Isometries and isomorphisms of spaces of continuous functions

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1 Overview of the Field

This is a report of the work of four colleagues as part of 'Research in Teams' at BIRS; the reference is 12rit146.

Our work has split into two rather distinct, but related, parts, and we summarize them separately.

The first part is directly concerned with isometries and isomorphisms of spaces of continuous functions.

Let K be a locally compact space. Then $C_0(K)$ denotes the collection of all the continuous, complex-valued functions on K that vanish at infinity, so that $C_0(K)$ is a linear space with respect to the pointwise operations. The *uniform norm* on K is denoted by $|\cdot|_K$, so that

$$|f|_K = \sup\{|f(x)| : x \in K\} \quad (f \in C_0(K)).$$

Then $(C_0(K), |\cdot|_K)$ is a Banach space. Further, with respect to the product given by pointwise multiplication of functions, $C_0(K)$ is a Banach algebra; it is a commutative C^*-algebra.

We shall be concerned in particular with the space $C(K)$ when K is totally disconnected, or Stonean; the latter means that K is compact and extremely disconnected, in the sense that pairs of disjoint open sets in K have disjoint closures.

We recall that the real Banach lattice $C(K)_{\mathbb{R}}$ is Dedekind complete if and only if the space K is Stonean.

Much of our work is formulated in the language of Boolean algebras. It is standard that the Stone space of a Boolean algebra B is the collection of ultrafilters on B, taken with the Stone topology; this gives the Stone space $St(B)$ of B. This space is always compact and totally disconnected, and it is Stonean if and only if B is complete. Important examples are the σ-field of Borel sets of a compact space and the space of regular-open subsets of a topological space.

We recall that two Banach spaces E and F are isomorphic, written $E \sim F$, if there is a continuous linear bijection from E onto F, and are isometric, written $E \cong F$, if there is an isometry from E onto F. We are concerned with the question when a Banach space E is isomorphic or isometric to a space of the form $C(K)$, and when two spaces of the form $C(K)$ and $C(L)$ are isomorphic. There is a substantial difference between the isomorphic and isometric theories. The classical Banach–Stone theorem tells us that $C(K) \cong C(L)$ if and only if K and L are homeomorphic, and Milutin's theorem states that $C(K) \sim C(L)$ when both K and L are compact, uncountable, metric spaces. However many basic questions remain open. For example, is it true that, for each compact space K, the Banach space $C(K)$ is isomorphic to $C(L)$ for some totally disconnected space L?

We are also concerned with the question when a space of the form $C(K)$ is isomorphically or isometrically the dual of another Banach space, and when it is the bidual of such a space. Many classical theorems discuss this situation. For example, it is well-known that a separable Banach space which is isomorphically a dual space has the Krein–Milman and Radon–Nikodym properties.

Indeed, it is very classical that $C(K)$ is isometrically a dual space (so that $C(K)$ is a von Neumann algebra) if and only if K is hyper-Stonean, in the sense that the union of the supports of the normal measures on K is dense in K; this notion goes back to Dixmier [2]. The normal measures on K are the measures on K that are order-continuous on $(C(K)_{\mathbb{R}}, \leq)$, and it is standard that the positive normal measures are those positive measures μ on K such that $\mu(L) = 0$ whenever L is a compact subspace of K such that $\mathrm{int}_K L = \emptyset$. The space $N(K)$ of normal measures is a closed subspace of $M(K)$, the Banach space of all measures on K. In the case where K is Stonean, $N(K)$ is the unique isometric predual of $C(K)$.

Important examples of spaces of the form $C(K)$ are the Baire functions. The Baire functions of order 0 are the continuous functions on $[0, 1]$, now denoted by $B_0(K)$. Let α be an ordinal with $\alpha > 0$. Given a definition of the Baire class of order β for each $\beta < \alpha$, we define B_α, the Baire class of order α, to be the space of bounded functions on $[0, 1]$ which are pointwise limits of sequences of functions in the union of the earlier classes; the recursive construction terminates at $\alpha = \omega_1$. The Baire functions on $[0, 1]$ are the members of the final class, B_{ω_1}. The spaces B_α are commutative C^*-algebras for the pointwise product, and so have the form $C(\Phi_\alpha)$ for a compact space Φ_α.

Let K be a locally compact space. Then the commutative C^*-algebra $C_0(K)$ has a second dual which is also a commutative C^*-algebra, and so this second dual has the form $C(\widetilde{K})$ for a certain compact space $\widetilde{K}$, called the hyper-Stonean envelope of K.

The second part of our work is concerned with weakly almost periodic functions on algebras of measures.

Let A be a Banach algebra. An element $\lambda \in A'$ is *weakly almost periodic* if the map $a \mapsto a \cdot \lambda,\ A \to A'$, is weakly compact. The space of these elements is denoted by $WAP(A)$.

Let G be a locally compact group, and let $(M(G), \star)$ be the measure algebra on G. Then we have the commutative C^*-algebra $C(\widetilde{G})$, as above. It is easy to see that we can identify $WAP(M(G))$ as a closed subspace of $C(\widetilde{G})$. It is a very striking theorem of Daws [1] that, in fact, $WAP(M(G))$ is a C^*-subalgebra of $C(\widetilde{G})$. However it is not easy to identify which elements of $C(\widetilde{G})$ belong to this algebra. Some partial results will be discussed below.

2 Recent Developments and Open Problems

The subject of isometries and isomorphisms of spaces of continuous functions was intensively studied some time ago; many strong results were obtained, but apparently basic open questions were left unresolved. There have been few recent advances, save perhaps in our own work. Here are some open questions.

Question 1 *Is it true that, for each compact space K, the Banach space $C(K)$ is isomorphic to $C(L)$ for some totally disconnected space L?*

Question 2 *Are any or all of the Banach spaces B_α and B_β pairwise isomorphic in the cases where $2 \leq \alpha, \beta < \omega_1$ and $\alpha \neq \beta$?*

Question 3 *Let X be a compact space. Suppose that $C(X)$ is isometrically isomorphic to the second dual of some Banach space. Does there always exist a locally compact space K such that $C(X)$ is isometrically isomorphic to $C(\widetilde{K}) = C_0(K)''$?*

Question 4 *Let E be an injective Banach space. Is E isomorphic to a 1-injective space? Let K be a locally compact space such that $C_0(K)$ is an injective Banach space. Is $C_0(K)$ isomorphic to $C(L)$ for some Stonean space L?*

Question 5 *Let K be a compact space such that $C(K)$ is isomorphically a dual space. Is K necessarily totally disconnected? Does there exist a Stonean space L such that $C(K)$ is isomorphic to $C(L)$?*

Question 6 *Let K be a compact space such that $C(K)$ is isomorphically a dual space, and suppose that L is a clopen subspace of K. Is $C(L)$ also always isomorphically a dual space?*

The main questions associated with the theory of weakly almost periodic functions on algebras of measures is the following.

Question 7 *Let $\mathbb{T}$ be the unit circle, a compact group, and let χ be the characteristic function of the spectrum Φ of $L^\infty(\mathbb{T})$, where Φ is regarded as a clopen subset of the hyper-Stonean envelope of $\mathbb{T}$. Does χ belong to $WAP(M(\mathbb{T}))$?*

Question 8 *Is there always a topological invariant mean on $WAP(M(G))$. If so, is it unique?*

3 Presentation Highlights

Since this was a workshop for four people assembled for 'Research in teams', there were no formal presentations.

4 Scientific Progress Made

We made progress in several areas, but we did not fully resolve any of the above open questions.

1) The standard discussion of properties defining hyper-Stonean spaces assume in advance constructed gave several new examples of spaces K for which $N(K) = \{0\}$ and $N(K) \neq \{0\}$. Some constructions involve the Gleason space G_K of K; some involve the Stone–Cech compactification of certain spaces. Further, we established the following theorem. Let K be a compact space. Then the union of the supports of the normal measures on K is dense in K if and only if the Gleason cover of K is hyper-Stonean.

2) We gave some new, more explicit, constructions of the hyper-Stonean envelope of a compact space.

3) We gave some new examples showing when $C(K)$ is isomorphically a dual space. For example, one can have this property even when $N(K) = \{0\}$.

4) Question 3, above, was already resolved by Lacey in an old paper [4] in the case where $C(X)$ is isometrically isomorphic to the second dual of a separable Banach space; see also [5]. The analogous question in the isomorphic theory of Banach spaces was resolved in a similar way by Stegall [8]; for related work, see [3]. However, all these works are rather complicated and difficult to follow. We have developed a different approach to this theorem, using background from topology and Boolean algebras: it provides a simpler proof of known results and some new results. We have hopes that it will resolve the general question.

5) We have compared the C^*-algebra $WAP(M(G))$ with the space $B^b(G)$ of bounded Borel functions on G, regarded as a subspace of $C(\widetilde{G})$.

6) Question 8, above, asks for an extension of an old result of Ryll-Nardzewski [7] given in the case where G is discrete. We obtained a partial result by showing that $WAP(M(G))$ always has a left-invariant mean; this used a fixed-point theorem from [6].

Our work is being written in two separate articles. The first, 'Isometries and isomorphisms of spaces of continuous functions' currently contains around 60 pages; the second, 'Weakly almost periodic functions on algebras of measures' currently contains around 14 pages.

5 Outcome of the Meeting

The four participants are continuing to work on preparing our work for publication. In particular, Dales, Dashiell, and Lau will meet again in Los Angeles in October, 2012, to attempt to make further progress

Lau will attend a conference on *Harmonic analysis* in Luminy, in October, 2012, and will discuss the second paper mentioned.

A proposal has been made to the *Fields Institute* in Toronto for a Thematic Program on *Banach algebras and harmonic analysis* in the first half of 2014; Dales and Lau are among the organisers of this programme. It is expected that questions related to our work, and related matters, will be discussed during this semester.

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