Quasi-isometric rigidity in low dimensional topology

Jason Behrstock (Lehman College, CUNY),
Michael Kapovich (University of California, Davis),
Walter Neumann (Columbia University)

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1 Overview of the field and recent developments

The early work of Mostow, Margulis and Prasad on rigidity of arithmetic lattices has evolved into a broad use of quasi-isometry techniques in group theory and low dimensional topology. The word metric on a finitely generated group makes it into a metric space which is uniquely determined up to the geometric relation called quasi-isometry, despite the fact that the metric depends on the choice of generating set. As for lattices in suitable Lie groups, where quasi-isometry of lattices implies commensurability, the general quasi-isometric study of groups aims to understand the remarkable extent to which this completely geometric notion often captures algebraic properties of the group.

The Milnor-Schwarz Lemma provides an equivalence between the geometry of the word metric on the fundamental group of a compact Riemannian manifold (or metric complex) with the geometry of its universal cover. So the quasi-isometry study of groups also returns information about the spaces. This relationship has proved particularly productive in low dimensional geometry/topology.

There are currently a large variety of groups whose quasi-isometric geometry is actively being studied by geometric group theorists. Many of these groups have close relations to objects studied by low dimensional topologists. Examples of these include automorphism groups of free groups, mapping class groups (Hamenstädt, Behrstock-Kleiner-Minsky-Mosher), 3-manifold groups (Gromov-Sullivan, Cannon-Cooper, Eskin-Fisher-Whyte, Kapovich-Leeb, Rieffel, Schwartz, Behrstock-Neumann), solvable Lie groups (Eskin-Fisher-Whyte, Dymarz; the 3-dimensional group Solv had long been the holdout in understanding geometric 3-manifold groups), Artin groups (Bestvina-Kleiner-Sageev, Behrstock-Neumann), relatively hyperbolic groups (Drutu-Sapir, Osin), and others.

The study of many of these groups had been completely out of reach until the flurry of activity which has occurred in recent years, bringing many of these groups within grasp.

An old theorem of Stallings, the Ends Theorem, can be reinterpreted as quasi-isometric invariance of splitting over a finite group. Quasi-isometric rigidity of group splittings has remained an active area of research (Papasoglou, Mosher-Sageev-Whyte, etc.).

Questions of quasi-isometric rigidity and classification are studied using a range of techniques. Indeed, Gromov's Polynomial Growth Theorem, which was one of the seeds of the modern study of quasi-isometric rigidity since it implies that virtual nilpotence is quasi-isometrically rigid, already employed a large number of tools: representation theory, differential geometry, Montgomery–Zippen's proof of Hilbert's 5th problem, etc. Since then a number of other tools have also come into use, including quasi-conformal analysis, asymptotic cones, $CAT(0)$ geometry, logic, etc. Several recent proofs of outstanding problems have added new tools, including coarse differentiation used to answer questions about solvable Lie groups (Eskin-Fisher-Whyte), applications of the Continuum Hypothesis to resolve non-uniqueness questions about asymptotic cones (Kramer, Thomas, Tent, Shalah), harmonic analysis in Kleiner's new proof of Gromov's theorem, representation theory in the work of Shalom, etc.

Despite recent major advances, very significant problems remain. For example, little is known about the outer automorphism group of the free group, one of the central groups in the intersection of low dimensional topology and geometric group theory. There have been several inroads into quasi-isometry for Artin groups, but the general case remains wide open. Despite the fact that nilpotent groups were the first to be shown to be quasi-isometrically rigid, their quasi-isometric classification remains a well-known difficult question. Similarly classification of hyperbolic and relatively hyperbolic groups remains open although rigidity is known (Gromov, Drutu). And there are several other areas of active study.

This conference brought together a range of specialists whose expertise in order to educate each other in the broad spectrum of techniques and problems in quasi-isometric rigidity. A number of graduate students actively participated in the conference as well.

2 Program

Monday	**March 8, 2010**
9:15–10:15	Mladen Bestvina, *The asymptotic dimension of mapping class groups is finite*
10:45–11:45	Christopher Cashen, *Line Patterns in Free Groups*
14:00–15:00	Jason Behrstock, *Quasi-isometric classification of right angled Artin groups*
15:30–16:30	Walter Neumann, *Quasi-isometry of 3-manifold groups*

Tuesday	**March 9, 2010**
9:00–10:00	Linus Kramer, *Coarse rigidity of euclidean buildings*
10:30–11:30	Anne Thomas, *Lattices in complete Kac-Moody groups*
14:00–15:00	Michael Kapovich, *Ends of groups and harmonic functions*
15:30–16:30	Mark Hagen, *LERF after Dani Wise*

Wednesday	**March 10, 2010**
9:00–10:00	Mark Sapir, *Dehn functions of groups and asymptotic cones*
10:30–11:30	Kevin Wortman, *Non-nonpositive curvature of some non-cocompact arithmetic lattices*

Thursday	**March 11, 2010**
9:00–10:00	Xiangdong Xie, *Quasiisometries Łof some negatively curved solvable Lie groups*
10:30–11:30	Tullia Dymarz, *Bilipschitz equivalence vs. quasi-isometric equivalence*
14:00–15:00	Eduardo Martinez-Pedroza, *Separation of Quasiconvex Subgroups in Relatively Hyperbolic Groups*
15:30–16:30	Genevieve Walsh, *Quasi-Isometry classes of hyperbolic knot complements*

3 Presentations

3.1 Jason Behrstock (Lehman College, CUNY) "QI classification of right-angled Artin groups"

This was the first of a two part talk involving joint work with Walter Neumann. This talk was on results from the papers [2] and [4], the second part was given by Walter Neumann and focused on the work in [3].

A *graph manifold* is an irreducible, non-geometric 3–manifold (possibly with boundary) for which every geometric piece is Seifert fibered. In the first half of the talk we discussed:

Theorem 3.1.1 (Behrstock–Neumann; [2])**.** *Let M, M' be graph manifolds (possibly with boundary). The following are equivalent:*

1. *$\widetilde{M}$ and $\widetilde{M}'$ are bilipschitz homeomorphic, here $\widetilde{M}$ denotes the universal cover.*
2. *$\pi_1(M)$ and $\pi_1(M')$ are quasi-isometric.*
3. *$BS(M)$ and $BS(M')$ are isomorphic as 2-colored trees, here $BS(M)$ is the Bass–Serre tree corresponding to the graph of groups decomposition of $\pi_1(M)$.*
4. *The minimal 2-colored graphs in the* bisimiliarity *classes of the colored decomposition graphs $\Gamma(M)$ and $\Gamma(M')$ are isomorphic.*

During this period we introduced the notion of *bisimiliarity* and gave a number of explicit examples. We sketched the proof of the above theorem, via a special case; showing that if M and M' are closed then their universal covers are bilipschitz homeomorphic, this answered an conjecture of Kapovich–Leeb from the early 90's.

Next we turned to applications of bisimilarity to the quasi-isometric classification of right-angled Artin groups. We introduced a family of such groups, n–tree groups which are the right-angled Artin groups associated to a family of n–dimensional simplicial complexes, namely the smallest family containing the n–simplex and with the property that the union of any two complexes in this class along a co-dimension one simplex is also in this class. For instance, for $n = 1$ this is the class of finite trees.

We then discussed:

Theorem 3.1.2 (Behrstock–Neumann; [2])**.** *Any two irreducible right angled 1–tree groups are quasi-isometric.*

N–tree groups admit an analogue of the geometric decomposition for 3–manifold groups, accordingly, some of the information in this geometric decomposition can be described via a finite bipartite colored graph, which we called $\Gamma(K)$, where K is the defining simplicial complex.

Theorem 3.1.3 (Behrstock–Januszkiewicz–Neumann; [4])**.** *Given two simplicial complexes K, K which yield n–tree groups. The groups A_K and $A_{K'}$ are quasi-isometric if and only if $\Gamma(K)$ and $\Gamma(K')$ are bisimilar after possibly reordering one of the color sets by an element of the symmetric group on $n + 1$ elements.*

3.2 Mladen Bestvina (University of Utah) "Asymptotic dimension of the mapping class group"

I started the talk with an introduction to asymptotic dimension. In particular, I recalled Gromov's proof that hyperbolic groups have finite asymptotic dimension and the Bell-Fujiwara argument that curve complexes associated to compact surfaces have finite asymptotic dimension.

Then I outlined a proof of the following theorem, joint with Bromberg and Fujiwara.

Theorem 3.2.1. *Mapping class groups have finite asymptotic dimension.*

The proof proceeds in three steps. The overall goal is to produce an action of a given mapping class group on the finite product $X_1 \times \cdots \times X_k$ with each X_i hyperbolic and of finite asymptotic dimension, such that an orbit map is a quasi-isometric embedding. Each X_i is obtained from a quasi-tree T_i by "blowing up" each vertex to the curve complex of a subsurface. The key step is the construction of T_i. This can be done "axiomatically", that is, whenever a group Γ acts on a set $\mathbb{Y}$ satisfying certain axioms, there is an induced action of Γ on a quasi-tree. A prototypical situation is that of a Kleinian group Γ acting on $\mathbb{H}^3$ with $\mathbb{Y}$ an orbit of axes of loxodromic elements. In our application, $\mathbb{Y}$ is a certain collection of isotopy classes of connected incompressible subsurfaces of the given surface Σ.

3.3 Christopher Cashen (University of Utah) "Line patterns in free groups"

Take a word w in a free group F of rank at least 2. Consider a tree T quasi-isometric to F. The cosets of $\langle w \rangle$ in F correspond to a pattern of lines in T. We study a space called the *decomposition space*, which is a quotient of the boundary of T related to the line pattern. We use the cut set structure of this space to prove quasi-isometric rigidity results for line patterns. In particular, we would like to determine when the group of quasi-isometries of the free group that preserves the line pattern is conjugate into an isometry group of some "nice" space.

We show that this is never true if the decomposition space is disconnected, has cut points or has cut pairs. We conjecture that these are the only cases that the line pattern fails to be rigid. With some hypotheses on the complexity of the line pattern, we show that the pattern is rigid and furthermore that we can take the "nice" space to be a finite valence tree.

These results have applications to quasi-isometric classifications for graphs of free groups, including mapping tori of some free group automorphisms.

This is joint work with Natasa Macura.

3.4 Tullia Dymarz (Yale University) "Bilipschitz equivalence is not equivalent to quasi-isometric equivalence for finitely generated groups"

A *quasi-isometric equivalence* between metric spaces is a map $f : X \to Y$ such that for some $K, C > 0$

$$-C + \frac{1}{K} d(x,y) \leq d(f(x), f(y)) \leq K d(x,y) + C$$

for all $x, y \in X$ and such that $nbhd_C(f(X)) = Y$. This is a generalization of the more common notion of a *bilipschitz equivalence*: a bijection between metric spaces that satisfies for some K

$$\frac{1}{K} d(x,y) \leq d(f(x), f(y)) \leq K d(x,y).$$

A natural question to ask is for which classes of metric spaces are these two notions equivalent. Burago-Kleiner and McMullen gave examples of a separated nets in $\mathbb{R}^2$ that are not bilipschitz equivalent to the integer lattice (but all nets are quasi-isometric). Our interest is in the class of finitely generated groups equipped with word metrics. For a finitely generated group Γ a choice of generating set S determines a Cayley graph Γ_S with metric d_S. The metric d_S depends on S but for any given group all Cayley graphs are bilipschitz equivalent. The examples of Burago-Kleiner and McMullen are not Cayley graphs of finitely generated groups. We prove the following Theorem:

Theorem 3.4.1. *Let F and G be finite groups with $|F| = n$ and $|G| = n^k$ where $k > 1$. Then there does not exist a bijective quasi-isometry between the lamplighter groups $G \wr \mathbb{Z}$ and $F \wr \mathbb{Z}$ if k is not a product of prime factors appearing in n.*

3.5 Mark Hagen (McGill University) "LERF after Dani Wise"

We discuss recent work of Dani Wise on fundamental groups of special cube complexes, focusing on applications to the subgroup separability and virtual fibering of closed hyperbolic Haken 3-manifolds. A *special cube complex* is a cube complex whose immersed hyperplanes do not exhibit certain pathologies; equivalently, a cube complex is special if it admits a local isometry to the cube complex associated to a right-angled Artin group. Special cube complexes generalize graphs in the sense that cubical local isometries to special cube complexes are virtual retracts, as is the case for immersions of graphs. This is used to prove that quasiconvex subgroups of virtually special groups are separable (QCERF). Moreover, the right-angled Artin group characterization of special cube complexes shows that virtually special groups are "residually finite rational solvable" (RFRS).

A *quasiconvex hierarchy* for a group G is a way of constructing G from a (finite) collection of trivial groups by a finite sequence of iterated HNN extensions and amalgams in such a way that the edge groups are all quasiconvex in G. Wise showed that groups admitting a *quasiconvex hierarchy* are virtually fundamental groups of special cube complexes, and thus enjoy the QCERF and LERF properties. In particular, the Haken hierarchy for a closed hyperbolic 3-manifold M with a geometrically finite incompressible surface yields a quasiconvex hierarchy for $\pi_1 M$. That $\pi_1 M$ is subgroup separable is immediate from local quasiconvexity and QCERF. Virtual fibering follows from RFRS, by a result of Agol.

3.6 Michael Kapovich (University of California, Davis) "Energy of harmonic functions and Gromov's proof of Stallings' theorem"

In his essay [9, Pages 228–230], Gromov gave a proof of the Stallings' theorem [26] on groups with infinitely many ends using harmonic functions:

Theorem 3.6.1 (Stallings)**.** *Let G be a finitely-generated group with infinitely many ends. Then G splits nontrivially as an amalgam $G = G_1 *_{G_3} G_2$ or HNN extension $G_1 *_{G_3}$ with a finite edge group G_3.*

The goal of this talk is to provide the details for Gromov's arguments.

Let M be a complete Riemannian manifold of bounded geometry, which has infinitely many ends. Suppose that there exists a number R such that every point in M belongs to an R*–neck*, i.e., an R-ball which separates M into at least three unbounded components. (This property is immediate if M admits a cocompact isometric group action.)

Let $\bar{M} := M \cup Ends(M)$ denote the compactification of M by its space of ends. Given a continuous function $\chi : Ends(M) \to \{0,1\}$, let

$$h = h_\chi : \bar{M} \to [0,1]$$

denote the continuous extension of χ, so that $h|M$ is harmonic. The uniqueness of h easily follows from the maximum principle, while the existence of h is nontrivial was independently established in [12] and [18].

Let $H(M)$ denote the space of harmonic functions

$$\{h = h_\chi, \chi : Ends(M) \to \{0,1\} \text{ is nonconstant}\}.$$

We give $H(M)$ the topology of uniform convergence on compacts in M. Let $E : H(M) \to \mathbb{R}_+ = [0,\infty)$ denote the energy functional.

Definition 3.6.2. *Given the manifold M, define its* energy gap $e(M)$ *as*

$$e(M) := \inf\{E(h) : h \in H(M)\}.$$

If M admits an isometric group action $G \curvearrowright M$, then G acts on $H(M)$ preserving the functional E. Therefore E projects to a lower semi-continuous functional $E : H(M)/G \to \mathbb{R}_+$, where we give $H(M)/G$ the quotient topology. Our main technical result is

Theorem 3.6.3. *1. $e(M) \geq \mu > 0$, where μ depends only on R, $\lambda_1(M)$ and geometry of M.*

2. If M admits a cocompact isometric group action, then $E : H(M)/G \to \mathbb{R}_+$ is proper in the sense that

$$E^{-1}([0,T])$$

is compact for every $T \in \mathbb{R}_+$.

Actually, it was observed by Bruce Kleiner that 1 easily implies 2.

We now sketch our proof of the Stallings' theorem. Since E is semicontinuous and proper mod G, E attains it minimum $e(M)$. Let $h \in H(M)$ be an energy-minimizing harmonic function. We then verify that the set $\Sigma := \{h(x) = \frac{1}{2}\}$ is *precisely-invariant* with respect to the action of G, i.e.. $g\Sigma \cap \Sigma \neq \emptyset$ iff $g\Sigma = \Sigma$. By choosing t sufficiently close to $\frac{1}{2}$ we obtain a smooth hypersurface $S = \{h(x) = t\}$ which is precisely-invariant under G and separates the ends of M. We then define *walls* if M to be the hypersurfaces $S_f = \{f = t\}, f = g^*(h)$ for some $g \in G$. We say that a hypersurface S_f *separates* points $x, y \in M$ if $f(x) < t$ while $f(y) > t$. We then define a graph T dual to the collection of walls in M: The edges of T are the walls, while the vertices of T are the "indecomposable" subsets of $M \setminus G \cdot S$, i.e., subsets which cannot be separated by one wall. We then verify that T is a tree. Clearly, G acts on T and the edge-stabilizers are finite since S is compact. Therefore, G splits over a finite group.

3.7 Linus Kramer (Universität Münster) "Coarse rigidity of Euclidean buildings"

In my talk I presented the following results. We prove coarse (i.e. quasi-isometric) rigidity results for trees (simplicial trees and $\mathbb{R}$-trees) and, more generally, for discrete and nondiscrete Euclidean buildings. For trees, a key ingredient is a certain equivariance condition. Our main results are as follows.

Theorem 3.7.1. *Let G be a group acting isometrically on two metrically complete leafless trees T_1, T_2. Assume that there is a coarse equivalence $f : T_1 \to T_2$, that T_1 has at least 3 ends and that the induced map $\partial f : \partial T_1 \to \partial T_2$ between the ends of the trees is G-equivariant. If the G-action on ∂T_1 is 2-transitive, then (after rescaling the metric on T_2) there is a G-equivariant isometry $\bar{f} : T_1 \to T_2$ with $\partial f = \partial \bar{f}$. If T_1 has at least two branch points, then $\bar{f}$ is unique and has finite distance from f.*

Theorem 3.7.2. *Let X_1 and X_2 be metrically complete nondiscrete Euclidean buildings whose spherical buildings at infinity $\partial_{cpl} X_1$ and $\partial_{cpl} X_2$ are thick. Let $f : X_1 \times \mathbb{R}^{m_1} \to X_2 \times \mathbb{R}^{m_2}$ be a coarse equivalence. Then $m_1 = m_2$ and there is a combinatorial isomorphism $f_* : \partial_{cpl} X_1 \to \partial_{cpl} X_2$ between the spherical buildings at infinity which is characterized by the fact that the f-image of an affine apartment $A \subseteq X_1$ has finite Hausdorff distance from the f_*-image of A.*

We remark that the boundary map f_* is constructed in a combinatorial way from f. In general, a coarse equivalence between CAT(0)-spaces will not induce a map between the respective Tits boundaries.

Theorem 3.7.3. *Let $f : X_1 \times \mathbb{R}^{m_1} \to X_2 \times \mathbb{R}^{m_2}$ be as in Theorem 3.7.2 and assume in addition that X_1 has no tree factors. Then there is (after rescaling the metrics on the irreducible factors of X_2) an isometry $\bar{f} : X_1 \to X_2$ with boundary map $\bar{f}_* = f_*$. Put $f(x \times y) = f_1(x \times y) \times f_2(x \times y)$. If none of the de Rham factors of X_1 is a Euclidean cone over its boundary, then $\bar{f}$ is unique and $d(f_1(x \times y), \bar{f}(x))$ is bounded as a function of $x \in X_1$.*

For a more general statement see our preprint [16]. Theorem 3.7.2 and Theorem 3.7.3 were proved by Kleiner and Leeb under the additional assumptions that the Euclidean buildings are thick (i.e. that the thick points are cobounded) and that the spherical buildings at infinity are Moufang [13, 1.1.3] or compact [17, 1.3]. (These results extended, in turn, Mostow-Prasad rigidity [23].) By Tits' extension theorem every thick irreducible spherical building of rank at least 3 is automatically Moufang. The spherical building at infinity of an irreducible 2-dimensional Euclidean building, on the other hand, need not be either Moufang or compact; see, for example, [5]. In contrast to [13], we construct the combinatorial boundary map f_* of Theorem 3.7.2 first and then use it to obtain a simpler approach to Theorem 3.7.3.

This is a joint work with Richard M. Weiss

3.8 Eduardo Martínez-Pedroza (McMaster University) "Separation of Quasiconvex Subgroups in Relatively Hyperbolic Groups"

A subgroup H of a group G is *separable* if for any $g \in G - H$ there is a homomorphism π onto a finite group such that $\pi(g) \notin \pi(H)$. A group is *residually finite* if the trivial subgroup is separable, is *LERF* if every finitely generated subgroup is separable, and is *slender* if every subgroup is finitely generated. For example, finitely generated abelian groups are LERF and slender.

Given a relatively hyperbolic group with peripheral structure consisting of LERF and slender subgroups, we study separability of relatively quasiconvex subgroups. This is connected to residual finiteness of hyperbolic groups. It is not known whether all hyperbolic groups are residually finite. In particular, the main result of [1] is the following.

Theorem 3.8.1. *[1] If all hyperbolic groups are residually finite, then every quasiconvex subgroup of a hyperbolic group is separable.*

We extended this result, answering a question in [1], as follows:

Theorem 3.8.2. *[19] Suppose that all hyperbolic groups are residually finite. If G is a relatively hyperbolic group with peripheral structure consisting of subgroups which are LERF and slender, then any relatively quasiconvex subgroup of G is separable.*

This extension together with deep results in 3-manifolds have some interesting corollaries.

Corollary 3.8.3. *[19] If all hyperbolic groups are residually finite, then all finitely generated Kleinian groups are LERF.*

Corollary 3.8.4. *[19] If all fundamental groups of compact hyperbolic 3–manifolds are LERF, then all fundamental groups of finite volume hyperbolic 3–orbifolds are LERF.*

Theorem 3.8.2 is proved by combining one of combination theorems for quasiconvex subgroups in [20] with Theorem 3.8.1 and the Dehn filling technique of [10, 21].

The main technical result is stated below.

Definition 3.8.5. *A relatively quasiconvex subgroup H of G is called* fully quasiconvex *if for any subgroup $P \in \mathcal{P}$ and any $f \in G$, either $H \cap P^f$ is finite or $H \cap P^f$ is a finite index subgroup of P^f. (Here $P^f = fPf^{-1}$.)*

Theorem 3.8.6. *Let G be hyperbolic relative to a collection of slender and LERF subgroups. For any relatively quasiconvex subgroup Q and any $g \in G - Q$, there is a fully quasiconvex subgroup H, and a surjective homomorphism $\pi : G \longrightarrow \bar{G}$ such that*

1. *$Q < H$,*
2. *$\bar{G}$ is a word-hyperbolic group,*
3. *$\pi(H)$ is a quasiconvex subgroup of $\bar{G}$,*
4. *$\pi(g) \notin \pi(H)$.*

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Definition 3.8.7. *A relatively quasiconvex subgroup H of G is called* fully quasiconvex *if for any subgroup $P \in \mathcal{P}$ and any $f \in G$, either $H \cap P^f$ is finite or $H \cap P^f$ is a finite index subgroup of P^f. (Here $P^f = fPf^{-1}$.)*

Theorem 3.8.8. *Let G be a torsion free group hyperbolic relative to a collection of slender and LERF subgroups. For any relatively quasiconvex subgroup Q of G and any element $g \in G$ such that $g \notin Q$, there is a fully quasiconvex subgroup H of G, and a surjective homomorphism $\pi : G \longrightarrow \bar{G}$ such that*

1. *$Q < H$,*

2. *$\bar{G}$ is a word-hyperbolic group,*

3. *$\pi(H)$ is a quasiconvex subgroup of $\bar{G}$,*

4. *$\pi(g) \notin \pi(H)$.*

We can prove Theorem 3.8.2 from Theorem 3.8.8 as follows:

Proof of Theorem 3.8.2. Let $Q < G$ be relatively quasiconvex, and let $g \in G \setminus Q$. By Theorem 3.8.8 there is a fully quasiconvex $H < G$ containing Q but not g, and a quotient $\pi : G \to K$ so that $\pi(g) \notin \pi(H)$, K is hyperbolic, and $\pi(K)$ is quasiconvex.

Assuming all hyperbolic groups are residually finite, Theorem 3.8.1 implies that there is a finite group F and a quotient $\phi : K \to F$ so that $\phi(\pi(g)) \notin \phi(\pi(H))$. Since $\phi(\pi(H))$ contains $\phi(\pi(Q))$, the map $\phi \circ \pi$ serves to separate g from Q. □

This is a joint work with Jason Manning.

3.9 Walter Neumann (Columbia University) "Quasi-isometries of 3-manifold groups"

The remaining case to be resolved for quasi-isometric classification of fundamental groups of compact 3–manifolds (allowing torus boundary components) is the case of irreducible 3-manifolds with non-trivial geometric decomposition in the sense of Thurston and Perelman.

The classification for non-geometric 3-manifolds with no hyperbolic pieces in their geometric decompositions was described in Jason Behrstock's talk (see subsection 3.1 and [2]). The general non-geometric case (also joint work with Behrstock, see [3]) is a combination of this case and the case when all pieces are hyperbolic, so my talk restricted to the all-hyperbolic case for simplicity. However, we assume that at least one piece is non-arithmetic, since if all pieces are arithmetic, then the manifold has very "arithmetic" behaviour, and the theory in this case is not yet fully worked out.

A non-geometric 3–manifold whose geometric decomposition decomposes it into hyperbolic pieces, at least one of which is non-arithmetic, is called an *NAH-manifold*.

The classification for graph-manifolds (no hyperbolic pieces) described in Behrstock's talk (subsection 3.1) was in terms of finite labelled graphs; the labelling consisted of a color black or white on each vertex and the classifying objects are such two-colored graphs which are minimal under a relation called *bisimilarity*. For NAH–manifolds the classification is again in terms of finite labelled graphs, and the the classifying objects are again given by labelled graphs which are minimal in a similar sense. The labelling is more complex: each vertex is labelled by the isomorphism type of a hyperbolic orbifold and each edge is labelled by a linear isomorphism between certain 2-dimensional $\mathbb{Q}$–vectorspaces. We call these graphs *NAH–graphs*. There is a natural morphism concept for such graphs, and the equivalence relation generated by existence of morphisms turns out to have a unique minimal object in each equivalence class. The main results are:

1. *These* minimal *NAH–graphs classify fundamental groups of NAH-manifolds up to quasi-isometry.*

2. *A minimal NAH–graph arises as the classifying graph for a quasi-isometry class of 3-manifold groups if and only if it is* balanced *(the product of determinants of the linear maps labelling edges along any closed path in the graph should be ± 1).*

3. *If the minimal NAH–graph is a tree and the vertex labels have no orbifold cusps then any two manifolds in the corresponding quasi-isometry class are commensurable.*

The third of these theorems and the "if" in the second are currently proved only under the assumption that the *Cusp Covering Conjecture* (CCC below) is true in dimension 3. They must therefore still be considered to be conjectural, although CCC is used in part for simplicity, and much less should be needed to prove these results.

CCC: *Any hyperbolic manifold M has a finite index subgroup of each of its cusp fundamental groups such that, for any choice of a smaller finite index subgroup P_i of each cusp fundamental group, there is a finite cover $\bar{M} \to M$ which restricts on each cusp of $\bar{M}$ to the covering given by the corresponding P_i.*

CCC (in all dimensions) is implied by the well known conjecture that any word-hyperbolic group is residually finite (RFCH). However the truth of RFCH is considered to be rather doubtful, while the Cusp Covering Conjecture is much more plausible.

3.10 Mark Sapir (Vanderbilt University) "On Dehn functions of groups"

I formulated and proved Gromov's theorem that a group with all asymptotic cones simply connected has polynomial Dehn function and linear isodiametric function. A partial converse was proved by Papasoglu: groups with quadratic Dehn functions have simply connected asymptotic cones. There are many different types of examples of groups with quadratic Dehn functions. Nevertheless they all seem to satisfy some strong algorithmic properties. In particular I formulated a conjecture due to Rips that all these groups have solvable conjugacy problem. I presented a quasi-proof of this conjecture (due to Olshanskii and myself). It is not known if this proof works for all groups with quadratic Dehn function It does work for multiple HNN extensions of free groups. In particular, this and the result of Bridson and Groves imply that free-by-cyclic groups have solvable conjugacy problem. Finally I formulated the main new result obtained jointly with A. Olshanskii

Theorem 3.10.1. *There exists a finitely presented group with undecidablde word problem and almost quadratic Dehn function (that is the Dehn function is smaller than Cn^2 on arbitrary long intervals).*

3.11 Anne Thomas (Oxford University) "Lattices in complete Kac–Moody groups"

Let G be a complete Kac–Moody group of rank 2 with symmetric Cartan matrix, defined over a finite field. An example is the "affine case" $G = SL_2(\mathbb{F}_q((t)))$, which is over $\mathbb{F}_q$. Such a group G is a totally disconnected locally compact group, which, apart from the affine case, is non-linear. The group G is obtained by completing a minimal or incomplete Kac–Moody group Λ with respect to some topology. For example, in the affine case $\Lambda = SL_2(\mathbb{F}_q[t, t^{-1}])$.

The group G acts on its Bruhat–Tits building X, a $(q + 1)$–regular tree, with quotient a single edge. We classify the cocompact lattices in G which act transitively on the edges of X. These lattices are given as graphs of groups, together with an embedding of the fundamental group of the graph of groups into G. Using this classification, we prove our main result:

Theorem 3.11.1. *Let G be a topological Kac–Moody group of rank 2 defined over the finite field $\mathbb{F}_q$, with symmetric generalised Cartan matrix $\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$, $m \geq 2$. Then for $q \geq 540$*

$$\min\{\mu(\Gamma\backslash G) \mid \Gamma \textit{ a cocompact lattice in } G\} = \frac{2}{(q+1)|Z(G)|\delta}$$

where $\delta \in \{1, 2, 4\}$ (depending upon the particular group G). Moreover, we construct a cocompact lattice $\Gamma_0 < G$ realising this minimum.

Here, $Z(G)$ is the centre of G, which is a finite group and is the kernel of the G–action on X. For q even or $q \equiv 3 \mod 4$ we find the minimum covolume among cocompact lattices in G by proving that the lattice which realises this minimum is edge-transitive. For $q \equiv 1 \mod 4$, there are in general no edge-transitive lattices in G, and Γ_0 in this statement has two orbits of edges on X.

A result of independent interest is the following analogue of the fact that lattices in semisimple Lie groups do not contain unipotent elements:

Proposition 3.11.2. *Let G be as in Theorem 3.11.1 above. If Γ is a cocompact lattice in G, then Γ does not contain p–elements.*

This is proved using the dynamics of the G–action on X. Our proofs also use covering theory for graphs of groups, the Levi decomposition for the parahoric subgroups of G and finite group theory.

3.12 Genevieve Walsh (Tufts University) "Quasi-isometry of hyperbolic knot complements"

A result due to R. Schwartz says that cusped hyperbolic 3-manifolds are rigid: they are quasi-isometric exactly when they are commensurable. Thus for hyperbolic knot complements, we have rigidity, and the remaining problem is to classify such knot complements up to commensurability. This is the goal of this work. An easier question is to understand how many knot complements are in a commensurability class. We say that $K \sim K'$ if the 3-manifolds $S^3 \setminus K$ and $S^3 \setminus K'$ are commensurable. Let $C(K) = \{K' | K \sim K'\}$. The following conjecture was made in [24].

Conjecture 3.12.1. *(Reid, Walsh) If $S^3 \setminus K$ is hyperbolic $|C(K)| \leq 3$.*

The following is substantial evidence for this conjecture.

Theorem 3.12.2. *(Boileau, Boyer, Walsh) Let K be a hyperbolic knot without hidden symmetries. Then $|C(K)| \leq 3$.*

For a Kleinian group Γ the commensurator $C^+(\Gamma)$ is the group of those elements g of $PSL(2, \mathbb{C})$ such that $\Gamma \cap g\Gamma g^{-1}$ is finite index in Γ and in $g\Gamma g^{-1}$. A hyperbolic knot admits hidden symmetries if the commensurator of Γ is strictly larger than the normalizer of Γ where $S^3 \setminus K = \mathbf{H}^3/\Gamma$. There are only two knots up to 12 crossings known to have hidden symmetries. In addition, there are restrictions on the shape on the cusp of a knot which has hidden symmetries. Thus we argue that not having hidden symmetries is a generic condition for knot complements. As a partial answer to the classification of hyperbolic knots up to commensurability, the proof yields that hyperbolic knot complements which do not admit hidden symmetries are commensurable exactly when they are cyclically commensurable. By this we mean that they admit a common cyclic cover. A more concrete classification is the following.

Theorem 3.12.3. *(Boileau, Boyer, Walsh) Let K by a periodic hyperbolic knot without hidden symmetries such that $|C(K)| > 1$. Then K is an unwrapped Berge-Gabai knot.*

3.13 Kevin Wortman (University of Utah) "Non-nonpositive curvature of some non-cocompact arithmetic lattices"

We show that irreducible non-cocompact arithmetic groups of type A_n, B_n, C_n, D_n, E_6 and E_7 have an isoperimetric inequality in some dimension that is bounded below by an exponential function. Consequently, such groups do not satisfy any reasonable definition of nonpositive curvature, including for example, combability or $CAT(0)$.

The proof proceeds by constructing an infinite family of cycles in a neighborhood of an orbit of the arithmetic group acting on its associated symmetric space. The volumes of the cycles grow polynomially.

The volumes of any family of chains filling the cycles that are contained in the same neighborhood of the orbit grows exponentially, even though there exist fillings of polynomial volume in the symmetric space.

The proof is a generalization of the proof of Thurston-Epstein that $SL_n(\mathbb{Z})$ is not combable if $n \geq 3$. The cycles we construct are contained on a "horosphere", and their most efficient fillings extend into a horoball that is disjoint from the orbit neighborhood of the arithmetic group.

The type restriction from the statement of the result ensures the existence of a maximal proper parabolic subgroup whose unipotent radical is abelian. It is this parabolic group that determines the horosphere mentioned above, and the unipotent radical being abelian simplifies computations.

3.14 Xiangdong Xie (Georgia Southern University) "Quasiisometries of some negatively curved solvable Lie groups"

Let A be an $n \times n$ matrix. Let $\mathbb{R}$ act on $\mathbb{R}^n$ by $(t, x) \to e^{tA}x$ ($t \in R$, $x \in \mathbb{R}^n$). Denote the corresponding semi-direct product by $G_A = \mathbb{R}^n \rtimes_A \mathbb{R}$. If the eigenvalues of A have positive real parts, then G_A admits left-invariant Riemannian metrics with negative curvature.

We classify all these G_A up to quasiisometry. We show that all quasiisometries between such manifolds (except when they are biLipschitz to the real hyperbolic spaces) are almost similarities and height-respecting.

Furthermore, we derive that such manifolds (except when they are biLipschitz to the real hyperbolic spaces) are not quasiisometric to any finitely generated groups.

Since two negatively curved spaces are quasiisometric if and only if their ideal boundaries are quasisymmetric, we prove these results by studying the quasisymmetric maps on the ideal boundary of these manifolds. We classify the ideal boundaries of negatively curved $G_A = \mathbb{R}^n \rtimes_A \mathbb{R}$ up to quasisymmetry, and show that every quasisymmetric map between the ideal boundaries (except when the solvable Lie groups are biLipschitz to the real hyperbolic spaces) are biLipschitz.

The results of Eskin-Fisher-Whyte, Dymarz and Peng yield quasiisometric classification and rigidity results for the class of groups $\{G_A\}$, where A has eigenvalues with positive real part and eigenvalues with negative real part, but has no eigenvalues with zero real part. Our results complement theirs. The proofs are also completely different.

4 Outcome of the Meeting

Due to late cancellations, the number of partipants was 17 instead of the planned 20, but there was a general consensus that the size of the group led to even closer interactions and more focussed discussion than is common at such meetings. Some progress was made on problems raised at the meeting, but it is too early to predict the full impact on new research and collaborations from the meeting. Suffice it to say that in conversations at the meeting and since, participants have described the meeting as having been unusually successful.

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