CLUSTER ALGEBRAS, REPRESENTATION THEORY, AND POISSON GEOMETRY

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1. Overview of the Field

1.1. **Cluster algebras.** Cluster algebras were introduced in 2000 by S. Fomin and A. Zelevinsky [26] as a tool for studying dual canonical bases and total positivity in semisimple Lie groups. They are constructively defined commutative algebras with a distinguished set of generators (cluster variables) grouped into overlapping subsets (clusters) of fixed cardinality. Both the generators and the relations among them are not given from the outset, but are produced by an elementary iterative process called seed mutation. This procedure appears to be closely related to constructions in various other fields, such as Poisson geometry, Teichmüller theory, representation theory of finite dimensional associative algebras and Lie theory and Coxeter groups. The theory of cluster algebras was further developed in the subsequent papers [27, 28, 3, 4, 29, 16, 17]. Remarkably, in the last two papers of this series superpotentials borrowed from mathematical physics play a prominent role. By now cluster algebras form a very active area of research which has obtained its own AMS classification number 13F60. A thematic semester in 2012 at the Mathematical Sciences Research Institute in Berkeley will be devoted to cluster algebras (more details on activities related to cluster algebras can be found on Fomin's cluster algebra portal at http://www.math.lsa.umich.edu/∼fomin/cluster.html).

1.2. **Quantum Cluster algebras.** In 2005 Berenstein and Zelevinsky [4] introduced quantum cluster algebras as non-commutative deformations (quantization) of cluster algebras and studied their basic properties. Note, that in the classical limit ($q = 1$) they give rise to a Poisson structure on the cluster algebra as introduced and studied in [31], see [4, Rem. 4.6]. The main motivation for this work was to start a general theory of canonical bases for cluster algebras. [4] finishes with some evidence that the quantised coordinate rings of double Bruhat cells carry have a natural quantum cluster structure. Apart from the somehow parallel developments by Fock-Goncharov [21], [22] and Kontsevich-Soibleman [42], quantum cluster algebras passed several years almost unnoticed. However, since 2010 there has been quite some activity in this direction. For example, the the following papers deal quite explicitly with quantum cluster algebras: [33], [34], [38], [45], [53], [52], [18], [19], [41], [46], [30], [54] (in chronological order, as the papers were posted on arXiv). Moreover, [48], [40] made clear the relevance of the work of Fock-Goncharov and Kontsevich-Soibelman to the further study of quantum cluster algebras.

Date: January 2012.

2. Recent Developments and Open Problems

2.1. **Additive Categorification of Cluster Algebras.** Cluster categories were introduced in [6], and, for type A_n also in [8], as a means for a better understanding of the acyclic cluster algebras. They are defined as orbit categories $\mathcal{C}_Q = D^b(\operatorname{mod} kQ)/\tau^{-1}[1]$ of the bounded derived categories of finitely generated kQ-modules, where Q are acyclic quivers. This construction has been generalized in [1] to quivers with non-degenerate potential whose Jacobian algebra is finite-dimensional, and the finiteness condition has been removed in [51].

The two classes of cluster algebras whose categorifications are currently best understood are the acyclic ones and the cluster algebras stemming from surfaces: Fomin, Shapiro and Thurston study in [23] the cluster algebra defined by a marked surface. They show that arcs (isotopy classes of curves without self-intersections) on the marked surface correspond to the cluster variables of this cluster algebra, and that mutations correspond to flips of arcs. In [2] it is shown for unpunctured surfaces that the Jacobian algebra of the associated quiver with potential is gentle. D. Labardini generalizes in [44] the definition of a potential to punctured surfaces, but the representation theory of the corresponding Jacobian algebras is not known at this moment.

Based on these results, in [5] the cluster category of [1] defined by an unpunctured surface is described concretely in terms of curves in the (unpunctured) marked surface. The curves without self-intersections correspond to objects in the category without self-extensions. It would be desirable to further develop these ideas: Earlier work of Chekhov and current work of Tumarkin and others points at a generalization of the above mentioned results from surfaces to orbifolds. They show, in particular, that all cluster algebras which are defined by a mutation-finite quiver are given by an orbifold construction. Moreover, string theorists became interested in mutation-finite quivers, since these classify BPS-states of $\mathcal{N} = 2$ complete 4d quantum field theories, as shown in [9].

2.2. **Bases of cluster algebras.** Since cluster algebras were introduced with the idea that they should help to understand canonical bases, it is very natural to consider the problem of finding a nice linear basis for cluster algebras: this basis should include the cluster monomials, and it should have good positivity properties.

An element of a cluster algebra is called *positive* if, when it is expressed as a Laurent polynomial in the cluster variables of *any* cluster, the numerator has positive coefficients. The Positivity Conjecture of Fomin and Zelevinsky can be viewed in this light as saying that the cluster variables are positive elements. The positive elements necessarily form a semiring in the cluster algebra (i.e., they are closed under sum and product).

A linear basis of a cluster algebra is called *atomic* if the positive linear combinations of the basis elements coincides with the semiring of positive elements. The existence of such a basis puts strong conditions on the positive semiring, so it is not at all clear that such a basis should exist. However, if it exists, it is unique.

In the case of cluster algebras of finite type, the answers are very satisfactory: the cluster monomials form a basis for the cluster algebra, they correspond to the dual canonical and semi-canonical bases (which agree), and this basis is atomic (proved by G. Cerulli Irelli [10]). Outside finite type, the cluster monomials no longer form a basis, and there are few results about the existence of atomic bases.

2.3. **The Open Orbit Conjecture.** Consider for a symmetric Kac-Moody group G the quantisation $\mathcal{O}_q(N)$ of the regular functions on the maximal, pro unipotent subgroup N of G. It is well known that this is isomorphic to $U_q(\mathfrak{n})$, the quantisation of the enveloping algebra of the corresponding pro-nilpotent Lie algebra $\mathfrak{n}$. By work of Lusztig and De Concini-Kac-Procesi, for each Weyl group element w, the algebra $\mathcal{O}_q(N)$ contains a subalgebra $\mathcal{O}_q(N(w))$ which can be considered as a quantisation of the coordinate ring of the unipotent subgroup $N(w) \subset N$ attached to w. Moreover, each $\mathcal{O}_q(N(w))$ is spanned by a subset of the dual of Lusztig's canonical basis of $U_q(\mathfrak{n})$, see for example [41]. Quite close to the conjecture [4, 10.10], it was shown in [30] that each of those quantised coordinate rings $\mathcal{O}_q(N(w))$ has a quantum cluster algebra structure with the initial seed given by certain quantum minors (which belong to the dual canonical basis). The open orbit conjecture in its sharpest version states that all quantum cluster monomials of this structure belong to the dual canonical basis. A slightly weaker version claims that this is true after specialisation to $q = 1$. In other words, the cluster monomials belong to the specialised dual canonical basis.

Note, that elements of (dual) canonical basis are notoriously difficult to come by. This conjecture would allow to obtain many of those elements by a relatively easy recursive procedure. An important application would be a proof of Conjecture 13.2 in [36] via quantum Schur-Weyl duality in type A: The Grothendieck rings of the finite dimensional modules over the quantum affine algebras $U_q(\widehat{\mathfrak{sl}}_n)$ which belong to the category $\mathcal{C}_l$ carry a cluster algebra structure, where the cluster monomials correspond to the classes of real simple representations see [36, 13.7] for more details. Here, $\mathcal{C}_l$ are subcategories where the roots of the Drinfeld polynomials of the simple composition factors fulfil certain integrality and boundedness conditions.

Special cases of this conjecture have been proven recently by several participants of this workshop, all centred around the case $w = c^2$, i.e. when the corresponding cluster algebra is acyclic.

3. Presentation Highlights

Below, we discuss some of the highlights of the presentations at the workshop. In the interest of bringing out certain themes of the workshop, not all the talks are mentioned individually.

3.1. **Cluster algebras and additive categorifications.** I. Reiten gave an overview on the construction of (additive) cluster categories. P.-G. Plamondon presented the construction of a "generic basis" S for a cluster algebra, using his version of a generalized cluster category given by a quiver with non-degenerate potential. In the setting studied by Geiss, Leclerc and Schröer, this basis S coincides with the dual semicanonical basis. Moreover, for affine Dynkin type $\tilde{\mathsf{A}}_n$ it yields the same basis obtained by Dupont.

However, in the general situation the generic basis S is only a basis for the upper cluster algebra, and it is not clear when the upper cluster algebra and the cluster algebra coincide. In Greg Muller's talk it was shown that the upper cluster algebra and the cluster algebra coincide under certain conditions — this result was obtained during the workshop.

3.2. **Bases of cluster algebras.** Gregg Musiker spoke about the problem of finding linear bases for cluster algebras arising from surfaces without punctures. His talk was based on joint work with Ralf Schiffler and Lauren Williams, both of whom were also in attendance. In [47] (which appeared shortly after the workshop), they provide two linear bases, denoted $\mathcal{B}$ and $\mathcal{B}^\circ$ for such cluster algebras, with coefficients allowed, under the hypothesis that the B-matrix is of full rank.

The basis $\mathcal{B}^\circ$ is conjectured to be atomic. This is known for disks (for which the associated cluster algebra is of finite type, and the two bases both coincide with the cluster monomials) and for annuli when the coefficients are set to 1, by [20].

The basis $\mathcal{B}$, on the other hand, is conjectured to agree with the generic basis, already mentioned above.

3.3. **Quantum cluster algebras and monoidal categorification.** In the quantum setting, Ph. Lampe showed for the cases $\tilde{\mathsf{A}}_1$ and A_n by rather direct computations that all quantum cluster variables belong to the dual canonical basis [45],[46]. Next, D. Hernandez and B. Leclerc [36] showed in the classical setting that for the cases A_n and D_4 indeed all cluster monomials belong to the dual canonical basis by a combinatorial approach via the characters of certain finite dimensional simple representations of the corresponding quantum affine algebras. This was then generalised in some sense to all acyclic types with sink-source orientation by H. Nakajima, using a monoidal categorification via perverse sheaves and Fourier-Deligne-Sato transformation for quiver varieties [49]. In fact, there the dual canonical basis of the decorated quivers has to be considered. Fan Qin reported in his talk about how to remove the sink-source hypothesis in Nakajima's result. Finally, D. Hernandez and B. Leclerc obtained, as an corollary of their recent work on the t-deformations of q-characters of finite dimensional representations of quantum affine algebras the quantum version of this conjecture for all (finite) Dynkin types. They presented this result during the conference, and one should note, that a first version of their paper [37] was uploaded to arXiv during the conference. Finally, A. Berenstein presented his recent work with A. Zelevinsky where they show, with a combinatorial approach, that in the acyclic cases the quantum cluster monomials belong to a differently defined "canonical basis". It should be very interesting to compare those results. Notice, that the impressive results by Hernandez-Leclerc and Nakajima are based on a monoidal categorification approach to cluster algebras, see [39, 4.4] for a concise description of this concept.

3.4. **Cluster Algebra and Poisson Geometry.** Poisson structures compatible with cluster algebra structure were introduced in [31]. Namely, there exists a Poisson structure such that cluster variables of any cluster form a log-canonical (or, log-constant) basis for this Poisson structure. The interplay between cluster and Poisson structure plays an important role for cluster theory. Of particular interest is the study of completely integrable systems associated with cluster dynamics. The Poisson part of the cluster theory was represented at the workshop by M.Gekhtman, Ph. Di Francesco, R. Kedem, P. Tumarkin, M. Yakimov, and S. Zwicknagl.

M. Gekhtman gave two talks: one a general introduction to the Poisson geometry and integrable systems, and the second one on the Poisson bracket compatible with cluster algebras mainly based on the paper [31].

R. Kedem reported on her joint research with Ph. Di Francesco that was published in the series of papers [11, 12, 13, 14, 15] where they studied from the cluster point of view Q- and T-systems and their quantizations that arise in various stages of the study of integrable quantum spin chain. These systems are particular cases of the so-called discrete Hirota equation which is known to be completely integrable. The cluster algebra reformulation allows the construction of explicitly conserved quantitites of the discrete dynamics. It leads also to continued fractions whose positivity implies positivity of solutions. These results can be partially generalized to the non-commutative case of quantum Q- and T-systems. Another approach to Q-systems from the point of view of directed networks in the annulus, cluster algebras, and Toda lattices was studied in [32].

S. Zwicknagl studied symplectic leaves of the cluster manifold. The main tool is the description of the structure of toric invariant Poisson ideals in upper cluster algebra [57]. M. Yakimov talked about interplay between cluster algebras and ring theory of quantum function algebra and the underlying Poisson Geometry [55, 56].

Finally, P. Tumarkin reported on the classification of cluster algebras of finite mutation type [24, 25]. This result allows to complete classification of cluster algebras by the growth rate (finite versus polynomial versus exponential). The classification in skew-symmetrizable case is associated with triangulation of two-dimensional bordered surfaces with orbifold points of order 2.

4. Scientific Progress Made

4.1. **Quantum Cluster Character.** Another important direction in the recent study of quantum cluster algebras is the search for a "quantum cluster" character. This would mean in particular closed formulae for the quantum-Laurent expansion of quantum cluster monomials with respect to any cluster, similar to the well known formulae of Caldero-Chapoton [7], Palu [50] and in the classical case. Note, that in those cases the relevant coefficients are given by the Euler-Characteristics of certain quiver Grassmannians.

First results in this direction where found by D. Rupel [53] and Fan Qin [52] who found formula for acyclic seeds, where the above mentioned Euler characteristics are replaced by Counting polynomials resp. Serre polynomials. However, elementary examples show, that this nice idea cannot be pushed further so easily. As K. Nagao explained in his lecture, from Kontsevich-Soibelman's work one should expect in the general case an answer in terms of motivic integration (which is not a topological invariant of the relevant quiver Grassmannians, but depends also on the potential associated to the seed). Unfortunately, this is for the quantum case still conjectural, since it depends then still on a deep conjecture by Kontsevich and Soibelman that the integration map from the corresponding motivic Hall algebra to the quantum torus is in fact an algebra homomorphism. So again, interestingly progress is confined for the moment to the acyclic situations.

4.2. **Integrability.** Study of integrable systems associated with cluster algebra was continued in the joint work [35]. The pentagram map associates to a projective polygon a new one formed by intersections of short diagonals. In [35] the pentagram map is included into a family of discrete completely integrable systems associated with some special cluster algebras. The main tool is Poisson geometry of weighted directed networks on surfaces. The ingredients necessary for complete integrability —invariant Poisson brackets, integrals of motion in involution, lax representation— are recovered from (cluster) combinatorics of the networks.

4.3. **Teichmüller theory.** Developing the approach outlined in the talk by P. Tumarkin a generalization of cluster algebra was constructed in the joint work by

L. Chekhov and M. Shapiro. In particular, a cluster algebra can be associated to the triangulation of a surface with orbifold points of any order. The results are in the process of preparation for publication.

5. Outcome of the Meeting

The workshop was very successful. It provided an opportunity to hear some of the newest work on cluster algebras. It also provided time for informal discussions among the participants. Some of the participants had not met before or had not previously had extensive opportunities to discuss mathematics; well-established teams of collaborators also had time to work on their ongoing projects.

We record here some comments which we received from participants.

Qin Fan wrote: "During that week, Prof. Hernandez and Leclerc posted a new paper on Arxiv. Thanks to the workshop, I could learn their results immediately from their talks and private conversations. Their work helps me finish the final part of one of my recent researches on cluster algebras."

Bernard Leclerc wrote: "For me the workshop was important in that it allowed me to present my new joint work with David Hernandez, on quantum Grothendieck rings and derived Hall algebras, in front of an ideal audience filled with experts in various aspects of this work: Nakajima, Keller, Nakanishi, Qin, Lampe, etc... This has been a strong stimulus for developing and writing up this work during the summer, in order to get it ready for the workshop. During the workshop Nakajima and Keller gave us important feedback, which was very appreciated."

Greg Muller wrote: "It would be an understatement to say that my participation in the conference was a tremendous boost to my research. At BIRS, I met dozens of researchers, and was able to talk with them about my research and present my results to them. My interactions were so fruitful, I have since been invited to visit several possible collaborators I met there. While I was at the conference, I developed the idea of 'locally acyclic cluster algebras', which I have already proved some exciting results about. I am currently finishing work on a preprint containing these results; and this might not be the only paper I write as a direct result of my participation there."

Gregg Musiker wrote: "Since it has only been a few months since the workshop, the most tangible research that I can point to is a recent paper "Bases for cluster algebras from surfaces" with Ralf Schiffler and Lauren Williams. Having time at Banff to discuss some finishing touches with my collaborators, as well as discuss questions that arose about related results with the other experts attending the workshop was quite useful. Several new contacts were also beneficial, including Demonet, Early, Muller, Plamondon, Tumarkin, among others. I had heard about the work of several of my new contacts (and in some cases been at the same conferences as them before) but this was the first opportunity to have long conversations with them about their research, which were quite enlightening to us both.

I am now in the process of pursuing some of the references and ideas that I learned about while at BIRS (both through lectures and discussions around the dining hall). Some of these projects include looking further at atomic bases, as discussed with Schiffler, Thomas, and Williams; investigating more connections to character varieties, as discussed with Muller; and studying (generalized) cluster categories and their relation to surfaces, as discussed with Brüstle, Keller, Plamondon, and Todorov."

Sebastian Zwicknagl wrote: “First, I learned from Greg Muller about the geometry of cluster varietes. It helped me understand the nature of the singularities and how to use Poisson structures to give criteria when such a variety is smooth.

Secondly, I learned from Milen Yakimov about ring theory and Allen Knutson about the algebraic geometry, e.g. the relationship between Frobenius splitting and Poisson structres, resp. cluster algebras.

Finally, I want to mention the lectures on quiver varieties and cluster algebras, as well as Rinat Kedem’s and Lauren Williams’ work which I found really interesting, and I think I now understand much better.”

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