REPORT ON THE BIRS WORKSHOP: "GEOMETRIC PROPERTIES OF SOLUTIONS OF NONLINEAR PDE'S AND THEIR APPLICATIONS"
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1. Overview of the field

The study of qualitative properties of solutions is one of the most important areas in the field of partial differential equations. Several directions of research can be included in this area and the workshop covered in various ways most of them.

Among these different directions, the aim of the workshop was to focus on convexity type properties of solutions to boundary value problems for elliptic and parabolic operators. Although many important results in this area were obtained in the '90, this is still a very active field and recent developments gave new impulse to this topic in the last years. The main target here is to determine structure conditions on the differential operator which allows to deduce convexity, or power convexity, or log–convexity, or quasi–convexity of the solution. The techniques used to this aim can be roughly divided in two types: microscopic and macroscopic techniques. It is remarkable that both techniques, quite different in nature, appeearently lead to the same conclusions. We note also that recently these techniques have been refined and adapted to more general situations including boundary–value problems for fully non–linear elliptic operators and to elliptic and parabolic nonlinear equations on Riemannian manifolds. One recent example is given by the breakthrough of the Christoffel–Minkowski problem. In this case establishing the convexity of the solution of a certain elliptic equation on the unit sphere is equivalent to prove the existence of a convex body having prescribed area measure of suitable order. Another recent development regards the discovery of the strong connection existing between convexity properties of solutions to elliptic and parabolic pde's and Brunn-Minkowski type inequalities for associated variational functionals.

Further than this main topic and included in the area of the workshop, there are other interesting arguments. For instance the study of overdetermined boundary value problems (or of free boundary problems; in some sense, two different ways to deal with the same question, which can be included also in the area of inverse problems). In this connection, we recall that in recent times, starting with a well–known result by Serrin, we assisted to the characterization of the sphere as the unique domain where some overdetermined boundary value problems can be solved (with radial solutions). Despite the number of extensions of Serrin's result, several interesting problems are still open in this subject.

Great interest is also devoted to the various symmetrization and rearrangement techniques and their applications, including the solution of geometric and physical problems through elliptic PDE's as well as isoperimetric type inequalities and identification of (best constant and) extremal functions in functional inequalities, like Poincaré or Sobolev inequalities

Other interesting questions regard the classification and the study of entire solutions of elliptic and parabolic equations, the study of the movement and the localization of hot spots (i.e. extrema of solutions to heat equation), the study of blow up solutions, curvature flows and their applications, etc.

Of course, many questions that were originally posed in the Euclidean space, naturally arise also in different context, as for instance in the hyperbolic space; this also falls in the interests of our workshop.

2. The Workshop

The workshop brought together leading experts and emerging young mathematicians on the subject in order to discuss recent developments, open problems and future lines of research. It is perhaps remarkable that major contributions to the field came from researchers from all over the world (Canada, China, France, Germany, Italy, Japan, USA, etc.). Thus the workshop provided a unique opportunity for interaction between different groups which would normally reside in several distinct continents. There were 33 participants and, in order to stimulate the discussion, most of them gave a talk about their recent results and connected open problems (see next section for the descriptions).

3. The topics

The workshop covered the recent developments in various areas of partial differential equations. They are closely inter-related, we divide them in subjects in order to provide a clear picture (to highlight the contributions of the participants of the workshop, we have given their names in italics).

1: **Eigenvalue problems**

Antoine Henrot investigated the two first eigenvalues of the Laplacian. In his talk, he discussed questions like the following ones. Let a, b be two positive numbers, does there exist an open set Ω with given volume such that a and b are the two first eigenvalues of the Laplacian with Dirichlet boundary conditions? Does there exist a convex one? What happen if we replace Dirichlet boundary conditions by Neumann ones? He also proposed some open problems.

Kazuhiro Ishige considered the following Cauchy problem for the heat equation with a potential

$$(P) \qquad \begin{cases} \partial_t u = \Delta u - V(|x|)u & \text{in} \quad \mathbf{R}^N \times (0,\infty), \\ u(x,0) = \phi(x) & \text{in} \quad \mathbf{R}^N, \end{cases}$$

where $\partial_t = \partial/\partial t$, $N \geq 3$, $\phi \in L^2(\mathbf{R}^N)$, and $V = V(|x|)$ is a smooth, nonpositive, and radially symmetric function having quadratic decay at the space infinity. Assuming that the Schrödinger operator $H = -\Delta + V$ is nonnegative on $L^2(\mathbf{R}^N)$, he gave the exact power decay rates of L^q-norm ($q \geq 2$) of the solution $e^{-tH}\phi$ of (P) as $t \to \infty$. Furthermore he studied the large time behavior of the solution of (P) and its hot spots.

Bernd Kawohl addressed several issues involving Dirichlet problems for the classical p-Laplacian operator for $p \in (1,\infty)$. First he looked at p-harmonic functions as $p \to \infty$ and $p \to 1$. Then he compared the p-Laplacian with its normalized version $\Delta_p^N := \frac{1}{p}|\nabla u|^{2-p}\Delta_p u$ and studied equations like $-\Delta_p = 1$ and $-\Delta_p^N u = 1$. He presented results and open problems on the eigenvalue problem $-\Delta_p u = \lambda |u|^{p-2}u$.

Rolando Magnanini presented a joint work with *Lorenzo Brasco* and *Paolo Salani* about some localization methods for the hot spot of a grounded convex conductor. A grounded heat conductor is a domain whose boundary is constantly kept at zero temperature. Suppose that the conductor's initial temperature is constant (and positive) and consider the evolution in time of the points where temperature takes its maximum — the hot spots. If the conductor is convex, there is only one hot spot, that starts from the set where the distance from the boundary takes its maximum and, as times grows, approaches the (unique) maximum point of the first Dirichlet eigenfunction of the Laplace operator. In his talk, Magnanini presented two methods to estimate the location of the hot spot in a convex conductor. The first is based on Aleksandrov's reflection principle and leads to a numerical algorithm to approximately locate the hot spot. The second is inspired by Aleksandrov-Bakelman-Pucci maximum principle and, by employing techniques of convex geometry, gives a lower bound for the distance of the hot spot from the boundary. This bound gives an answer to a problem raised in a paper by Gidas, Ni and Nirenberg. He also presented some open problems in connection with the argument of the talk.

Cristina Trombetti, in collaboration with *Barbara Brandolini*, Pedro Freitas and *Carlo Nitsch*, investigated a Dirichlet eigenvalue problem for a linear operator with a non-local term. Varying the coefficient of the non-local term they looked at those domains which, for a given measure, achieve the least first eigenvalue. It turns out that there exists a threshold value of such a coefficient above which the geometry of the optimal domains changes radically.

2: **Convexity of solutions**

Baojun Bian spoke about convexity and partial convexity for solutions of partial differential equations. In his talk, he presented a joint work with *Pengfei Guan*, where they establish the microscopic (partial) convexity principle for (partially) convex solution of nonlinear elliptic and parabolic equations. As application, he discussed the preservation of (partial) convexity in parabolic equations.

Xinan Ma gave an overview about the so-called constant rank technique and its applications and presented a constant rank theorem for the spacetime Hessian of solutions of heat equation. He showed how to apply this tool to study the spacetime convexity of solutions of parabolic equations. He also showed possible applications to other geometrical flow. His talk was partially referred to some results obtained in a recent collaboration with *Paolo Salani*.

Wolfgang Reichel discussed an electrostatic characterization of spheres, which takes to an overdetermined elliptic problems, and related conjectures. He presented recent developments obtained in collaboration with *Andrea Colesanti* and *Paolo Salani*.

3: **Qualitative study of solutions and blow-ups**

Barbara Brandolini presented recent results, obtained in collaboration with Francesco Chiacchio and *Cristina Trombetti*, about symmetrization for singular elliptic equations. Precisely, they prove some comparison results for the solution to a Dirichlet problem associated to a singular elliptic equation and they study how the summability of such a solution varies depending on the summability of the datum.

Xavier Cabré discussed the uniqueness and stability of saddle–shaped solutions to the Allen–Cahn equation. In a recent work, he established the uniqueness of a saddle-shaped solution to the diffusion equation $-\Delta u = f(u)$ in all of $\mathbb{R}^{2m}$, where f is of bistable type, in every even dimension $2m \geq 2$. In addition, he proved its stability whenever $2m \geq 14$. Saddle-shaped solutions are odd with respect to the Simons cone $C = \{(x^1; x^2) \in \mathbb{R}^m \times \mathbb{R}^m : |x^1| = |x^2|\}$ and exist in all even dimensions. Their uniqueness was only known when $2m = 2$. On the other hand, they are known to be unstable in dimensions 2, 4, and 6. Their stability in dimensions 8, 10, and 12 remains an open question. In addition, since the Simons cone minimizes area when $2m \geq 8$, saddle-shaped solutions are expected to be global minimizers when $2m \geq 8$, or at least in higher dimensions. This is a property stronger than stability which is not yet established in any dimension.

Filippo Gazzola discussed the Emden-Fowler equation $-\Delta u = |u|^{p-1}u$ on the hyperbolic space and investigated radial solutions, namely solutions depending only on the geodesic distance from a given point. The critical exponent for such equation is the same as in the Euclidean setting, but the properties of the solutions show striking differences with the Euclidean case. While previous papers consider finite energy solutions, in a recent paper he deal with infinite energy solutions and determine the exact asymptotic behavior of wide classes of finite and infinite energy solutions.

Chang-Shou Lin discussed about counting topological degree formulas for a class of generalized Liouville systems. In his talk, he considered a generalized Liouville system with exponential nonlinear terms: he proved an uniform bound for non-critical parameters and obtained a degree counting formulas. He also proved that any bubbling solutions should blow up fully, i.e. all components of solutions should simultaneously blow up at any blow-up points.

Gerard Philippin considered a class of initial boundary value problems for the semilinear heat equation with time dependent coefficients. Using a first order differential inequality technique, he investigated the influence of the data on the behavior of the solutions (blow-up in finite or infinite time, global existence). Lower and upper bounds are derived for the blow-up time when blow-up occurs.

Shigeru Sakaguchi and Tatsuki Kawakami considered the bounded nonnegative solution of the Cauchy problem for Fisher's equation with initial data having compact support, and introduced some property such that the solution has a sequence of similar level sets as time tends to infinity. They showed that this property characterizes the spatially radial solutions in some class of initial data. He also considered the case where the support of initial data is unbounded and suggested similar problems for the heat equation.

Jie Xiao presented a recent work in collaboration with D. R. Adams, where they used the newly discovered facts about the embedding of a Morrey space into the Zorko spaces and the Morrey-Hausdorff iso-capacitary inequality to control the Hausdorff dimension of the singular set of a Morrey potential, and consequently, to produce some Hausdorff dimension estimates for the singular sets of: weak solutions of nonlinear elliptic systems of Meyers-Elcrat's type; weak solutions of the second order quasilinear elliptic systems with quadratic gradient growth; stationary harmonic maps; weak W-harmonic maps; stationary bi-harmonic maps; stationary admissible Yang-Mills connections.

4: **Sobolev type inequalities**

Andrea Cianchi described a general principle yielding sharp higher-order Sobolev type embeddings via first-order ones. This principle, obtained in a joint work with Lubos Pick, is applied to Euclidean Sobolev embeddings in (possibly) irregular domains, to trace Sobolev embeddings, and to Sobolev embeddings in product probability spaces, of which the Gauss space is a classical instance. As a consequence, the validity of Sobolev inequalities of arbitrary order for rearrangement invariant norms is reduced to inequalities for one-dimensional integral operators involving the same norms.

Jean Dolbeault spoke about improved Sobolev inequalities, relative entropy and fast diffusion equations, presenting a joint work with G. Toscani. The difference of the two terms in Sobolev's inequality (with optimal constant) is known to measure a distance to the manifold of the optimal functions. They give a precise expression of the remainder term, by establishing an improved inequality, with explicit norms and fully detailed constants. Their approach is based on nonlinear evolution equations and improved entropy - entropy production estimates along the associated flow: optimizing a relative entropy functional with respect to a scaling parameter, or handling properly second moment estimates, turns out to be the central technical issue, although it is by no mean trivial in nonlinear evolution equations. The method by Dolbeault and Toscani also applies to other interpolation inequalities of Gagliardo-Nirenberg-Sobolev type.

Carlo Nitsch spoke about the longest–shortest fence and sharp Poincaré–Sobolev inequalities. The shortest curve bisecting the area of a given planar set provides a sharp estimate of its best constant in Poincaré-Sobolev type inequalities. He discussed two fencing conjectures raised more than fifty years ago. In particular, in collaboration with Luca Esposito, *Vincenzo Ferone*, *Bernhard Kawohl* and *Cristina Trombetti*, he showed that among all the planar convex sets of given measure the disk, and only the disk, has the longest shortest bisecting curve.

Ritva M. Hurri-Syrjanen explored the $(1, p)$-Poincaré inequality in irregular domains. Her talk was based on a joint work with P. Harjulehto and A. V. Vahakangas, in which they show that s-John domains support the $(1, p)$-Poincare inequality for all finite $p > p_0$ where p_0 is sharp.

5: **Geometric equations**

Lei Ni proved a sharp estimate on the expansion modulus of the gradient of the parabolic kernel to the Schrödinger operator with convex potential, which improves an earlier work of Brascamp-Lieb. He also included alternate proofs to the improved log-concavity estimate, and to the fundamental gap theorem of Andrews-Clutterbuck via the elliptic maximum principle. Some applications of the estimates are also obtained, including a sharp lower bound on the first eigenvalue.

Joel Spruck surveyed some boundary value problems at infinity and their curvature flows in Hyperbolic space. Precisely, let $\mathbb{H}^{n+1}$ be the $n+1$ dimensional hyperbolic space and $\partial_\infty \mathbb{H}^{n+1}$ its boundary at infinity. A *boundary value problem at infinity* (or an asymptotic Plateau problem) is a solution to the following problem: let $\omega \subset \partial_\infty \mathbb{H}^{n+1}$ be a smooth bounded domain, $\Gamma = \partial\Omega$ and a smooth symmetric function f of n variables be given. It is interesting to seek a complete hypersurface Σ of constant curvature" in $\mathbb{H}^{n+1}$ satisfying

$$f(\kappa[\Sigma]) = \sigma \tag{0.1}$$
$$\partial\Sigma = \Gamma \tag{0.2}$$

where $\kappa[\Sigma] = (\kappa_1, \ldots, \kappa_n)$ denotes the hyperbolic principal curvatures of Σ and $\sigma \in (0, 1)$ is a constant. Classical choices for the curvature function $f(\kappa[\Sigma])$ are the mean curvature H, the Gauss curvature K and the curvature quotient K/H. Corresponding to this "stationary" problem with prescribed boundary, it is natural to study a corresponding curvature flow (oriented by the outward hyperbolic unit normal N

to the complete hypersurface $X(t)$), starting from a complete hypersurface Σ_0 with asymptotic boundary $\partial\Sigma_0 = \Gamma$:

$$\dot{X} = (f(\kappa[X(t)])N\,, \quad \partial X(t) = \Gamma\,, \quad X(0) = \Sigma_0\,. \tag{0.3}$$

In his talk, Spruck presented his work on existence and uniqueness of solutions of the stationary problem and that of his student Ling Xiao on the convergence of the corresponding curvature flow to the stationary solution.

Deane Yang discussed an Orlicz Minkowski problem. The classical Minkowski problem is one of the centerpieces of the classical Brunn-Minkowski theory and it concerns the existence of convex hypersurfaces whose Gauss curvature (possibly in a generalized sense) is prescribed as a function of the outer unit normal. The L_p Minkowski problem extends this classical question and was critical for the proofs of affine Sobolev inequalities. The Orlicz Minkowski problem is a generalization of the L_p Minkowski problem which is part of a new Orlicz-Brunn-Minkowski theory developed by Yang and other authors, as Cristoph Haberl, Erwin Lutwak and Gaoyong Zhang.

6: **Geometric optics**

Cristian Gutierrez presented a joint work with Henok Mawi about geometric optics and Monge–Ampère type equations. The physical phenomena of refraction and reflection occur simultaneously: if a light ray strikes a boundary separating to media with different refractive indices, then the ray splits into an internally reflected ray and a refracted (or transmitted) ray, each one having certain intensity. A precise description of these intensities is given by the Fresnel formulas, a consequence of Maxwell's equations. In his talk, he gave some physical background to understand this phenomena and presented a new model taking into account the splitting of energy, discussing the existence of surfaces separating two homogeneous materials transmitting radiation in a prescribed way. Notice that the problem gives rise to a Monge-Ampère type equation.

Other contributions came from *Qun Li*, *Jyotshana Prajapat*, *Nguyen Phuc* and *Robert Stanczy*, that gave short talks on their research works.

During the workshop, all the participants actively and lively participated to the discussions. Several problems generated high interests.

(1) **Hot spots.** Magnanini's talk stimulated further discussion of the problem of estimating the hot spots.

(2) **The Orlicz Minkowski problem.** After D. Yang's presentation of his joint work with Lutwak and Zhang, D. Yang and P. Guan initiated the discussion of higher regularity of the problem. This was continued recently in New York, may pave some new collaboration of Lutwak, Yang, Zhang and Guan.
(3) **Blow-up analysis of systems.** C.S. Lin's lecture on Liouville systems provided some discussions on how to carry out blow-up analysis for general nonlinear systems. People believe this is one of the important directions of nonlinear PDEs.

4. Participants

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5. Talks

It follows the list of talks, in the order of time they were given.

R. Magnanini, *The location of the hot spot in a grounded convex conductor*
S. Sakaguchi, *Reaction–diffusion with similar level sets*
G. Philippin, *Blow–up phenomena for solutions of some non–linear parabolic problems*
K. Ishige, *L^p norms of non–negative Schrödinger heat semigroup and the large time behavior of hot spots*
C. Nitsch, *The longest–shortest fence and sharp Poincaré–Sobolev inequalities*
X. Ma, *The convexity of the solutions of heat equation and its geometry applications*
C.–S. Lin, *The counting topological degree formulas for a class of generalized Liouville systems*
B. Bian, *Convexity and partial convexity for solutions of partial differential equations*
R. M. Hurry–Syrjanen, *On the $(1, p)$–Poincaré inequality*
J. Dolbeault, *Improved Sobolev inequalities, relative entropy and fast diffusion equations*
J. Xiao, *Singularities for Morrey potentials with applications to some non–linear elliptic systems*
R. Stanczy, *Uniqueness and multipilicity results for some local and nonlocal ellpitic equations under geometrical assumptions on the domain and the set of solutions*
X. Cabré, *Uniqueness and stability of saddle–shaped solutions to the Allen–Cahn equation*
C. Gutierrez, *Geometric optics and the Monge–Ampère type equations*
D. Yang, *The Orlicz Minkowski problem*
Q. Li, *Complex Monge-Ampère equations on Hermitian manifolds*
A. Henrot, *About the two first eigenvalues of the Laplacian*
C. Trombetti, *Sharp estimates for a non–local eigenvalue problem*
J. Spruck, *Boundary value problems at infinity and their curvature flows in Hyperbolic space*
N. Phuc, *A nonlinear Calderón-Zygmund theory arising from the p-laplacian and its applications*
B. Brandolini, *Symmetrization for singular elliptic equations*
F. Gazzola, *Radial solutions to the Emden–Fowler equation on the Hyperbolic space*
W. Reichel, *Electrostatic characterization of spheres*

J. Prajapat, *On Chern-Simons system of equations:Existence, asymptotic behaviour and uniqueness*
A. Cianchi, *A sharp iteration principle for higher-order Sobolev-type embeddings*
J. Prajapat, *On Chern-Simons system of equations:Existence, asymptotic behaviour and uniqueness*
L. Ni, *Estimates on the modulus of expansion for vector fields solving non–linear equations*
B. Kawohl, *Variations on the p–Laplacian*