then for any action of G on X, the crossed product $C_0(X) \rtimes G$ has finite nuclear dimension. (A result of Hirshberg and Wu, presented in the workshop.)
- For any countable amenable group G, the generic action α of G on the Cantor set has a $\mathcal{Z}$-stable crossed product [4]. This crossed product is also known to satisfy the other hypotheses of Theorem A, so it is classifiable.
- If G is an inductive limit of residually finite groups and A is a classifiable C*-algebra (i.e., satisfying the hypotheses of Theorem A) for which the trace space $T(A)$ is a Bauer simplex, and α is an action of G on A which is pointwise strongly outer, and which acts trivially (or at least periodically) on $T(A)$, then the crossed product $A \rtimes G$ has finite nuclear dimension (a result of Gardella–Hirshberg and Gardella–Phillips–Wang, presented in the workshop). If the crossed product also satisfies the Universal Coefficient Theorem, then it is classifiable.
- Kerr recently defined a notion of “almost finite” actions, and proved that the crossed products of such actions are $\mathcal{Z}$-stable.

Another natural problem to turn to is the classification of actions of groups on C*-algebras. A group action consists of a homomorphism from the given group to the group of *-automorphisms of the C*-algebra; in case the group is equipped with a topology, one asks that this homomorphism be continuous, using the point-norm topology on the set of automorphisms. There are several notions of equivalence which one may consider; although this is not immediately apparent, *cocycle conjugacy* turns out to be one of the most natural. However, the classification problem is believed to be intractable for general actions, and generally it is natural to restrict to actions with the Rokhlin property.

This problem is at an early stage, and there is much to do. A few highlights:

- Actions with the Rokhlin property of finite groups on a classifiable Kirchberg algebra or on a classifiable C*-algebra with tracial rank zero have been classified by Izumi [15, 16].
- Actions of the circle $\mathbb{T}$ on classifiable Kirchberg algebras with the continuous Rokhlin property have been classified by Gardella.

- Actions of the real numbers, $\mathbb{R}$, also called *flows*, on classifiable Kirchberg algebras with the Rokhlin property have been classified by Szabó [40], an exciting new result that was presented at the workshop.

3 Open Problems

All participants contributed to the following list of open problems. Our thanks to Hannes Thiel for recording the list.

(i) The UCT-problem: Does every nuclear C*-algebra satisfy the universal coefficient theorem (UCT)?

(ii) How bad can crossed products be? When is $C(X) \rtimes G$ classifiable?

(iii) Let D be a strongly self-absorbing (s.s.a.) C*-algebra with $D \not\cong \mathcal{O}_2$. Is there a unital embedding $D \hookrightarrow \mathcal{O}_\infty \otimes \mathcal{Q}$, where $\mathcal{Q}$ denotes the universal UHF-algebra? (This can be considered as an infinite version of the quasidiagonality problem [52].)

(iv) Develop applications of classification. Establish Giordano–Putnam–Skau type (orbit equivalence) theorems ([13]) for spaces that are not Cantor spaces.

(v) More examples of actions of $\mathbb{Z}^d$ on Cantor sets ([11, 12]).

(vi) Sell classification. Simplify the proof and make it accessible to other areas of mathematics.

(vii) Is conjugacy of shifts of finite type (SFT) decidable? Can this be rephrased in terms of C*-algebras?

(viii) Do simple C*-algebras with real rank zero necessarily have strict comparison of positive elements? Are they $\mathcal{Z}$-stable?

(ix) Range results for Cuntz semigroups.

(x) Which rigid C*-tensor categories embed into the representation theory of $\mathcal{Z}$?

(xi) Let D be a s.s.a. C*-algebra, let G be a torsion-free amenable group. Does there exist a unique, strongly outer action of G on D?

(xii) Develop a theory of subfactors for inclusions of classifiable C*-algebras ([17, 30]).

(xiii) Let G be an étale, amenable groupoid with $G^{(0)}$ a Cantor set such that $C^*(G)$ is simple. Is $C^*(G)$ $\mathcal{Z}$-stable?

(xiv) Let A be a simple, classifiable C*-algebra, and let G be a (finite) group of automorphisms of the Elliott invariant $\mathrm{Ell}(A)$. Does G lift to an action on A? Does the map $\mathrm{Aut}(A) \to \mathrm{Aut}(\mathrm{Ell}(A))$ split?

(xv) Let A be a simple, classifiable C*-algebra, let $a \in A$, and let $\pi\colon A \to B(H)$ be an irreducible representation. Does H have an $\pi(a)$-invariant subspace?

(xvi) Is every simple, classifiable C*-algebra a groupoid C*-algebra ([5, 34])? Is every simple (exact) C*-algebra a groupoid C*-algebra?

(xvii) Develop analogies of Popa's rigidity theory (for actions of property (T) groups on s.s.a. C*-algebras) ([32, 28, 33]). Is there a theory of intertwining by bimodules for C*-algebras?

(xviii) Classify non-simple TAF algebras.

(xix) Is every real rank zero ASH algebra with slow dimension growth TAF?

(xx) Develop a better understanding of completely positive approximations. In particular, when is an inductive limit, in the category of operator systems, of finite-dimensional C*-algebras, order-isomorphic to a C*-algebra?

(xxi) What are the possible Cowling–Haagerup invariants for separable, simple, exact C*-algebras?

4 Scientific Progress Made

The workshop brought together experts in various aspects of C*-algebra theory, and created an environment which fostered collaboration and the generation of new ideas. Here are a few examples of specific progress made during the meeting.

- Francesc Perera, Leonel Robert, and Hannes Thiel made advances in understanding the Cuntz semigroup, and specifically addressing the question of when infima exist in this ordered C*-algebra invariant.
- Søren Eilers, Jamie Gabe, and Takeshi Katsura made a decisive breakthrough on the problem of determining when an extension of two simple graph C*-algebras is again a graph C*-algebra.
- Gábor Szabó benefitted from discussions with experts in C*-algebra classification, leading to an insight that will likely lead to new results.
- José Carrión, Chris Schafhauser, Aaron Tikuisis, and Stuart White initiated a new collaborative project on aspects of C*-algebra classification.

5 Outcome of the Meeting

There was a lot of sharing of expertise and learning at this meeting. This led to new collaborations, and inspired serious discussions of future directions for researchers in the classification of C*-algebras, and more generally, directions for the field of C*-algebras itself. It is expected that there will be a number of future meetings which follow up on these directions, including Noncommutative Geometry and Operator Algebras 2018: C*-algebras and Dynamics (Münster).

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