Strong asymptotics for Cauchy biorthogonal polynomials

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1 Overview of the Field

The main objective of the workshop was to complete a project on the strong asymptotics of Cauchy biorthogonal polynomials [1] and an associated Cauchy two-matrix model introduced in [2].

The model consists of two random Hermitean positive-definite matrices M_1, M_2 of size $n \times n$ equipped with the probability measure

$$d\mu(M_1, M_2) = \frac{1}{\mathcal{Z}_n} \frac{dM_1 dM_2}{\det(M_1 + M_2)^n} \mathrm{e}^{-N\mathrm{Tr}(U(M_1))} \mathrm{e}^{-N\mathrm{Tr}(V(M_2))}$$

where U, V are scalar functions defined on $\mathbb{R}$. The model was termed the *Cauchy matrix model* because of the shape of the coupling term. Similarly to the case of the Hermitean one-matrix models for which the spectral statistics is expressible in terms of appropriate orthogonal polynomials [3], this two-matrix model is solvable with the help of a new family of biorthogonal polynomials named *the Cauchy biorthogonal polynomials* [4]. The Cauchy biorthogonal polynomials are two sequences of monic polynomials $(p_j(x))_{j=0}^{\infty}, (q_j(y))_{j=0}^{\infty}$ with deg p_j =deg $q_j = j$ that satisfy

$$\iint_{\mathbb{R}_+ \times \mathbb{R}_+} p_j(x) q_k(y) \frac{\mathrm{e}^{-N(U(x)+V(y))}}{x+y} d\,x d\,y = h_k \delta_{jk} \ , \ \ \forall j, k \geq 0 \ , h_k > 0 \ .$$

These polynomials appeared initially in an inverse problem for the nonlinear dispersion model (the Degasperis-Procesi equation) [5] and were further developed in [6] in relation with the spectral theory of the cubic string as well as applied to other nonlinear partial differential equations [7].

2 Recent Developments

One of the problems we set out to solve was the large $N = n + r$ asymptotics of the Cauchy biorthogonal polynomials $(p_j(x))_{j=0}^{\infty}, (q_j(y))_{j=0}^{\infty}$. This required first the formulation of an appropriate Riemann-Hilbert problem and that was accomplished in [1]. With the help of suitable equilibrium potentials, called below ρ_1 and ρ_2 and a sequence of deformation we formulated the outer and inner parametrix problems essentially following the machinery laid out by Deift and Zhou [8] with an important modification that our RH problem deals with 3×3 matrices, rather than 2×2, and it exhibits a different asymptotic behaviour at infinity. For example

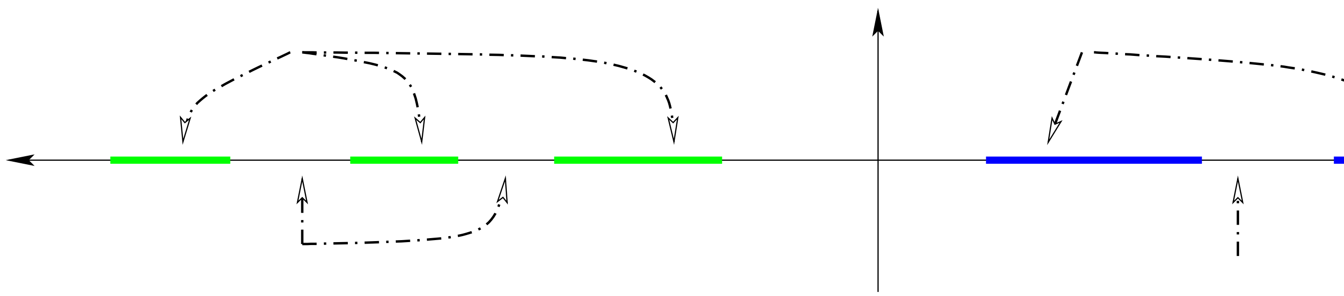

Figure 1: The RHP for the outer parametrix

Problem 2.1 (Outer parametrix)**.** *Find a 3×3 matrix $\Psi(z)$, analytic in $\mathcal{D}_0 := \mathbb{C}\setminus((-\infty, b_0]\cup[a_0,\infty))$ and with the following properties*

(i) the jumps indicated in Figure 1 with specific values

$$\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}\ \textit{on the left (green) cuts}\qquad \begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}\ \textit{on the right (blue) cuts}$$

$$\begin{pmatrix}1&0&0\\0&e^{-2\pi i N\sigma_l}&\\0&0&e^{2\pi i N\sigma_l}\end{pmatrix}\ \textit{on the left l-th gap}\qquad \begin{pmatrix}e^{2\pi i N\epsilon_l}&0&0\\0&e^{-2\pi i N\epsilon_l}&0\\0&0&1\end{pmatrix}\ \textit{on the right l-th gap}$$

(ii) the growth conditions at $z=\infty$ and near an endpoint are, respectively

$$\Psi(z)\sim\left(\mathbf{1}+\mathcal{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix}z^r&&\\&1&\\&&z^{-r}\end{pmatrix}\ ,\ \Psi(z)=\mathcal{O}\left((z-a)^{-\frac{1}{4}}\right)\ ,\ a\in\{a_i,b_i\}_{i=1,\dots}$$

We solved completely the RH problem 2.1 by giving an explicit solution in terms of the Riemann Theta functions associated to certain Riemann surface $\mathcal{L}$ which can be realized as a double of the bordered Riemann surface obtained by gluing together three Riemann spheres slit along the support of one of the equilibrium measures ρ_1 and glued there with the middle Riemann sphere which is subsequently slit also along the support of the second equilibrium measure ρ_2 and glued across it with the third Riemann sphere, cut along the support of ρ_2. More details can be found in [9].

3 Outcome of the Meeting and Open Problems

The original project was completed and in fact during the workshop the paper reporting the results was submitted as well as posted on the archives [9]. As expected, all spectral statistics, including gap probabilities, were proved to follow the standard universality results for the one-matrix Hermitean model. However, it is expected that new universal behavior will appear at the zero eigenvalue where two interacting matrices (a positive-definite M_1 and a negative-definite $-M_2$) in some sense come "close" one to another. We started investigating this scaling regime by looking at the concrete example, which is of independent interest as it involves certain classical special functions. More concretely, during the workshop we made a significant progress on the explicit construction of both Cauchy biorthogonal polynomials as well as the accompanying two-matrix model for the case of two Laguerre-type measures $d\mu = x^a e^{-x}, d\nu = x^b e^{-x}, x>0$.

Theorem 3.1. *Given two Laguerre-type measures $d\mu, d\nu$ specified above let us set $\alpha := a + b$. Then the associated Cauchy biorthogonal polynomials p_n, q_n are expressed as (they are normalized so that the leading coefficient is the same)*

$$p_n(z) = (-1)^n \sqrt{2n+\alpha+1} \sqrt{\frac{\Gamma(a+n+1)}{\Gamma(b+n+1)}} \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)\Gamma(a+1)} {}_2F_2(-n, \alpha+n+1; a+1, \alpha+1; z)$$

$$q_n(z) = (-1)^n \sqrt{2n+\alpha+1} \sqrt{\frac{\Gamma(b+n+1)}{\Gamma(a+n+1)}} \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)\Gamma(b+1)} {}_2F_2(-n, \alpha+n+1; b+1, \alpha+1; z)$$

and they satisfied 3*-rd order differential equations:*

$$\left[z\left(\Delta-n\right)\left(\Delta+\alpha+n+1\right)-\Delta\left(\Delta+a\right)\left(\Delta+\alpha\right)\right]p_n=0,$$
$$\left[z\left(\Delta-n\right)\left(\Delta+\alpha+n+1\right)-\Delta\left(\Delta+b\right)\left(\Delta+\alpha\right)\right]q_n=0,$$

where $\Delta = z\frac{d}{dz}$.

The fact that the Cauchy biorthogonal polynomials are expressed in this case by hypergeometric functions of type ${}_2F_2$ is a welcoming sign that the theory of these polynomials is an extension of "classical function theory". Moreover, the scaling limit points to a new universal behaviour:

Theorem 3.2. *The orthonormal $p_n(z), q_n(z)$ in the scaling regime behave as*

$$q_{n+r}(zn^{-2}) = (-1)^{n+r+1} \left(1 + \frac{r(\alpha+2\Delta) + \frac{(\alpha+1)(\alpha+2\Delta)}{2}}{n} + \frac{1}{n^2}\left(\frac{(\alpha+2\Delta)(\alpha+2\Delta-1)r^2}{2} + rC_1 + C_2\right)\right) G(z)$$

where C_1, C_2 are two operators that are independent of r (and n) while $G(z)$ is the Meijer G function

$$G(z) \equiv G^{1,0}_{0,3}\left(z\middle|0,-\alpha,-a\right) = \frac{1}{2\pi i}\int_\gamma \frac{\Gamma(u)}{\Gamma(1+\alpha-u)\Gamma(1+a-u)} z^{-u} du.$$

Conjecture 3.1. *All correlation functions in the scaling regime $\frac{z}{n^2}, n \to \infty$, can be expressed as differential expressions in the the scale invariant Δ applied to the Meijer G function.*

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