(0,2) Mirror Symmetry and Heterotic Gromov-Witten Invariants

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1 Overview of the Field

Shortly after the construction of the ten-dimensional heterotic string theories, it was realized that a compactification of these theories on Calabi-Yau manifolds could yield four-dimensional supersymmetric Poincaré-invariant vacua with the massless spectrum consisting of minimal supergravity coupled to a chiral non-abelian gauge theory. This was a remarkable development in theoretical physics, as it connected a heterotic string theory—believed to be a consistent theory of quantum gravity—to a chiral gauge theory remarkably similar to the Standard Model.

Despite this beautiful relation, it was understood that a number of issues remained to be addressed. For example, it was difficult to produce either the Standard Model gauge group or a grand unified model with couplings that would lead to the Standard Model. Moreover, the construction was perturbative in two senses: the results required small string coupling and the large radius limit, the former being a statement about string perturbation theory, while the latter requiring the compactification geometry to be a smooth manifold with volume large compared to the string length. Both of these issues are intimately tied to the existence of moduli—parameters introduced by the compactification, such as Kähler and complex structures on the Calabi-Yau manifold. What is the structure of this moduli space? How do physical quantities depend on the moduli? Are there heterotic compactifications without a large radius limit? Can one obtain a Standard Model gauge group or a favorable grand unified theory? Does an understanding of these issues teach us something about non-perturbative effects in the heterotic string?

The answers to these questions inevitably lead to new mathematical structures. Broadly speaking, the purpose of the workshop was to bring together researchers who are developing the mathematical structures and applying them to the physical questions. Major themes of the workshop were:

- a generalization of the notion of mirror symmetry to heterotic theories;
- new constructions of four-dimensional vacua from the heterotic string.

In what follows, we will review these areas in a little more detail.

2 Generalizations of mirror symmetry

Mirror symmetry—a proven ground for rich and mutually beneficial interactions between mathematicians and physicists [1]— is an isomorphism of two superconformal field theories (SCFTs) defined on a genus g Riemann surface, with one theory associated to a Calabi-Yau manifold M, and the other to its mirror W. Already at genus zero, the isomorphism yields a precise relation between generating functions of genus zero Gromov-Witten invariants of M and "classical" algebro-geometric period computations on W. The study of mirror symmetry led to physically and mathematically significant insights into geometry, including the clarification of the moduli spaces of the SCFTs and the Calabi-Yau manifolds, the notion of quantum cohomology, computations of Gromov-Witten invariants, an explicit combinatoric construction of mirror pairs as complete intersections in toric varieties, and the homological mirror symmetry conjectures.

The SCFTs typically considered in mirror symmetry possess $(2, 2)$ world-sheet supersymmetry, a property with a number of important ramifications:

- in the case of $(2, 2)$ SCFTs associated to Calabi-Yau manifolds, the moduli space has a familiar local structure, splitting into the moduli space of the complexified Kähler form and the moduli space of complex structures;
- a $(2, 2)$ SCFT has a chiral ring—a set of local operators with a well-defined product;
- correlators of chiral operators are independent of world-sheet metric and are computed by a Topological Field Theory (TFT).

The correlators transform as sections of certain bundles over the moduli space and may be determined by working with a topologically twisted (2,2) non-linear sigma model—a field theory of maps from a Riemann surface to the Calabi-Yau manifold. The correlators of the resulting TFT provide a path integral representation for the Gromov-Witten generating functions. The twisting procedure may be refined in an important way when M is a hypersurface in a toric variety V [2]. The result is a "quantum restriction formula" that relates correlators in the M SCFT to correlators in a vastly simpler TFT associated to V, known as the gauged linear sigma model (GLSM) [2, 3]. Together with the mirror map—an isomorphism between the complexified Kähler moduli space of M and the complex structure moduli space of W—these completely determine the correlators.

Despite these remarkable features, the moduli space of $(2, 2)$ SCFTs typically constitute a surprisingly unremarkable locus in a larger moduli space of SCFTs preserving $(0, 2)$ supersymmetry [4, 5, 6, 7, 8]. This larger moduli space has a geometric interpretation: the defining data of a (torsion-free) $(0, 2)$ non-linear sigma model is a Calabi-Yau manifold M and a stable holomorphic vector bundle $E \to M$, with

$$c_1(E) = 0 \text{ and } \mathrm{ch}_2(E) = \mathrm{ch}_2(TM).$$

A $(2, 2)$ point corresponds to $E = TM$. For example, the familiar $(2, 2)$ SCFT associated to a quintic hypersurface in $\mathbb{C}P^4$ has 224 deformations that only preserve $(0, 2)$ supersymmetry, and all of the familiar quantities like the moduli space metric, Yukawa couplings, and quantum cohomology are expected to vary smoothly across the $(2, 2)$ locus. In addition, there are $(0, 2)$ theories without a $(2, 2)$ point in the moduli space. From the physical point of view, these $(0, 2)$ SCFTs provide the basic building blocks for a large class of phenomenologically interesting compactifications of the heterotic string.

2.1 Recent developments

Topological rings and quantum cohomology. A revival of interest in world-sheet (0,2) models came with the observation [9] that (0,2) theories seemed to have a ground ring akin to the chiral ring of (2,2) models. This was first observed by an application of Hori-Vafa duality, and then was confirmed by direct computations [10]. In [11] a proof was given that such structures indeed exist for all theories based on a holomorphic bundle of rank less than 8 over a Calabi-Yau manifold. It was also shown that massive theories, such as those defined by a bundle over a toric variety possess the ring structure. Techniques were developed for computing these "topological heterotic rings" for (0,2) models constructed by deforming the tangent bundle over compact Kähler toric varieties [12, 13], for (0,2) Landau-Ginzburg models [14, 15], as well as for Calabi-Yau hypersurfaces in toric varieties via quantum restriction [16]. These developments were presented in the talks of Sharpe, Guffin, and McOrist.

The results of these computations are interesting from mathematical and physical perspectives alike. On the physics side, they encode non-trivial information about a strongly-coupled (0,2) quantum field theory and lead to (un-normalized) Yukawa couplings of space-time fields. Mathematically they lead to a deformation of the usual quantum cohomology, as well as a notion of a "quantum sheaf"—an object whose properties depend on both the complex structure and Kähler moduli of the underlying manifold.

A (0,2) mirror map. The concrete computations motivated a search for a generalization of the mirror map. On physical grounds, it is clear that since the (2,2) SCFTs associated to a mirror pair M and W are isomorphic, there must be a map of parameters for the deformations of TM to a bundle $E \to M$ to the deformations of TW to a bundle $F \to W$. Given a (2,2) GLSM for M inside a toric variety V, certain deformation are naturally realized as holomorphic parameters of the GLSM. Experience with (2,2) mirror symmetry suggested the hypothesis that a (0,2) mirror map should exchange the holomorphic parameters of the GLSM for $E \to M$ with those of the GLSM for $F \to M$. However, a counting of holomorphic parameters in mirror pairs showed that for most hypersurfaces $M \subset V$ this is not the case [17]. A priori this does not constitute a failure of mirror symmetry, as some of the deformations may be realized by more complicated operators, but it does make it difficult to find an explicit mirror map. Nevertheless, as reported in the lecture by Plesser, for a special class of GLSMs, corresponding to "reflexively plain polytopes" [17], where the number of GLSM parameters match on the two sides of the mirror, an explicit (0,2) map can be constructed [18]. It was shown that the proposed isomorphism exchanged the singular loci of the mirror theories—sub-varieties in the moduli space where the topological heterotic rings become singular. This constitutes a simple test of the proposal.

Towards higher genus results. The world-sheet methods described above are all, in one way or another, tied to the gauged linear sigma model. The relation between (2,2) gauged linear sigma model correlators and Gromov-Witten invariants at genus zero has been understood for some time; however, an extension beyond genus zero remained elusive. In his talk Diaconescu described a recently developed mathematical construction [19, 20]. If it is possible to generalize this to (0,2) models, it would provide a higher genus version of (0,2)-deformed Gromov-Witten theory.

Special geometry and Kähler potentials. While many aspects of the moduli space of (2,2) theories and (2,2) mirror symmetry have a straightforward (0,2) generalization, at least in the case where the bundle is a deformation of the tangent bundle, there are some notable exceptions. First, there is no longer a correspondence between the moduli fields and matter charged under the unbroken gauge group. More importantly, the moduli space of a (2,2) theory is constrained to be a special Kähler manifold. This is most straightforward to see in the context of type II string theory, where this follows from the requirements of $N = 2$ space-time supersymmetry, but it may also be determined directly in the heterotic string [21] by using Ward identities of the underlying (2,2) SCFT.

This is a remarkably powerful constraint, since it means that the Kähler potential, a real function of the moduli, is determined in terms of a holomorphic prepotential. The various techniques available to compute exact quantities as functions of the moduli are typically powerful enough to determine holomorphic quantities, such as superpotential couplings and prepotentials, but extending the methods to compute real quantities seems difficult, if not impossible.

A generic (0,2) compactification preserves $N = 1$ space-time supersymmetry. This requires the moduli space to be a Kähler Hodge manifold (i.e. a manifold X with a Kähler form with integral class in $H^2(X)$.), but does not impose more stringent conditions. At the same time, lacking the additional Ward identities of (2,2) SCFT, there are no other "obvious" constraints on the moduli space geometry. Thus, with the current state of affairs, the available computational techniques are seemingly not powerful enough to determine the Kähler potential, and hence the moduli space geometry.

The moduli space metric is an important lacuna in the understanding of heterotic string theory. For instance, it is needed to compute properly normalized physical couplings in the effective four-dimensional theory. In addition, a knowledge of the metric is needed to determine which (possibly singular) points in the moduli space are a finite distance away, and which correspond to infinite distance "de-compactification" limits.

Despite such a gloomy perspective from $N = 1$ supergravity, there is some reason to be optimistic. The key idea is that an $N = 1$ supergravity theory that arises from a string compactification may not be generic, and thus may have additional properties such as some notion of special geometry. Evidence for this has been found in the context of type II compactifications involving D-branes. It was shown some time ago in [22] that such a theory, although it possesses only $N = 1$ space-time supersymmetry, does seem to have a preferred set of holomorphic coordinates. While originally this was developed for D-branes on non-compact Calabi-Yau manifolds, the results have since been extended to F-theory and heterotic compactifications [23, 24, 25]. This work, presented by Jockers, offers a hope that the extra structure does give additional control on four-dimensional physics. In particular, a concrete proposal is made for the Kähler potential involving certain complex structure and bundle moduli.

Another perspective on this was offered by Quigley, who reported on ongoing work in collaboration with Anguelova and Sethi. The idea was to explicitly evaluate certain perturbative corrections to the Kähler potential for the complex moduli. They too found encouraging signs suggesting a decoupling between the different moduli beyond leading order results. It would be very interesting if this could be strengthened into a full non-renormalization theorem for the Kähler potential of complex and bundle moduli.

2.2 Open problems

The structure of (0,2) theories and their moduli spaces has been elucidated by a number of new results presented and discussed during the workshop. However, many exciting and crucial problems remain. A very incomplete list might be:

1. What is the mathematical framework appropriate to describe the quantum bundles and deformed quantum cohomology? Can recent results mathematical results on higher genus GLSM invariants be generalized to the (0,2) setting?

2. There is a proposed mirror map for models where the bundle is a deformation of TM. Can it be shown to exchange topological heterotic rings? Can the proposal be extended to other examples, say with bundles of rank 4 or 5? Does it relate the (0,2) mirror pairs found in [26, 27]?

3. Can the results be extended to the class of deformations that are not readily identified with holomorphic parameters of the GLSM? For instance, are such additional (0,2) deformations obstructed by world-sheet instantons?

4. Is there an $N = 1$ special geometry structure intrinsic to (0,2) half-twisted theories? F-theory/heterotic duality suggest that the answer is likely yes, but it would be instructive to find this structure directly.

5. Are there non-renormalization theorems for the Kähler potential? Can it be determined in terms of some holomorphic quantities?

3 New heterotic constructions

The preceding section dealt with the world-sheet properties of heterotic theories constructed in a rather standard fashion: the base manifold is a Calabi-Yau three-fold, and the bundle is a deformation of the tangent bundle. As already mentioned, such a construction has been known for some time, and while it may teach us some general lessons about (0,2) theories and their moduli spaces, it would be nice to get a handle on more generic theories. However, before one tackles issues of moduli spaces and stringy geometry of these more generic theories, it is necessary to produce the examples themselves.

At the level of perturbative heterotic string, in order to build a string vacuum with four-dimensional Poincaré invariance and minimal supersymmetry, we must choose a modular-invariant (0,2) SCFT with central charges $(c, \bar{c}) = (22, 9)$ and a supersymmetric GSO projection. Unfortunately, a classification of such objects is well out of the reach of current technology, and we must be content with a more specific constructions. Some of these, such as asymmetric orbifolds, have the advantage of being exactly solvable theories, while others have a closer connection to geometry.

The geometric models are defined by some anomaly-free (0,2) NLSM corresponding to a compactification geometry $E \to M$ as described above. Supersymmetry and anomaly cancellation require that M is a complex manifold with $c_1(TM) = 0$, while the bundle satisfies the topological conditions $c_1(E) = 0$

$\mod 2$ and $\mathrm{ch}_2(E) = \mathrm{ch}_2(TM)$. There are, however, additional requirements in order for the NLSM to be a superconformal theory at the quantum level. At leading order in the NLSM coupling α', these require E to admit a Hermitian Yang-Mills connection. In addition, the Hermitian form ω and the non-vanishing holomorphic three-form Ω defined on M must satisfy [28, 29]

$$4i\partial\bar{\partial}\omega = \alpha'(\mathrm{Tr}R \wedge R - \mathrm{Tr}F \wedge F), \qquad d(||\Omega||\omega \wedge \omega) = 0,$$

where R is the Ricci form and F is the curvature of the bundle E.

In general these are complicated equations, and the existence of solutions for general $E \to M$ satisfying the topological requirements is difficult to establish by a direct analysis. One standard approach is to consider solutions that have a large radius limit. In this limit the Hermitian form is closed, so that M is a Calabi-Yau manifold. As long as E is chosen to be a stable bundle, the Donaldson-Uhlenbeck-Yau theorem guarantees existence of a Hermitian Yang-Mills connection. This limit can be used a starting point for constructing a moduli space of solutions; moreover in the limit many properties of the effective four-dimensional theory can be determined by the algebraic geometry underlying $E \to M$.

A more ambitious approach is to look for a more general class of solutions, for instance on M that does not admit a Kähler structure, and thus cannot possess a large radius limit. This is the realm of heterotic flux compactifications. A priori, the resulting NLSMs must be taken with a large grain of salt, since the quantum corrections are large, and it is not obvious what the corrected string equations are, nor that a solution to the system above implies a solution to the full equations. Remarkably, duality arguments show that such theories do exist as bona fide heterotic vacua [30].

3.1 Recent developments

Bundles over Calabi-Yau manifolds. Heterotic compactifications over Calabi-Yau manifolds have been studied for almost the entire history of the heterotic string itself. In principle, many of the physical quantities are determined by specific algebraic geometry computations. For instance, the massless spectrum is determined by certain Dolbeault cohomology groups valued in the bundle and related sheaves. Similarly, Yukawa couplings of the matter theory may be computed by studying a holomorphic Chern-Simons theory on M.

These computations are, however, quite difficult in practice, and the technology is still being developed. In the workshop Donagi reviewed the spectral cover methods for constructing explicit bundles, computing spectra and Yukawa couplings. These techniques [31], based on Atiyah's construction of the moduli space of vector bundles over an elliptic curve, allow an explicit construction of phenomenologically interesting stable holomorphic bundles over elliptically fibered Calabi-Yau manifolds. These techniques have led to a heterotic construction of a theory with the charged matter spectrum of the minimal supersymmetric extension of the Standard Model [32], as well as explicit computations of certain Yukawa couplings, e.g. [33, 34].

In addition to the bundle data, there is also the choice of a base Calabi-Yau manifold. New manifolds are still being found, some with quite desirable (from a phenomenological point of view) properties. Candelas described one such construction, where a Calabi-Yau manifold with Euler number -6 is constructed as a quotient of a complete intersection in $(CP^2)^4$ by a freely acting group of order 12.

Heterotic flux backgrounds. Substantial progress has also been made in the study of compactifications where M does not admit a Kähler structure. It was shown in [35] that a class of M constructed as a non-trivial T^2 principal bundle over $K3$ does not admit a Kähler structure. This was just the sort of compactification identified by the duality argument of [30]. It was proven in [36] that such an M admits a solution to the NLSM equations of motion.

A substantial generalization of this construction [37] was discussed by Becker, who showed that there exists a much larger class of flux backgrounds. A particularly interesting technical point that arose in that investigation is a choice of preferred connection in defining the Chern-Simons terms appearing in the heterotic H-field. This choice is natural from the point of view of space-time supersymmetry and may substantially simplify a direct analysis of the existence of solutions.

Another important generalization, presented in Sethi's talk, concerns "non-geometric" compactifications of the heterotic string. The construction discussed the class of backgrounds obtained by fibering the heterotic string on a T^2 over some non-trivial base manifold. Such a compactification will be non-geometric provided that the Kähler and complex structure of the T^2 both undergo monodromies over the base space [38]. By using F-theory/heterotic duality it is possible to show that under certain conditions such a non-geometric compactification should be a consistent string vacuum.

There has also been progress on constructing a gauged linear sigma model for heterotic flux backgrounds [39, 40]. As discussed in Lapan's talk, the current constructions can only accommodate the rather special compactifications preserving $N = 2$ space-time supersymmetry. However, they do allow computations of exact spectra at Landau-Ginzburg points, and it is to be hoped that a further exploration of their properties will help to describe the moduli space of heterotic flux backgrounds, perhaps even leading to further generalizations of mirror symmetry and stringy geometry.

3.2 Open problems

Much remains to be understood in these large classes of backgrounds. We list just a few of the outstanding issues.

1. Are there world-sheet instanton corrections to the DUY stability conditions?
2. Can the methods based on half-twisted theories be applied to the phenomenologically promising compactifications?
3. Currently, there are few techniques for exploring the moduli space of heterotic flux backgrounds. Can we at least have a method for counting the moduli?
4. What are the topological heterotic rings in the linear sigma model for flux vacua?
5. Is there a world-sheet description of at least some of the non-geometric heterotic vacua?

4 Outcome of the meeting

The workshop brought together mathematicians and physicists who have been pursuing several quite distinct approaches to the heterotic string. The atmosphere created by BIRS—with simple organization, wonderfully efficient staff, excellent facilities, and of course incredible views—helped us to learn about the progress made, the technical issues, and the problems that remain in these different research directions. The small size of the workshop allowed the lectures to be fairly informal, often leading to an extended discussion with the speaker. These conversations would continue in the cozy atmosphere of Corbett Hall or while enjoying the great food in the dining hall. The workshop brought together several collaborations, allowing the members to work together, sometimes by grabbing a non-member for a consultation. A number of the problems defined at the workshop are now under active investigation, and we are sure that the collaborations either continued or begun at the workshop will open many new directions for progress. Several participants have voiced the opinion that the workshop was "one of the most useful I have ever attended," and we are sure this is sentiment that would be seconded by most, if not all of the attendees.

It is a pleasure to thank BIRS for this wonderful opportunity!

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