REPORT OF FOCUSED RESEARCH GROUP:
NONLINEAR DISCRETE OPTIMIZATION

BANFF JULY 2010

This is a report on the focus research group 10frg140 *Nonlinear Discrete Optimization*, which met at BIRS during the week of July 18 to July 25, 2010. The participants were Jesus De Loera (UC Davis, Mathematics) Raymond Hemmecke (Darmstadt, Mathematics) Matthias Koeppe (UC Davis, Mathematics) Jon Lee (IBM Research, Mathematical Sciences) Shmuel Onn (Technion, Industrial Engineering) Robert Weismantel (Magdeburg, Mathematics) The participants looked at the general problem of optimizing nonlinear functions over a finite space of feasible solutions. We often assumed that the feasible solutions are the integer solutions of integer programs defined by linear equations and inequality constraints (a convex polyhedron). Without loss of generality one can assume that the optimization problem has the form:

$$max/min f(x) \text{ subject to } Ax = bx \geq 0,\ x \in \mathbb{Z}^n,$$

and thus the type of matrix A we have as input is quite important to define the difficulty of the problem. Although we only managed to prove minor lemmas and results, we had a very exciting week, full of questions, discussion, and ideas that will carry us through for a long collaboration. Here are expand on three areas of discussion we had. Two are about finding new extensions of very successful techniques, the third is about a generalization of the hot area of compressed sensing:

(1) We are interested on the use of test sets for solving problems. a *test set* is a finite collection of integral vectors with the property that every feasible non-optimal solution of an integer program can be improved by adding a vector in the test set [11]. Two of the most famous sets are Graver and Gröbner test sets. Although test sets are normally very very large and impractical, there is nice special case in which the linear system is of N-fold type. N-fold systems are nice models and include multidimensional transportation and assignment problems with mild restrictions [9, 3, 4]. The members of the team had in prior work proved that convex integer minimization and maximization is possible for N-fold problems. The transpose of an N-fold integer program gives rise to a model of 2-stage stochastic integer programming. In the latter case, algorithms for convex integer minimization and maximization are significantly more complicated than for ordinary N-fold systems, but were achieved by members of the team too. Due to their very nice properties is desirable to generalize the concept of N-fold systems and gain structural insight that will allow us to find similar computational efficiency for larger classes of problems. Although our discussions yielded so far no new major result, we produced several leads as to what possible generalizations could exist. An important property of test sets is that they have an integral Carathéodory property says that each integral vector in

the positive cone generated by the columns of a matrix can be written as a nonnegative integer combination of at most dimension many elements of the test set. This has been useful to prove efficient termination of some convex minimization problems for which a Graver test set is known. One of the main open questions we discussed is whether the integral Carathéodory property holds for other test sets besides Graver test sets.

Finally, and also related to generalizing N-fold matrices, we discussed a question raised by Peter Gritzmann: It is well-known that totally unimodular matrices A have the property that their linear integer optimization problem can be solved in polynomial time by linear programming. Thus it is natural to wonder whether matrices that are obtained from a unimodular matrix from the operation of replacing its entries by a fixed matrix block still give a nicely behaved matrix. We were able to prove this is not the case, we found that some nasty family of NP-hard integer programs lies within this construction.

(2) Because of applications to image processing a very exciting direction of modern research is *compressed sensing*. In those areas one needs to solve an underdetermined system of linear equations maximizing the number of zero coefficients in the solution. Searching a solution with this constraint is NP-hard, and so is computationally infeasible, but it was recognized (see [10]) that solving a linear program (which runs very very efficiently) already approximates nicely the desired sparseness solution. Our observation is that one can consider We observed that support functions maximizing the number of zeros is a special case of a *quasi-convex function* (the inverse image of any set of the form $(-\infty, a)$ is a convex set) and tried to understand several variations of optimizations problems with linear constraints and quasi-convex objective function $q(x)$ and how to include integrality conditions which complicate all these problems much more. Methods we discussed include Diophantine approximation and dynamic programming, but the ultimate goal is to find a linear or convex program that again approximates well the behavior of $q(x)$. For example, we proved

Theorem 1. *Let $D_i = \{l_i, \ldots, u_i\}$ and $D = D_1 \times \cdots \times D_N$. Let U be an upper bound on $u_i - l_i + 1$. For any separable convex function $f\colon D \to \mathbb{R}$, $f(\mathbf{x}) = f_1(x_1) + \cdots + f_N(x_N)$, there exists a separable convex function $g\colon D \to \mathbb{Z}$, $g(\mathbf{x}) = g_1(x_1) + \cdots + g_N(x_N)$ such that*

(i) *$f(\mathbf{x}) \leq f(\mathbf{y})$ if and only $g(\mathbf{x}) \leq g(\mathbf{y})$,*

(ii) *the functions g_i take values on their domain $\{l_1, \ldots, u_1\}$ whose* binary encoding size *is bounded by a polynomial in N and U.*

In fact, there exists an (inefficient) algorithm that computes this.

There are of course many open questions: Can we improve the above result to be useful for binary-encoded upper bounds as well? Suppose now the objective function is a separable quasi-convex function Does there exist a convex function that refines the induced by semiorder of the quasi-convex function. In the above "compression LP", we could just leave out the equations $g(x) = g(y)$ for the x, y with $f(x) = f(y)$ to construct a separable convex function g that refines the semiorder if it exists. It is open whether it exists though, i.e., whether this LP is always feasible. We have several ideas for attacking this problem.

(3) Finally, the members of the team have been very active on using generating function techniques (see [1, 2]), they contributed a fully polynomial approximation scheme for the problem of optimizing

$$max f(x) \text{ subject to } Ax = bx \geq 0,\ x \in \mathbb{Z}^n.$$

when n is a fixed constant and f is a non-negative polynomial function, and even with multiple objective functions (see [7, 8, 6, 5]). They also proved that the already optimizing a degree four polynomial and $n = 2$ is NP-hard, but the tantalizing question still remains what is the complexity of the same maximization problem when $f = x^T Qx$, Q is a 2×2 matrix. Is this problem NP-hard? We think this is a fascinating simple problem that needs to be answered.

References

[1] Alexander I. Barvinok and James E. Pommersheim. An algorithmic theory of lattice points in polyhedra. In Louis J. Billera, Anders Björner, Curtis Greene, Rodica E. Simion, and Richard P. Stanley, editors, *New Perspectives in Algebraic Combinatorics*, volume 38 of *Math. Sci. Res. Inst. Publ.*, pages 91–147. Cambridge Univ. Press, Cambridge, 1999.

[2] Alexander I. Barvinok and Kevin Woods. Short rational generating functions for lattice point problems. *Journal of the AMS*, 16(4):957–979, 2003.

[3] Yael Berstein and Shmuel Onn. The Graver complexity of integer programming. *Ann. Comb.*, 13(3):289–296, 2009.

[4] J. A. De Loera, R. Hemmecke, S. Onn, U. G. Rothblum, and R. Weismantel. Convex integer maximization via Graver bases. *J. Pure Appl. Algebra*, 213(8):1569–1577, 2009.

[5] Jesús A. De Loera, David Haws, Raymond Hemmecke, Peter Huggins, and Ruriko Yoshida. A computational study of integer programming algorithms based on Barvinok's rational functions. *Discrete Optimization*, 2:135–144, 2005.

[6] Jesús A. De Loera, Raymond Hemmecke, and Matthias Köppe. Pareto optima of multicriteria integer linear programs, 2009.

[7] Jesús A. De Loera, Raymond Hemmecke, Matthias Köppe, and Robert Weismantel. Integer polynomial optimization in fixed dimension. *Mathematics of Operations Research*, 31(1):147–153, 2006.

[8] Jesús A. De Loera, Raymond Hemmecke, Matthias Köppe, and Robert Weismantel. FPTAS for optimizing polynomials over the mixed-integer points of polytopes in fixed dimension. volume 115, pages 273–290, 2008.

[9] Jesús A. De Loera, Raymond Hemmecke, Shmuel Onn, and Robert Weismantel. n-fold integer programming. *Discrete Optim.*, 5(2):231–241, 2008.

[10] David L. Donoho. Compressed sensing. *IEEE Trans. Inform. Theory*, 52(4):1289–1306, 2006.

[11] Jon Lee, Shmuel Onn, and Robert Weismantel. On test sets for nonlinear integer maximization. *Oper. Res. Lett.*, 36(4):439–443, 2008.