Rigidity Theory: Progress, Applications and Key Open Problems (12w5069), 5 day workshop, July 2012

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1 Overview of the Field

The rigidity and flexibility of a structure, either man-made in buildings, linkages, and lightweight deployable forms, or found in nature ranging from crystals to proteins, is critical to the form, function, and stability of the structure. The mathematical theory of 'rigidity and flexibility' is developing methods for the analysis and design of man-made structures, as well as predictions of the behavior of natural structures such as proteins. We live in 3-dimensions, and a fundamental problem is to develop results for 3-dimensions which are as good and as efficient as the recently developed theory for structures in 2-dimensions.

Rigidity theory probes some of the fundamental geometry of space, distance and metrics. With the multiple applications of this theory, results which yield new descriptions, new constructions, new analyses and new algorithms spread out to impact on our understanding, our designs, our predictions and our connections within and among these areas of application.

Rigidity theory has both geometric and combinatorial aspects: what happens at particular configurations, and what 'almost always' happen as we search over all configurations. The methods used now draw on a wide range of geometric and combinatorial techniques, ranging from matroid theory, through positive semi-definite programming, representation theory of symmetry groups, and functional analysis to algebraic and projective geometry.

The well-developed theory for plane frameworks highlights the central 100 year old problem of of characterizing the graphs of generic infinitesimally rigid bar and joint frameworks in 3-D. The characterization of graphs which are generically globally rigid in 3-space has recently risen to be another of these key unsolved problems. These were central topics to the talks and the discussions during the workshop.

2 Recent Developments and Open Problems

Over the last four years, since the previous BIRS workshop, recent work from a new generation of contributors to rigidity theory have developed new methods, and probed new problems connected to the rigidity and flexibility of new classes of frameworks, under symmetry and periodicity conditions, for specialized families of examples, as well as for broader problems in CAD constraints. Along with the new focus on such topics,

several key problems have been resolved - notably the Molecular Theorem, confirming that methods developed for body-bar frameworks apply directly to the study of generic molecular structures arising in material science and biochemistry. These results have potential extensions and applications to the study of built structures, protein structures with symmetry (such as protein dimers) or repetitive structures such as beta sheets or crystals.

The week-end after our workshop was devoted to a specific focus on inductive constructions in the framework of a 2-day workshop. This combination of a 5-day workshop and a 2-day workshop has proven very fruitful for our community.

Some key factors in the recent accelerated progress have been: (i) the arrival of a new generation of researchers, at multiple centers around the world; (ii) insights and new questions coming from fields of application; (iii) the evolving community and interactions of people from multiple communities which has been supported by a series of small and medium sized gatherings. Two of the gatherings contributing to these developments were previous meetings at BIRS.

The objectives for the workshop included:

1. With the rapid progress outlined above, including: global rigidity; body-bar and molecular rigidity; periodic rigidity; rigidity under symmetry; transfer of results among metrics and to other surfaces, it was a central activity to share what is known and identify key open problems that should be addressed.

2. With the range of applications, one objective was to make connections of current results, the key questions, and the answers that might matter in these areas, such as algorithmic and inductive characterizations; decompositions (Assur Graphs, rigid components); stability (global rigidity); short time scale motions (fixed lattice) versus slower deformation of materials (flexible lattice, fixed or variable volume);

3. The workshop highlighted key conjectures: sufficient connectivity in 3-space for generic rigidity and generic global rigidity (12 connectivity may be sufficient); proposed inductive constructions for all isostatic frameworks in 3-space; proposed characterization of all globally rigid frameworks in 3-space; and some new conjectures in all the areas were born;

4. The workshop helped the community develop new larger scale mathematical programs for further work on key questions such as: 3-space rigidity; rigidity of infinite frameworks, with and without periodicity; flexibility of surfaces in higher dimensions; the impact of symmetry on global rigidity and applications.

3 Presentation Highlights

For this meeting, the proposed focus was on developments within rigidity theory itself over the range of concepts and structures outlined above. Some connections were drawn to a variety of applications, and insights into rigidity theory brought from ongoing work in these areas of application with a few key people from these areas invited to the workshop. The list of invitees included key people from a number of these interdisciplinary collaborations, as well as all the core researchers in the many facets of rigidity theory, across all the generations.

The workshop started by a special session, effectively moderated by one of the organizers, Ileana Streinu. In this session each participant gave a short (4-5 minutes) talk, in which he or she could offer a glimpse of an upcoming longer presentation, or pose an open problem, or introduce a topic of interest. This unusual start supported immediate interactions, from the first lunch, as people immediately identified some areas of shared interest and possible questions they might be able to answer.

After this session the following talks were scheduled for Monday, to present a range of connections to unusual topics from old-hands, and to introduce some new researchers into our larger community.

Helmuth Stachel: Self-motions of the Kokotsakis tesselation,

Ileana Streinu: Rigidity and Origami

Hans-Peter Schroecker: New Results in Algebraic Kinematics: Linkages, Factorization, Discrete Mathematics

Andrea Micheletti: Engineering Tensegrities: Some Recent Studies

Oleg Karpenkov: Strata of Tensegrity Frameworks.

The program for the rest of the week was organized around some focus areas which featured new results, as well as opportunities to hear some overview talks.

Tuesday was devoted to Global Rigidity, Semidefinite Programming and related topics with an open discussion session on problems and connections and several lectures. The lectures:

Steven Gortler: A survey of Global and Universal Rigidity, with new results

Anthony Man-Cho So: Graph Realization via Schatten p-Quasi Norm Minimization

Igor Pak: Global Rigidty and Graph Colouring

Abdo Alfakih: Universal Rigidity of bar frameworks in General Position: Trilateral and Chordal graphs

Michael Thorpe: Why is the Maxwell count so good for the Rigidity Threshold in Homogeneous Networks?

Overall, this area of global rigidity generated a number of lively discussions which continued on through the workshop and afterwards. The paper [1] was refined through the continuing conversations over the week (and beyond).

Wednesday included some shorter updates on recent work - along with afternoon walks through the wonderful countryside around BIRS.

Alexander Gaifullin: Sabitov polynomials for polyhedra in four dimensions

Tiong-Seng Tay and Shin-ichi Tanigawa: Body-bar-hinge-rod frameworks

Stephen Power: Frameworks on Surfaces

Wendy Finbow-Singh: Isostatic almost spherical frameworks via disc decomposition

John Owen: Specialisation of Generic Frameworks

Shisen Luo: Lower Bound For The Rank of Rigidity Matrix of 4-Valent Graphs (participation via Skype)

Thursday was devoted to Periodic and Symmetric Frameworks :

Ciprian Borcea: Periodic Rigidity

Shin-ichi Tanigawa: Matroids of Gain Graphs in Applied Discrete Geometry

Viktória Kaszanitzky: Gain-sparsity and Symmetric Rigidity in the Plane

There was an extended discussion following the first talk, during which the group explored number of possible definitions of rigidity and flexibility for infinite frameworks with and without initial periodic structure. As a companion, a set of counterexamples were also explored, refining the choices and resulting in several documents posted to the workshop wiki over the next few weeks by Stephen Power and Walter Whiteley (see below).

We had a long discussion and open problem session on Friday morning. Before the workshop and throughout the week, a wiki site was used to post a number of conjectures and open problems. There were regular updates, including posting of presentations and revised conjectures. This was quite fruitful and continued to be available for some months following the workshop.

4 Scientific Progress Made and Outcome of the Meeting

Many collaborative groups were formed during the workshop. It is notable that many of these involved the younger researchers who were at BIRS for the first time. Some groups have reported progress on projects whose origin, direction or momentum can be traced back to the workshop.

- Anthony Man-Cho So reports: “For the topic I spoke at the workshop (Graph Realization via Schatten p-Quasi Norm Minimization), the paper has just been accepted to INFOCOM 2013 (acceptance rate is 17 percent).”
- Audrey Lee-St.John reports: “Louis Theran and I continued working on matroids and pebble games for body-and-cad rigidity (discussion started in Toronto at Fields in 2011). Also, Shin-ichi Tanigawa and I started discussing an algorithm for optimizing rigid systems (motivated by applications to control theory).”

- Brigitte Servatius reports: "I'd like to state that it was good to be able to interact with the young researchers, in particular Oleg Karpenkov, Viktoria Kaszanitzky and Shi-Ichi Tanigawa. I talked to them about several open problems for which I do have some partial results such as the Jackson - Jordán - Szabadka conjecure [2, Conjecture 5.9] or the generalization of the molecular theorem which would allow more than two bodies to share a hinge. The discussion surrounding Conjecture 5.9 finally enabled Herman Servatius and me to answer the following question of Jackson and Jordán [3] on mechanisms: is it true that for a generic realization of a mechanism the operation of 1-extension may be performed without restricting the motion? We worked out a counterexample to this question. This example is mentioned in the latest version of [3] and we are writing up a short paper giving not just a counterexample but a more general answer to the question."

- Tony Nixon reports: "Bernd Schulze and I have been looking at symmetric frameworks on surfaces and so far we have focused on orbit graph constructions for half-turn symmetric frameworks on the cylinder.

  There will be a workshop on geometric and topological graph theory in Bristol from 15th-19th April, 2013 where a number of the participants at the BIRS workshop have been invited to speak. In this workshop we anticipate that many of the ongoing rigidity problems arising from the BIRS meeting will be presented and developed."

- Meera Sitharam reports:

  "Having understood affine rigidity (recent results of Steve Gortler et al.) better from the workshop, I am now using it towards the open problem I posed in the 5-minute talk in the workshop that arose from a problem in machine learning called dictionary learning. Collaboration with Steve Gortler is likely both on this problem and the more general problem of characterizing rigidity and global rigidity when the equivalence and congruence used in defining these properties is based on general groups rather than the Euclidean group.

  Jialong Cheng and I have almost worked out a purely combinatorial, explicit algorithmic characterization of a matroid (for independence and closure) and shown that it is an abstract rigidity matroid in 3D. We know that the rigidity matroid in 3D is at least as restrictive than this matroid. If it turns out to be strictly less restrictive, that disproves a conjecture about abstract rigidity matroids. If not, it gives a combinatorial 3D rigidity characterization."

- Ciprian Borcea reports: "My visit to BIRS, for the workshop on Rigidity Theory: Progress, Applications and Key Open Problems was useful, stimulating and pleasant at the same time. I believe that the workshop was successful in addressing and disseminating recent results in rigidity theory. I welcomed the chance to survey some of the new developments on periodic frameworks. This area has reached a certain "critical mass" and has interesting outstanding open problems. I mentioned in my talk the wider relation between rigidity, periodicity and sparsity and the topic of ultrarigidity. It is a matter of satisfaction to report that, under the direct stimulus of the workshop, I obtained - in collaboration with Ileana Streinu - new results on periodic volume and symplectic frameworks."

- Stephen Power reports:

  "The workshop facilitated my collaboration with John Owen and Tony Nixon. It lead ultimately to an improvement of work that was in progress in July 2012 and which is now submitted to a research journal and is on the ArXiv [6].

  The workshop was also important in enabling conversations for rapid and informed progress concerning new rigidity theory projects with my postdoctoral research associate, Dr Derek Kitson. This is a new collaboration.

  The workshop was also invaluable in providing myself, and also Derek Kitson, with an up-to-date view from specialists on a range of topics, such as periodic rigidity, global rigidity and universal rigidity. I now expect these themes to feature in my future work."

- Robert Connelly reports:

"I think the talk by Igor Pak about applying global rigidity ideas to unique colorability problems was especially interesting and I think the discussions at BIRS were a help to him."

- Walter Whiteley reports: "To me, the several key outcomes were:

  (i) the morning discussion of infinite frameworks / the talk of Ciprian Borcea and the discussion of ways to the rigidity approach infinite frameworks. Stephen Power and myself uploaded a sequence of documents to the wiki of the workshop to show the ongoing development, including a summary of definitions and connections for infinite frameworks with incidental (and forced) periodic structure. This focuses on infinitesimal rigidity and infinitesimal flexibility.

  From these Stephen Power created an example showing that there is a countably infinite bar-joint framework which is continuously flexible (has a finite flex) yet has no infinitesimal flex. We also explored simple 3-d examples which illustrate the distinction between strongly sequentially rigid infinite frameworks and weakly sequentially rigid frameworks. These tubes continued to be explored in the lounge, with physical models made of polyhedron, over the rest of the week. See some related simulations at: $http://wiki.iri.upc.edu/index.php/Symmetric_linkages$.

  (ii) the discussions around incidental symmetry - frameworks which are initially symmetric but may have motions which break the symmetry, or go to a reduced symmetry. There are some new collaborations involving Bernd Schulze, Louis Theran and others on this topic. There has also been a growing collaboration between Bernd, Adnan Sljoka (another of my students who just graduated) and the group represented by Lluis Ros from Barcelona. They have been using their path continuation techniques to explore the finite motions / configuration space of some of the structures (most recently the tubes I was presenting at the workshop as examples). "

- Tibor Jordán reports:

  "With my co-authors, who were also present at the workshop, I have made substantial progress on some joint projects and papers during the meeting. These include new results on globally linked pairs in generic two-dimensional frameworks (joint work with Bill Jackson [3]), frameworks with dihedral symmetry (joint work with Viktória Kaszanitzky and Shin-ichi Tanigawa [4]) as well as universally rigid graphs and frameworks in one-dimensional space (joint work with Viet-Hang Nguyen [5])."

- Louis Theran reports: "Tony Nixon and I have been working on the question of generic rigidity of frameworks supported by surfaces with no isometries. An inductive approach seems promising.

  Also, Audrey Lee-St John and I have been working on some things relating to body-CAD and matroids. Bernd Schulze and I have also been looking a bit at generic incidental symmetry in the plane."

- Bill Jackson reports: " Viet Hang Nguyen and myself began discussing the rigidity of d-dimensional body-direction-length frameworks in Banff. We shall continue this research. We have used a recursive construction to characterize rigidity in the cases when the bodies are either rigid or direction rigid and are now working on the case when the bodies are length rigid."

It is clear that, overall, the workshop supported new collaborations, and supported the development of all the graduate students and post doctoral fellows who participated. The series of workshops, about four years apart, has played a major role in expanding the community. In the Spring of 2013 a workshop in Bristol, England will provide follow up for a number of the mathematical topics which arose during this five day workshop and the follow-on two day workshop.

References

[1] Steve Gortler and Dylan Thurston. Generic global rigidity in complex and psuedo-euclidean spaces. *arXiv:1212.6685*.

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[4] Tibor Jordán, Viktória Kaszanitzky, and Shin ichi Tanigawa. Gain-sparsity and symmetry-forced rigidity in the plane. *Tech. report: Egerváry Research Group, Budapest, TR-2012-17*, 2012.

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[6] Anthony Nixon, John Owen, and Stephen Power. A Laman theorem for frameworks on surfaces of revolution. *arXiv:1210.7073*, 2012.