REPORT ON BIRS WORKSHOP 12W5075 "TORSION IN THE HOMOLOGY OF ARITHMETIC GROUPS: GEOMETRY, ARITHMETIC, AND COMPUTATION"

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This report briefly recalls the main objectives of the workshop, gives a detailed list of the talks presented there, and finally some responses from participants.

1. Summary of the objectives of the workshop

The goal of the workshop was to bring together topologists, geometers, and number theorists with the intent of exploring connections between topology, analysis, and number theory, with the special focus on topics described in the section below. We had hoped that the balance between the lectures and free time as well as the intimate setting of the Banff Research Station will stimulate many informal discussions and collaborations. We were delighted to see that these objectives were fulfilled.

We would like to thank BIRS staff for their hospitality and efficiency.

To briefly recall, the main point of the workshop was the remarkable fact — whose significance is only beginning to be appreciated and analyzed — that the cohomology of arithmetic groups contain a lot of torsion, and the conjectured relationships of the Langlands program extend to describe this torsion.

The workshop played a significant role in popularizing this circle of ideas and questions, and the topic has attracted even more attention since the completion of the workshop – on the arithmetic side, the work of Peter Scholze; on the analytic side, much more work on extending the use of analytic torsion to the noncompact case by Jonathan Pfaff and others. Indeed, we believe that a sequel workshop would be appropriate in view of these developments.

2. Summary of talks delivered at the meeting

The talks at the workshop reported on many new results related to the above list of topics. There were also some expository talks from experts, on both the automorphic and representation theory sides, designed to help build bridges between the different topics. Finally, there was an evening problem session that both summarized open questions mentioned in the talks and also generated new ideas. In the following we summarize the contents of each talk. Junior speakers are indicated by a bullet (•).

Frank Calegari (Northwestern) spoke on *Torsion in the cohomology of arithmetic groups*. He gave an overview of how torsion in the cohomology of arithmetic groups interacts with questions in the Langlands program, with a particular emphasis on the (mostly conjectural) relationship between cohomology and Galois representations.

Simon Marshall (IAS) (•) presented *Multiplicities of cohomological automorphic forms on* GL_2 [9]. He discussed some ideas related to the theory of p-adically completed cohomology developed by Frank Calegari and Matthew Emerton. If F is a number field which is not totally real, he showed how to use these ideas to prove a strong upper bound for the dimension of the space of cohomological automorphic forms on GL_2 over F which have fixed level and growing weight.

Nicolas Bergeron (Jussieu) (•) gave the talk *Growth of torsion homology for arithmetic groups.* His talk addressed the question, "When does the amount of torsion in the homology of an arithmetic group grow exponentially with the covolume?" Presenting joint work with Akshay Venkatesh [1], he gave many examples where this is so, and conjectured precise conditions under which this is expected to be true.

John Voight (Dartmouth) spoke on *Minimal isospectral, nonisometric orbifolds.* By revisiting a construction due to Vignéras, he exhibited the minimal pairs of orbifolds and manifolds of dimension 2 and 3 arising from arithmetic Fuchsian and Kleinian groups that are isospectral but nonisometric. This was joint work with Peter Doyle and Benjamin Linowitz.

Werner Müller (Bonn) gave a talk entitled *Analytic torsion and the growth of torsion in the cohomology of arithmetic groups* (cf. [10]). Analytic torsion is an invariant of a compact Riemannian manifold and a flat vector bundle over this manifold. It is defined in terms of the spectrum of the Laplace operators on forms twisted by the flat bundle. Bergeron and Venkatesh used the analytic torsion to study the growth of the torsion in the cohomology of towers of congruence subgroups with coefficients in fixed strongly acyclic flat bundle. His talk addressed the complementary case, where the lattice is fixed and the flat bundle varies. He also discussed the finite volume case.

Akshay Venkatesh (Stanford) spoke on *A torsion Jacquet–Langlands correspondence.* This talk presented joint work with Frank Calegari [3] that investigates relationships between torsion classes in the cohomology of split arithmetic groups and their inner forms.

Toby Gee (Imperial College) (•) gave the talk *p-adic Hodge-theoretic properties of etale cohomology with mod p coefficients, and the cohomology of Shimura varieties* [6]. He presented new results about the etale cohomology of varieties over a number field or a p-adic field with coefficients in a field of characteristic p, and gave some applications to the cohomology of unitary Shimura varieties. This was a report on joint work with Matthew Emerton.

Jean Raimbault (Jussieu) (•) spoke on *Asymptotics of analytic and homological torsion for congruence subgroups.* The goal of the talk was to describe work extending the cocompact computations of Bergeron–Venkatesh to Bianchi groups, which have $\mathbf{Q}$-rank 1 [12].

Nathan Dunfield (UIC) presented the talk *Integer homology* 3*-spheres with large injectivity radius.* Conjecturally, the amount of torsion in the first homology group of a hyperbolic 3-manifold must grow rapidly in any exhaustive tower of covers (see Bergeron-Venkatesh and F. Calegari-Venkatesh). In contrast, the first betti number can stay constant (and zero) in such covers. Here "exhaustive" means that the injectivity radius of the covers goes to infinity. In this talk, Dunfield explained how to construct hyperbolic 3-manifolds with trivial first homology where the injectivity radius is big almost everywhere by using ideas from Kleinian groups.

He then related this to the recent work of Abert, Bergeron, Biringer, et. al. This was a report on joint work with Jeff Brock [2].

Thomas Church (Stanford) (•) spoke on *Stability and instability in the homology of arithmetic groups*. He surveyed both classical and recent results on stability in the homology of arithmetic groups, including the rational stability proved by Borel, and the more complicated central stability in the mod-p homology of congruence subgroups proved by Putman. He described a conjecture, joint with Benson Farb and Andrew Putman [4], that predicts a certain stability in the unstable homology of $\mathrm{SL}_N(\mathbf{Z})$ and its congruence subgroups.

Philippe Elbaz-Vincent (Grenoble) gave a talk entitled *Computations of the cohomology of modular groups and applications*. The focus of the talk was explicit techniques to compute the cohomology groups of $\mathrm{GL}_N(\mathbf{Z})$ and $\mathrm{SL}_N(\mathbf{Z})$ for $N = 5, 6, 7$, and their homology groups with coefficient in their Steinberg modules. He gave applications to algebraic K-theory and number theory. Part of the talk was based on joint work with H. Gangl (Durahm) and C. Soule (CNRS & IHES) [5].

Aurel Page (Bordeaux) (•) spoke on *Presentations of Arithmetic Kleinian Groups*. He presented an algorithm to compute a fundamental domain and presentation for arithmetic Kleinian groups. This was a report on his Ph.D. thesis [11].

Mehmet Sengun (Warwick) (•) presented the talk *On the integral cohomology of Bianchi groups*. In the first part of his talk, he presented numerical data investigating the asymptotic growth of torsion in the homology of cofinite Kleinian groups. In the second part, He presented some data which suggests a speculative connection between even Galois representation of Q and torsion in the homology of Bianchi groups [13].

Alexander Rahm (Galway) (•) gave the talk *Accessing the Farrell-Tate cohomology of discrete groups*. He introduced a method to explicitly determine the Farrell-Tate cohomology of discrete groups. His method allows one to show that the Farrell-Tate cohomology of the Bianchi groups is completely determined by the numbers of conjugacy classes of finite subgroups, and allowed him to give a conceptual description of the integral cohomology ring structure of the Bianchi groups.

Dan Yasaki (UNC Greensboro) (•) spoke on *Some explicit $\delta = 1, 2$ computations*. This was a preliminary report on some recent computations (joint with Paul Gunnells) of torsion in the integral cohomology for $\Gamma_0(N) \subset \mathrm{GL}_2(\mathcal{O}_F)$, where F is the complex cubic field of discriminant -23 ($\delta = 1$) and where F is the cyclotomic field of fifth roots of unity ($\delta = 2$) (cf. [7, 8]). The results provided computational evidence in support of conjectures of Bergeron–Venkatesh about the growth of torsion with covolume.

In addition to these talks, a discussion session **Computational challenges** was held.

3. Feedback and response from the conference

After the completion of the workshop, the organizers received many positive comments from both seniore and junior participants. The junior participants in were happy about being able to present their work to more senior colleagues. All participants expressed excitement about being together in an environment that gave many opportunities to interact with experts from different fields, all with a common interest in the geometry, topology, arithmetic, and analysis surrounding torsion phenomena in the cohomology of arithmetic groups. and gives some flavor of the

energy of the conference. Based on these comments, we expect new colloborations to arise from the workshop.

To give some flavor of the energy of the conference we include a link to a blog writeup by one of the participants:

`https://quomodocumque.wordpress.com/2012/07/19/torsion-in-the-homology-of-arithmetic-groups-and-an-iwasawa-algebra-puzzle/`

At the completion of the workshop, it was agreed that a similar workshop should be held in the near future. This is especially pressing in light of Scholze's recent breakthrough in connecting Galois representations to torsion classes. The organizers plan to submit an application for such a follow-up workshop in the near future.

References

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3. F. Calegari and A. Venkatesh, *A torsion Jacquet-Langlands correspondence*, submitted.
4. Thomas Church, Benson Farb, and Andrew Putman, *A stability conjecture for the unstable cohomology of $SL_n(Z)$, mapping class groups, and $Aut(F_n)$*, to appear.
5. Philippe Elbaz-Vincent, Herbert Gangl, and Christophe Soulé, *Perfect forms and the cohomology of modular groups*, to appear.
6. M. Emerton and T. Gee, *p-adic Hodge-theoretic properties of tale cohomology with mod p coefficients, and the cohomology of Shimura varieties*, preprint.
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9. Simon Marshall, *Bounds for the multiplicities of cohomological automorphic forms on* GL_2, Ann. of Math. (2) **175** (2012), no. 3, 1629–1651. MR 2912713
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