Recent Advances in Transversal and Helly-type Theorems in Geometry, Combinatorics and Topology

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A prominent role in combinatorial geometry is played by Helly's theorem, which states the following:
Theorem: [22] Let $\mathcal{A}$ be a finite family of at least $d+1$ convex sets in the d-dimensional euclidean space $\mathbb{R}^d$. If every $d+1$ members of $\mathcal{A}$ have a point in common, then there is a point common to all members of $\mathcal{A}$.

Helly's theorem has stimulated numerous generalization and variants. There are many interesting connections between Helly's theorem and its relatives, the theorems of Radon, of Caratheodory and of Tverberg, theorems that have been the object of active research, and inspired many problems in the field. To see a sample of numerous problems associated to Helly's theorems, see the paper that now a days is one of the most cited papers in discrete geometry: "Helly's Theorem and Its Relatives" [9].

In the past ten years, there has been a significant increase in research activity and productivity in the area. (For an excellent survey in the area, see [42].) Notable advances have been made in several subareas including the development of the theory of transversals (see [24]); topological versions of Helly Theorem; the proofs of interesting colorful theorems generalizing classical results (see [43]); and many others such as the problem of finding a line transversal to a family of mutually disjoint congruent disks in the plane.

This workshop brought together senior and junior researchers in the area with the objective of interchanging ideas and assessing recent advances, of fostering awareness of the inter-disciplinary aspects of the field such as geometry, topology, combinatorics, and computer science, and of mapping future directions of research.

The workshop combined and interesting mixture of talks, problem sessions and many time for discussions in groups. During the week the academic interest was mainly centerer about 4 topics:

- Helly-type Theorems, Piercing and (p, q)-Theorems,
- Variations and Generalizations of Tverberg's theorem,
- Transversal Theorems,

  and
- Finite sets of points and finite sets of convex sets.

Although many of the talks at this workshop on transversals and Helly-type theorems deal essentially with classical subjects related to this area, many of the talks also had deep relationships with other areas of discrete and non-discrete mathematics, such as algebraic topology, algebraic geometry and probability.

Applications of algebraic topology to discrete geometry was an especially interesting topic. The common theme of several of the talks given in this workshop relate algebraic topology to discrete geometry, transversals and Helly-type theorems using the topology of the space of Grassmannians and its canonical vector bundle together with the structure of the cohomology ring of spaces, was used several times during the workshop. The relations with the Algebraic Geometry were very interesting, specially those concerning linear space transversals to secant linear spaces in $\mathbb{R}^d$ and tangent lines to four balls in $\mathbb{R}^3$.

1 Helly-type Theorems, Piercing and (p,q)-Theorems

Given a universe (a set) $\mathcal{U}$ and a property $\mathcal{P}$, (closed under inclusions, for subsets of $\mathcal{U}$). Results of the type "if every subset of cardinality μ of a finite family $\mathcal{F} \subset \mathcal{U}$ has property $\mathcal{P}$, then the entire family $\mathcal{F}$ has property $\mathcal{P}$ " are called Helly type theorems The minimum number μ for which the result is true is called the Helly number of the Helly type theorem $(\mathcal{U}, \mathcal{P}, \mu)$.

If the Universe $\mathcal{U}$ consists of a special family of sets and the property $\mathcal{P}$ is to be pierced with k elements, or equivalently, to have a transversal of cardinality k, then we have a Helly-Gallai type theorem. These theorems have been widely studied for different settings (see for instance surveys such as [9] ,[15]). In fact such theorems are in general not easy to find, see, for instance, Danzer and Grünbaum [11] where they show that even for the case of families of Boxes in $\mathbb{R}^d$ such theorems does not always exist. During theworkshop several discussions about this subject where obtained. For example, L. Montejano and D. Oliveros obtained a Helly-Gallai theorem when $\mathcal{U}$ is the family of closed intervals in $\mathbb{R}^d$ showing that μ can be bounded by a function of k which is a polynomial of degree 5.

In 1930 Helly realized that a finite family of sets in $\mathbb{R}^d$ has nonempty intersection if for any subfamily of size at most $d+1$, its intersection is homeomorphic to a ball in $\mathbb{R}^d$. In fact, the result is true if we replace the notion of topological ball by the notion of acyclic set, see [7] and [26]. In 1970, Debrunner [12] proved that a finite family of open sets in $\mathbb{R}^d$ has nonempty intersection if for any subfamily of size j, $1 \leq j \leq d+1$, its intersection is $(d-j)$-acyclic.

During the workshop, L. Montejano spoke about a new Topological Helly theorem that follows the same spirit, except that instead of $\mathbb{R}^d$, it is require a topological space X in which $H_*(U) = 0$ for $* \geq d$ and every open subset U of X. Moreover, instead of the hypothesis $(d-j)$-acyclic, he just require that the $(d-j)$-dimensional reduced homology group is zero.

That is, for a topological space X with the property that $H_*(U) = 0$, for $* \geq d$ and every open subset U of X, a finite family of open sets in X has nonempty intersection if for any subfamily of size j, $1 \leq j \leq d+1$, the $(d-j)$-dimensional reduced homology group of its intersection is zero, where $H_{-1}(U) = 0$ if and only if U is nonempty.

The fact that this is a non-expensive topological Helly theorem —in the sense that it does not require the open sets to be simple— from the homotopy point of view (we only require its $(d-1)$-dimensional homology group to be zero), allows Montejano to prove some new results concerning transversal planes to families of convex sets like the following:
Let F be a pairwise disjoint family of at least 6 smooth, convex bodies in $\mathbb{R}^3$ with the property that for any subfamily $F' \subset F$ of cardinality 5, F' admits a transversal line and for any subfamily $F' \subset F$ of cardinality 4, the space of transversal lines is connected. Then, F' admits a transversal line.

During this workshop, it was discussed the following conjecture stated by Xavier Goaoc: Let $F = \{A_1, ..., A_n\}$ be a pairwise disjoint collection of convex sets in $\mathbb{R}^d$, $n \geq 2$. Then space of transversal lines consists of a finite collection of acyclic sets. Known true for $n = 2$ and 3, $d = 3$. During the workshop the conjecture was proved for $d = 4$, $n = 4$. If the conjecture turns out to be true, this will give interesting connections and generalizations with the Montejano's transversal results stated in the above paragraph.

About Helly type theorems and Piercing (p, q) problems, Deborah Oliveros spoke about About piercing numbers of families of planes, lines and intervals. She presented some bounds for the (p, q) problem and for piercing numbers of some families of affine hyperplanes, lines and intervals following the spirit of Erdös-Gallai, joint work with M. Huicochea, J. Jeronimo and L. Montejano.

About colourful and fractional (p,q)-problems, Ferenc Fodor considered colourful and fractional versions of the classical (p, q)-problem for systems of intervals in the real line. This was a preliminary report of ongoing research with I. Bárány, L. Montejano, and D. Oliveros. In fact, several discussions during the workshop allow the authors to finish this project.

Concerning piercing number, Juergen Eckhoff in [14] stated the following problem: let $N(p, q)$ be the piercing number of axis-parallel boxes in the plane having the (p, q)-property. Wegner (1965) conjectured that $N(p, 2) < 2(p-1)$ for all $p > 1$. This is true (and best possible) for $p = 2, 3, 4$ and would imply, among other things, that $N(p, q) = p - q + 1$ for all $q > 3$.

In his talk, Jürgen Eckhoff spoke about *The teasing strip problem*. The τ-strip problem consists in proving (or disproving) the following conjecture: If a finite set of points in the plane is such that every three of the points lie in some strip of width 1, then all points lie in some strip of width τ;. (Here $\tau = 1.6180...$ is the golden number.) The conjecture is more than 40 years old and, despite considerable progress, still unsolved. Jürgen Eckhoff talk described a new approach, based on numerical evidence, which may help to tackle the conjecture.

Helge Tverberg proposed the following problem on $(1, k)$-separation (H.T.1979). That is, let k be a positive integer. Then there is a positive integer $f(k)$ so that for every family of $f(k)$ pairwise disjoint compact convex sets in the plane there is a line separating at least one of the sets from at least k of the other sets. The best upper estimate for $f(k)$ so far is ca. $7.2(k-1)$, obtained by M.Novick in [36] while the best lower estimate $3k - 1$ is given by K.Hope and M.Katchalski in Math.Scand.66 (1990),44-46.

In statistics, there are several measures of the depth of a point p relative to a fixed set S of sample points in dimension d. One of the most intuitive is the simplicial depth of p introduced by Liu (1990), which is the number of simplices generated by points in S that contain p. In general obtaining a lower bound for the simplicial depth is a challenging problem. In fact, in terms of simplicial depth, Carathéodory Theorem can be restated as follows: If p belongs to the convex hull of S then the simplicial depth is at least 1.

In 1982 Bárány showed that the simplicial depth is a least a fraction of all possible simplices generated from S. Gromov (2010) improved the fraction via a topological approach. Bárány's result uses a colourful version of Carathéodory Theorem leading to the associated colourful simplicial depth.

During the work shop we also have an interesting combinatorial, computational, and geometric approaches to the colourful simplicial depth by A. Deza. where he provide a new lower bound for the colourful simplicial depth improving the earlier bounds of Bárány and Matoušek and of Stephen and Thomas. Computational approaches for small dimension and the colourful linear programming feasibility problem introduced by Bárány and Onn were discussed. All these results based on join works with Frédéric Meunier (ENPC Paris), Tamon Stephen (Simon Fraser), Pauline Sarrabezolles (ENPC Paris), and Feng Xie (Microsoft)

2 Generalizations and Variations of Tverbergs Theorem

The workshop include several very interesting developments about generalizations and variations of Tververg Theorem, one of the most beautiful theorems in combinatorial convexity is Tverberg's theorem, which is the r-partite version of Radon's theorem, and it is very closely connected with the multiplied, or colorful versions of the theorems of Helly, Hadwiger and Caratheodory. The first of these colorful versions was discovered by Barany and Lovasz and has many applications (see [3]).

First, Pablo Soberón spoke about Equal coefficients in coloured Tverberg partitions. He analyze a variant of the coloured Tverberg partitions where the convex hulls of the colourful sets are required to intersect using the same coefficients. He give a theorem of this kind with an optimal number of colour classes and points, and extend it to intersections with tolerance.

Ricardo Strausz spoke about A generalization of Tverberg's theorem. In his lecture the following generalization of Tverberg's theorem was presented: every set of $(t+1)(k-1)(d+1)+1$ points in the euclidian d-space admits a k-Tverberg partition with tolerance t. That is, there is a k-Tverberg partition such that, whenever t points are removed from the configuration, the partition of the remaining points is still intersecting. This is a joint work with Pablo Soberon and answers positively a conjecture of Natalia Garcia-Colin.

In fact, Juergen Eckhoff proposed the following open Problem that is variation on Tverberg's Theorem:

Let A be a set of at least $(k-1)(d+1)+2$ points in d–dimensional space. Is it true that A admits a "balanced" Tverberg k–partition, that is, a partition into k subsets whose convex hulls intersect and whose cardinalities differ by at most 1? This is false if A has $(k-1)(d+1)+1$ points. See Discrete Math. 221 (2000) 61-78.

Concerning Radon partitions Ricardo Strausz stated the following problem. How many minimal Radon Partitions determines a configuration? Wold be possible to be $n-d-1$?.

Given $d \geq 1, r \geq 1, k \geq 2$, let $\nu(r,k,d)$ be the maximum number such that there is a set $X \subset \mathbb{R}^d$ of size ν such that for every partition $A_1, A_2, \ldots, A_k$ of X into k parts, there is a subset $C \subset X$ of size r such that

$$\bigcap_{i=1}^{k} \langle A_i \backslash C \rangle_{conv} = \emptyset.$$

The following values of ν are known $\nu(1,2,d) = 2d+2$ for $d = 1,2,3$ done by Larman, $\nu(1,2,4) = 10$, by Las Vergnas, Forge, Schuchert, $\nu(1,2,d) \geq \lceil \frac{5d}{3} \rceil + 3$ for all d by Ramirez-Alfonsín, $\nu(r,2,d) \geq 2d+r+3$ for $r > 1$ of García-Colín, and $\nu(r,k,d) \geq k(\lfloor \frac{d}{2} \rfloor + r + 1) - 1$ by Soberón, and $\nu(r,k,d) \leq (r+1)(k-1)(d+1)$ by Soberón and Strausz.

Then Pablo Soberon conjecture that $\nu(r,k,d) = (r+1)(k-1)(d+1)$.

The discrete center point theorem states that for any finite set $X \subset \mathbb{R}^d$ there exists a *center point* $c \in \mathbb{R}^d$ such that any closed half-space $H \ni c$ contains at least $\left\lceil \frac{|X|}{d+1} \right\rceil$ points of X. where as the dual center point theorem states that for any family of n hyperplanes in general position in $\mathbb{R}^d$ there exists a point c such that any ray starting at c intersects at least $\left\lceil \frac{n}{d+1} \right\rceil$ hyperplanes.

In this set up, Roman Karasev spoke about Projective center point and Tverberg theorems, were he present projective versions of the center point theorem and Tverberg's theorem, interpolating between the original and the so-called "dual" center point and Tverberg theorems. Furthermore he give a common generalization of these and many other known (transversal, constraint, dual, and colorful) Tverberg type results in a single theorem, as well as some essentially new results about partitioning measures in projective space, and focusing on two classical topics in discrete geometry: the center point theorem from Neumann and Rado and Tverberg's theorem.

Many deep generalizations of these classical results have been made in the last three decades, starting from the topological generalization by Bárány, Shlosman, and Szűcz. A good review on this topic and numerous references are given in Matoušek's book. After this book was published, new achievements were made by Hell, Engström and Engström–Norén, K., and Blagojević–M.–Ziegler , establishing "constrained", "dual", and "optimal colorful" Tverberg type theorems.

Here the use of the adjective "dual" is rather descriptive, it does not refer to projective duality. Thus it is interesting to dualize it once more projectively and compare it with the original center point theorem.

Then, the projective dual of the "dual center point theorem" can be stated as follows. Assume that X is a family of n points in $\mathbb{RP}^d$ and $c \in \mathbb{RP}^d$ is another point such that the family $X \cup c$ is in general position. Then there exists a hyperplane $W \subseteq \mathbb{RP}^d$ such that together with any hyperplane $H_1 \ni c$ it partitions $\mathbb{RP}^d$ into two parts each containing at least $\left\lceil \frac{n}{d+1} \right\rceil$ points of X.

From the proof of this theorem Roman Karasev can assure that W does not contain c; however if we omit the general position assumption then the theorem remains true by a compactness argument but W may happen to contain c.

Now he is going to interpolate between the original center point theorem and the latter "dual to dual" version (they appear as special cases when V is the hyperplane at infinity or when V is a point):

He also interpolate between Tverberg's theorem and its dual, and generalize further and state a very general theorem incorporating almost all is know about (dual, transversal, constrained, colorful) Tverberg type theorems.

3 Transversal Theorems

In 1955 Hadwiger [20] posed the problem of determining the smallest number k with the property that if every collection of k members of the family of $n \leq k$ pairwise disjoint unit disks in the plane are met by a line, then all the disks are met by a line; that is, he proposed to find a Helly number for the problem of finding a line transversal to a family of disjoint unit disks in the plane. There is an example proposed in the same paper by Hadwiger, consisting of 5 almost touching disks centered at the vertices of a regular pentagon in such a way that every four of them have a line transversal but the set of all of them does not. Hadwiger's problem was solved by Danzer [10], showing that a Helly number does exist for $k = 5$. In 1989 Tverberg [40] gave a generalization of Danzer's theorem on unit disks for disjoint translations of a compact convex set in the plane.

Denote by $\mathcal{F}$ a family of *ovals* (compact convex sets with non-empty interior) in the euclidean plane and let say that $\mathcal{F}$ has property T if there is a line that intersects all members of $\mathcal{F}$. If there is a line that meets not all but at most k members of $\mathcal{F}$, then $\mathcal{F}$ has the property $T - k$. Finally, if each k-element subfamily of $\mathcal{F}$ has a transversal line, then $\mathcal{F}$ has property $T(k)$. With this notation Danzer theorem cited above, says that $T(5)$ imply T for families of disjoint unit disks, however, it is known that for congruent ovals satisfying $T(5)$ does not imply T in general.

In this workshop Aladar Heppes spoke about an extension of Danzer's theorem for families of moderately overlapping unit discs. Particularly he spoke about finite family of at least five $\frac{2}{3}$-disjoint unit discs. If any 5-tuple if the discs has a line transversal then there is a line meeting all discs. (Joint work with T. Bisztriczky and K. Böröczky).

On this same order of ideas D. Oliveros propose the investigation of the existence of (p, q)–theorem or $T(p, q)$–theorem for transversals, that ensures a property T with some tolerance, That is $T(p, q)$–property will imply, that if out of p discs q of them have a transversal line if this implies that all but k of them have a transversal, the firs natural number to ask is 5 due to Danzer theorem, problem that has been answer negative for A. Holmsen in the case $(5, q)$ for every $1 < q < 5$, but still open for the more general cases, in fact to to D. Oliveros and L. Montejano [34] show the existence of a constant p such that $T(p, p - 1)$ implies $T - k$ (for some k).

The workshop broth also the opportunity to discuss interesting problems of F. Sottile about of linear space transversals to secant linear spaces in $\mathbb{R}^d$ that has deep connections with algebraic geometry.

Fix positive integers $k < d$. For $t \in \mathbb{R}$, let $\gamma(t) := (t, t^2, \ldots, t^d) \in \mathbb{R}^d$. Then $\gamma := \gamma(\mathbb{R})$ is the *moment curve* in $\mathbb{R}^d$.

Let $I \subset \gamma$ be an interval (image of an interval in $\mathbb{R}$). (An affine) linear space L is secant to γ along I if L is affinely spanned by its intersections (necessarily exactly $\dim L + 1$) with I. Such a linear space does not meet $\gamma \smallsetminus I$.

Algebraic geometry (Schubert calculus) together with a result of Mukhin, Tarasov, and Varchenko [35] tells us that if we take general $(d{-}k)$-planes $L_1, \ldots, L_{(k+1)(d-k)}$ secant to γ, then there are finitely many complex k-planes that are transversal to (meet) each L_i. The actual number $\delta_{k,d}$, is huge, it is

$$\delta_{k,d} \ := \ [(k{+}1)(d{-}k)]! \cdot \frac{0!1!2!\ldots k!}{(d{-}k)!(d{-}k{+}1)!\ldots d!} \,.$$

Then F. Sottile problem is concern about conditions on the L_i which force these common transversals to be real. The following conjecture was made in [18], based on extreme (more than 1 tera-Hertz year of computing) experimental evidence and some theoretical justifications.

Secant Conjecture:If the linear spaces $3L_1, \ldots, L_{(k+1)(d-k)}$ are secant to γ along disjoint intervals, then there are exactly $\delta_{k,d}$ real k-planes transversal to each of the L_i.

There are some special cases of this that are known.

First of all, if an interval I of secancy shrinks to a point $\gamma(t)$, then the secant plane becomes an osculating plane. If we replace secant by osculating in this conjecture, we recover the conjecture of Shapiro and Shapiro, which has been proven.

The case $k = d{-}2$ was proven by Eremenko and Gabrielov in a paper in the Annals of Mathematics [16]. It is equivalent to the following statement: A rational function, all of whose critical points lie on a circle, maps that circle to a circle. Their proof used the Uniformization Theorem in complex analysis and was quite difficult. They later gave a proof that used only elementary complex analysis and some algebraic geometry [17]. Neither proof is elementary in any sense.

The general case of the Conjecture of Shapiro and Shapiro has been given two proofs by Mukhin, Tarasov, and Varchenko [35]. These proofs used differential equations, representation theory of quantum groups, hyperplane arrangements, and mathematical physics, and are extremely sophisticated. A consequence of this result is the statement that general secant planes have the expected number of complex transversal k-planes.

The Secant Conjecture is also true if $k = d-2$. In this case it is a statement about rational functions which take the same value at each of $2d - 2$ pairs of real points. This proof relies on the results of [16, 17] and uses a fixed point theorem from topology. The Secant Conjecture is also true if the points of secancy of each linear space form an arithmetic sequence (in the domain $\mathbb{R}$ of the moment curve γ), with the same step size for each linear space [35]. That result of Mukhin, Tarasov, and Varchenko used a similar mix of methods from mathematical physics.

There ample scope for new, elementary ideas. Here are some questions to focus.

Problem 3. In the case $k = 1$ and $d = 3$, the Secant Conjecture asserts that given four lines secant to the moment curve in $\mathbb{R}^3$ along disjoint (think consecutive) intervals, then the two (a priori complex) lines that meet all four are in fact real. Can one find an elementary proof of this fact?

Of the 17 combinatorial configurations of quadruples of secant lines along the projective closure of γ, four can have non-real transversals while the other 13 can only have real transversals. For twelve of the thirteen with only real transversals there is an elementary argument for this reality, and the only one which does not yet have an elementary proof is the configuration of the Secant Conjecture. See § 4 of [18].

Problem 4. Give an elementary proof that there is one real secant k-plane when the linear spaces $L_1, \ldots, L_{(k+1)(d-k)}$ are secant along disjoint intervals? Is there an elementary proof of the Secant Conjecture in any family of subcases?

The Secant Conjecture is much wider than described above. Another class of problems are as follows. Let $a_1, \ldots, a_n$ be positive integers with $a_1 + \ldots + a_n = (k + 1)(n - k)$. Then we have planes $L_1, \ldots, L_n$ secant to γ along disjoint intervals where $\dim L_i = d-k+1 - a_i$, for each i. (The conjecture given above has each $a_i = 1$.) For example, when $k = 2$ and $d = 2a + 1$, we set $n = 4$ and each $a_i = a$. Then there are $a + 1$ real lines meeting four a-planes that are secant along disjoint intervals of γ. It is possible to show this with an elementary argument?

Jorge Ramirez Alfonsin spoke on a problem closely related with the Kneser Theorem about transversals to the convex hull of all subsets of size k of a finite collection of points in $\mathbb{R}^d$. The Kneser Conjecture proved by Lovasz concerns the computation of the chromatic number of the Kneser graphs, which is a purely combinatorial problem. In this talk there is a relation of this problem with the following geometric problem: What is the maximum number n such that any finite set $N \subset \mathbb{R}^d$ of size n has a hyperplane transversal to the family of all convex hulls of k-set of N? It turns out that this number is related to the chromatic number of the Kneser graph $G^2(n, k)$.

In his talk, Ramirez-Alfonsin defined $M(k, d, \lambda)$ = the maximum positive integer n such that every set of n points in $\mathbb{R}^d$ has the property that the convex hull of all k-sets have a transversal $(d - \lambda)$-plane, and he introduced a special *Kneser hypergraph* establishing a close connection between its *chromatic number* and $M(k, d, \lambda)$. In fact, he defined the *Kneser hypergraph* $KG^{\lambda+1}(n, k)$ as the hypergraph whose vertices are $\binom{[n]}{k}$ and a collection of vertices $\{S_1, \ldots, S_\rho\}$ is a hyperedge of $KG^{\lambda+1}(n, k)$ if and only if $2 \leq \rho \leq \lambda + 1$ and $S_1 \cap \cdots \cap S_\rho = \phi$. He remarked that $KG^{\lambda+1}(n, k)$ is the Kneser graph when $\lambda = 1$. Furthermore he noted that the Kneser hypergraph defined by him is different from that defined in [1] and using the cohomology structure of the space of Grasmannians and following the spirit of Dolnikov [13]. It is possible to prove that

$$\chi(KG^{\lambda+1}(n, k)) \leq d - \lambda + 1, \text{ then } n \leq M(k, d, \lambda).$$

Finally, he conjectured that $M(k, d, \lambda) = (d - \lambda) + k + \lceil \frac{k}{\lambda} \rceil - 1$.

In his talk he also discussed recent progress toward the validity of this conjecture in the case when $k = 4$. During the workshop important discussions concerning the valitidity of this conjecture took place. Several important new ideas were developed which hopefully will give rise to the solution of the conjecture. This is a join work with J. Arocha, J. Bracho and L. Montejano.

In this same order of ideas J. Eckhoff proposed a another problem about fractional transversals. Let $\mathcal{F}$ stand for a finite family of convex sets in the plane. What is the smallest number $\alpha > 0$ such that, if $\mathcal{F} \in T(3)$, then some subfamily $\mathcal{G}$ of $\mathcal{F}$ with $\mathcal{F} \geq \alpha|\mathcal{F}|$ has a common transversal?.

Katchalski conjetured that $\alpha \sim \frac{2}{3}$ but Holmsen (2010) showed that $\frac{1}{3} \leq \frac{1}{2}$ and believes that $\alpha = \frac{1}{2}$.

Furthermore, if $N(m, k)$ denote the smallest number n such that, if $|\mathcal{F}| = n$ and $\mathcal{F} \in T(k)$, then some m members of $\mathcal{F}$ have a common transversal. Wegner (unpublished) showed that $N(4, 3) = 6$, and Eckhoff (2008) conjetured that $N(k+1, k) = k+2$ if $k \geq 4$, that was probed by Novick in (2012) for $k \geq 8$. What about the cases $4 \leq k < 8$.

Alfredo Hubard proposed the following problem: Given K and L smooth convex bodies, with the property that $bd(K)$ and $bd(L)$ intersect transversally and assume you know $bd(K) \cap bd(L)$ what can you say about $\tau(K) \cap \tau(L)$? Where $\tau(K)$ is the space of tangent hyerplanes to K.

In 2001, Macdonald, Pach, and Theobald [31] proved that four spheres in $\mathbb{R}^3$ in general position have 12 common complex tangent lines, four unit spheres centered at the vertices of a regular tetrahedron with edge length e satisfying $\sqrt{3} < e < 2$ will have exactly 12 common real tangent lines. And Megyesi considered when the four spheres have coplanar centers [33]. That four unit spheres can have at most 8 common real tangents.

It is not hard to find four unequal spheres with coplanar centers having 12 common tangents. Three spheres of radius 4/5 centered at the vertices of an equilateral triangle with side length $\sqrt{3}$ and one of radius 1/3 at the triangles center have 12 common real tangents.

At the workshop, F. Sottile stated a very interesting set of problems concerning tangent lines to four spheres with exactly 12 common real tangents. Let $\mathcal{C}$ be the set of configurations of four spheres with 12 common real tangents.

Problem 1. Determine the topology of the configuration space $\mathcal{C}$. Is $\mathcal{C}$ connected? Is it possible to continuously transform the tetrahedral configuration into the one with coplanar centers, staying within $\mathcal{C}$? Are there any other (essentially different) configurations of four spheres with 12 common tangents?

All known examples of unit spheres in $\mathcal{C}$ have at least one pair overlapping. Fulton asked if it were possible to find four disjoint unit spheres with 12 common tangents. Theobald and I [39] gave an example of four disjoint spheres with 12 common tangents:

Problem 2. Do there exist four disjoint unit spheres with 12 common tangents? What is the maximum number of isolated real tangent lines to four disjoint unit spheres?

Sottile believe that the answer to the first question is no, but that it would be extremely hard to show that. There are also examples of four disjoint unit spheres with 8 common isolated real tangent lines. For more on this problem of line transversals to spheres, see the survey [39]

4 Finite sets of points and finite sets of convex sets

The Erdös-Szekeres theorem states that every sufficiently large set of points in general position in the plane contains a large subset which is convexly independent. During this workshop there were a bast number of talks focusing in generalizations of Erdös- Szekeres Theorem, for instance, there are several results and conjectures on possible extensions to pseudo-line arrangements or convex sets, and at this respect, Andreas Holmsen presented his joint work with Michael Dobbins and Alfredo Hubard, about a several generalizations of the Erdos-Szekeres theorem, and presented a unified viewpoint and report of their progress on some of these questions.

Alfredo Hubard, spoke about the topology and geometry of the realization spaces by families of convex bodies. He say that two families of convex bodies have the same combinatorial type if there is a selfhomeomorphism of the cylinder $S^{d-1}xR$ that maps the graphs of the support functions of one family to the the graphs of the support functions of the other one. He metrize the space of families of convex bodies with the Hausdorff metric. This talk was about results on the topology and geometry of all families with a fixed combinatorial type.

In fact A. Holmsen proposed the following topological problem: Consider 5 pairs of points in $\mathbf{R}^4$ and let S denote the union of the 32 distinct 4-dimensional simplices obtained by choosing a point from each pair and taking their convex hull. It is known that S is *simply connected*. Show that S is *contractible*.

Furthermore, Xavier Goaoc stated the following problem of finite sets of points

Two ordered n-point sets in the plane are χ-equivalent if for any $1 \leq i, j, k \leq n$, the orientations of the triples points with indices i, j and k are the same in both sets. A chirotope of size n is an equivalence class for that equivalence relation. What is the probability that in a chirotope chosen uniformly at random, the first four points are in convex position?

The following problem on squares (T.Rado 1928) was stated by H. Tverberg. T.Rado asked for the best constant c such that given a finite set of closed axis-parallel squares in the plane,one can find a subset ,consisting of disjoint squares, such that its area is at least c times the original area. He conjectured that $c = 1/4$. It is known that 1/4 works (and is best possible) if the squares are congruent,but M.Ajtai showed by an example (1973) that 1/4 does not work in general. L.Mirsky asked about the special case when the squares have sidelenghts 1 and 2. In that case a fairly simple argument shows that one may reduce the problem to the case when each small square is a square on a (generalized) chessboard while each large one is formed by 2 white and 2 black squares on the board. Does c=1/4 work then? A good set of references is found in a paper by S.Bereg et al. in Algorithmica 57 (2010),538-561.

Edgardo Roldan stated a problem on Partitions related with the Yao-Yao Theorem. Consider the smallest number $N(d, k)$ such that the following holds: For any "nice" measure in $\mathbf{R}^d$ there is a partition of $\mathbf{R}^d$ into $N(d, k)$ convex pieces of equal μ-measure such that every hyperplane avoids at least k of these pieces.

In [41], A. C. Yao and F. F. E. Yao showed that $N(d, 1) \leq 2^d$. This is known as the Yao-Yao Theorem.

B. Bukh asked if $N(d, 1) = O(d)$. A construction was given in [38] that implies $N(d, 1) \geq C2^{d/2}$ for some fixed constant C, however there is still no better upper bound.

Another question is what happens when $k > 1$. One can split $\mathbb{R}^d$ into two pieces of equal measure and construct a Yao-Yao partition in each, this gives a total of 2^{d+1} pieces. Since every hyperplane avoids 2 of them, then $N(d, 2) \leq 2^{d+1}$. Another bound is obtained by iterating the Yao-Yao partition method, after m steps we obtain 2^{md} pieces and every hyperplane avoids $2^{md} - (2^d - 1)^m$. This gives $N(d, 2^{md} - (2^d - 1)^m) \leq 2^{md}$.

These bounds on $N(d, k)$ are rather rough. In [38] it is shown that $N(d, 2) \leq 3 \cdot 2^{d-1}$, but the method used fails for $k > 2$. It would be interesting to find better bounds for $N(d, k)$ than those obtained from simple iterations of this kind.

Remark. The polynomial ham sandwich theorem gives another way to partition a measure in $\mathbb{R}^d$ (see [27] for example). The number of pieces a hyperplane intersects is well controlled but the convexity of the pieces is lost.

5 Conclusions

The workshop was successful in many ways, bringing together old and new colleagues from all over the world. We had participants from many countries including Russia, Germany, France, USA, Mexico, Korea, Canada, Hungary, and Denmark, among others. The talks were far from being the only academic activity of the workshop. We had many formal and informal mathematical discussions and all these activities have given rise to many new research projects and new collaboration.

We appreciate and would like to thank the support we have received from BIRS. The excellent facilities and environment that it provides are perfect for creative interaction and the exchange of ideas, knowledge, and methods within the Mathematical Sciences. We would like to thank programme coordinator Wynne Fong and Station Manager Brenda Williams for all their support in the organization of the conference. We would like to thank as well all the participants of the Recent Advance in Transversal and Helly-type Theorems in Geometry, Combinatorics and Topology Workshop for all their enthusiasm and the productive, enjoyable environment that was created.

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