Probabilistic and Extremal Combinatorics*

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1 Overview

Combinatorics, sometimes also called Discrete Mathematics, is a branch of mathematics focusing on the study of discrete objects and their properties. Although Combinatorics is probably as old as the human ability to count, the field experienced tremendous growth during the last fifty years and has matured into a thriving area with its own set of problems, approaches and methodology.

Extremal and Probabilistic Combinatorics are two of the most central branches of the modern combinatorial theory. Extremal Combinatorics deals with problems of determining or estimating the maximum or minimum possible cardinality of a collection of finite objects satisfying certain requirements. Such problems are often related to other areas including Computer Science, Information Theory, Number Theory and Geometry. This branch of Combinatorics has developed spectacularly over the last few decades. Probabilistic Combinatorics can be described informally as a (very successful) hybrid between Combinatorics and Probability, whose main object of study is probability distributions on discrete structures. Although probabilistic arguments have proven to be extremely powerful when applied in problems from adjacent fields in Combinatorics and Theoretical Computer Science, Probabilistic Combinatorics can undoubtedly considered an independent discipline with its own methodology and objects of study, most notably random graphs.

Roughly speaking, Probabilistic Combinatorics comprises three main topics, for each of which we give a short description. Naturally, there are considerable overlaps between these topics.

The first topic is the application of probability to solve combinatorial problems. Typical examples are the "existence" proofs in which one generating an appropriate probabilistic space to show existence of certain object. The last twenty years or so have witnessed significant progress in this topic. The development of new and powerful techniques, such as the semi-random method and various sharp concentration inequalities, has enabled researchers to attack many famous open problems, which were considered intractable not so long ago, with considerable success. The area in which this has been strikingly successful is Extremal Combinatorics.

The second topic is the analysis of properties of random structures, mainly random graphs and hypergraphs. This study was initiated by Erdős and Rényi around 1960 and by now there is a rich and beautiful theory of random graphs, and many models of random graphs are fairly well understood. These include the classical models of Erdős and Renyi, the investigation of graph processes and hitting times, the well studied models of random regular graphs, and various less studied and more recent models based on preferential attachment in which the intention is to explain the behavior of real world networks, like the graph of the Internet. Other closely related models in which there have been some recent exciting developments and yet much less is known

*Report on BIRS Workshop 09w5035 Probabilistic and Extremal Combinatorics held at Banff International Research Station (BIRS), Banff Center, August 23-28, 2009.

deal with random subgraphs of given graphs, various percolation models and the study of the random k-SAT problem.

The third topic is the study of randomized algorithms. Here the main question is either to design randomized algorithms for a certain goal or to analyze natural algorithms given special inputs. While this topic can also be considered as a topic in Computer Science, it has turned out quite recently that it also has much to do with Statistical Physics. For instance, there is a natural algorithm (motivated by problems from statistical physics) for generating a random coloring of a graph. A tantalizing question is to know when this algorithm runs in polynomial time, and a proper bound would have amazing consequences in Physics.

The subject of Extremal Combinatorics is perhaps less structured than Probabilistic Combinatorics, for this reason we will confine ourselves here to describing in brief few of its most important topics.

The first topic of Extremal Combinatorics we would like to mention is Extremal Graph Theory. There, the subject of study is extremal problems on graphs. Problems and results such as the maximum possible number of edges in a planar graph with a given number of vertices (with possibly some additional restrictions added) or the maximum possible number of edges in a graph of given order not containing a copy of a forbidden graph (so called Turán-type problems) fall into this category. This is one of the most important branches on modern Graph Theory, with a variety of methods and arguments (linear algebraic arguments, probabilistic considerations, ad hoc proofs) applied. The development of this subject was very instrumental in turning Graph Theory into modern, deep and versatile field of Combinatorics.

The second subject is Extremal Set Theory. There, extremal problems are usually formulated and studied for families of sets satisfying given restrictions. Good examples include the famous Sperner theorem about the maximum cardinality of a family of subsets of the ground set of size n with no two family members containing each other, or the Erdős-Ko-Rado theorem about the maximum size of a family of subsets of 1,...,n, in which every two members share at least one common element. Problems of this type are especially appealing, in part due to the fact that quite a few of them arise in a variety of applications in diverse fields of Mathematics, Computer Science, Coding and Information Theory.

Ramsey Theory is undoubtedly one of the most central branches of modern mathematics, studying quantitatively the following phenomenon: every large object, chaotic as it may be, contains a sub-object who is guaranteed to be well structured, in certain appropriately chosen sense. This phenomenon is truly ubiquitous and manifest itself in a variety of ways, ranging from the most basic Pigeon Hole Principle to intricate statements from Set Theory. Quite a few questions from Ramsey theory, including estimates on the so called Ramsey numbers, can be cast and viewed as problems in Extremal Graph or Set Theory. Probabilistic arguments are essential here, and their importance and applicability can not be overestimated.

There are now fields of Graph Theory and Combinatorics that combine both extremal and probabilistic mindsets in the most natural ways. Extremal Theory of Random Graphs is just one such subject, there one studies typical behavior of basic graph theoretic parameters over the probability space of random graphs. This topic has served as a catalyst for developing new deep combinatorial tools, the so called Sparse Regularity Lemma being one of them. Another hybrid subject is the theory of pseudo-random graphs, where one tries to capture quantitatively (sometimes quite elusive) properties of random graphs, and to suggest deterministic models of random graphs; this theory has had quite a few important applications to Computer Science and Coding Theory.

The workshop specifically focused on several major research topics in modern Combinatorics. These topics include Extremal Problems for Graphs and Hypergraphs, Ramsey Theory, Random Graphs, Quasi-random Graphs, Additive Combinatorics and Probabilistic Methods. One aim of the workshop was to foster interaction between researchers in these rather diverse fields and to discuss recent progress and to communicate new results. We've also put an emphasis on the exchange of ideas, approaches and techniques between Probabilistic and Extremal Combinatorics.

In the remainder of the report we describe in detail some of the advances presented at the workshop.

2 Erdős-Rényi model of random graphs

CRITICAL RANDOM GRAPHS: LIMITING CONSTRUCTIONS AND DISTRIBUTIONAL PROPERTIES

Louigi Addario-Berry *joint with N. Broutin and C. Goldschmidt*

Since its introduction by [10], the model $G(n,p)$ of random graphs has received an enormous amount of attention [14, 3]. In this model, a graph on n labeled vertices $\{1,2,\dots,n\}$ is chosen randomly by joining any two vertices by an edge with probability p, independently for different pairs of vertices. This model exhibits a radical change in structure (or *phase transition*) for large n when $p = p(n) \sim 1/n$. For $p \sim c/n$ with $c < 1$, the largest connected component has size (number of vertices) $O(\log n)$. On the other hand, when $c > 1$, there is a connected component containing a positive proportion of the vertices (the *giant component*). The cases $c < 1$ and $c > 1$ are called *subcritical* and *supercritical* respectively. This phase transition was discovered by Erdős and Rényi in their seminal paper [10]; indeed, they further observed that in the *critical* case, when $p = 1/n$, the largest components of $G(n,p)$ have sizes of order $n^{2/3}$. For this reason, the phase transition in random graphs is sometimes dubbed the *double jump.*

Understanding the critical random graph (when $p = p(n) \sim 1/n$) requires a different and finer scaling: the natural parameterization turns out to be of the form $p = p(n) = 1/n + \lambda n^{-4/3}$, for $\lambda = o(n^{1/3})$ [4, 16, 17]. In this talk, we will restrict our attention to $\lambda \in \mathbb{R}$; this parameter range is then usually called the *critical window.* One of the most significant results about random graphs in the critical regime was proved by [1]. He observed that one could encode various aspects of the structure of the random graph (specifically, the sizes and surpluses of the components) using stochastic processes. His insight was that standard limit theory for such processes could then be used to get at the relevant limiting quantities, which could, moreover, be analyzed using powerful stochastic-process tools. Fix $\lambda \in \mathbb{R}$, set $p = 1/n + \lambda n^{-4/3}$ and write Z_i^n and S_i^n for the size and surplus (that is, the number of edges which would need to be removed in order to obtain a tree) of $\mathcal{C}_i^n$, the i-th largest component of $G(n,p)$. Set $\mathbf{Z}^n = (Z_1^n, Z_2^n, \dots)$ and $\mathbf{S}^n = (S_1^n, S_2^n, \dots)$.

Theorem 1 ([1])**.** *As $n \to \infty$.*

$$(n^{-2/3}\mathbf{Z}^n, \mathbf{S}^n) \xrightarrow{d} (\mathbf{Z}, \mathbf{S}).$$

Here, the convergence of the first co-ordinate takes place in $\ell^2_{\searrow}$, the set of infinite sequences $(x_1, x_2, \dots)$ with $x_1 \geq x_2 \geq \dots \geq 0$ and $\sum_{i\geq 1} x_i^2 < \infty$. (See also [17, 15].) The limit $(\mathbf{Z}, \mathbf{S})$ is described in terms of a Brownian motion with parabolic drift, $(W^\lambda(t), t \geq 0)$, where

$$W^\lambda(t) := W(t) + t\lambda - \frac{t^2}{2},$$

and $(W(t), t \geq 0)$ is a standard Brownian motion. The limit $\mathbf{Z}$ has the distribution of the ordered sequence of lengths of excursions of the reflected process $W^\lambda(t) - \min_{0\leq s\leq t} W^\lambda(s)$ above 0, while $\mathbf{S}$ is the sequence of numbers of points of a Poisson point process with rate one in $\mathbb{R}^+ \times \mathbb{R}^+$ lying under the corresponding excursions. Aldous's limiting picture has since been extended to "immigration" models of random graphs [2], hypergraphs [12], and most recently to random regular graphs with fixed degree [18].

The purpose of this work is to give a precise description of the limit of the sequence of *components* $\mathcal{C}^n = (\mathcal{C}_1^n, \mathcal{C}_2^n, \dots)$. Here, we view $\mathcal{C}_1^n, \mathcal{C}_2^n, \dots$ as metric spaces $M_1^n, M_2^n, \dots$, where the metric is the usual graph distance, which we rescale by $n^{-1/3}$. The limit object is then a sequence of compact metric spaces $\mathbf{M} = (M_1, M_2, \dots)$. The appropriate topology for our convergence result is that generated by the Gromov–Hausdorff distance on the set of compact metric spaces, which we now define. Firstly, for a metric space (M, δ), write d_H for the Hausdorff distance between two compact subsets K, K' of M, that is

$$d_H(K, K') = \inf\{\epsilon > 0 : K \subset F_\epsilon(K') \text{ and } K' \subset F_\epsilon(K)\},$$

where $F_\epsilon(K) := \{x \in M : \delta(x, K) \leq \epsilon\}$ is the ϵ-fattening of the set K. Suppose now that X and X' are two compact metric spaces, each "rooted" at a distinguished point, called ρ and ρ' respectively. Then we define the *Gromov–Hausdorff* distance between X and X' to be

$$d_{GH}(X, X') = \inf\{d_H(\phi(X), \phi'(X')) \vee \delta(\phi(\rho), \phi(\rho'))\}$$

where the infimum is taken over all choices of metric space (M, δ) and all isometric embeddings $\phi : X \to M$ and $\phi' : X' \to M$. (We consider G to be rooted at its vertex of smallest label.) We then have the following result.

Theorem 2. *As $n \to \infty$,*

$$(n^{-2/3}\mathbf{Z}^n, n^{-1/3}\mathbf{M}^n) \xrightarrow{d} (\mathbf{Z}, \mathbf{M}),$$

for an appropriate limiting sequence of metric spaces $\mathbf{M} = (M_1, M_2, \ldots)$. Convergence in the second co-ordinate here is in the metric specified by

$$d(\mathbf{A}, \mathbf{B}) = \left(\sum_{i=1}^{\infty} d_{\mathrm{GH}}(A_i, B_i)^4\right)^{1/4} \tag{1}$$

for any sequences of metric spaces $\mathbf{A} = (A_1, A_2, \ldots)$ and $\mathbf{B} = (B_1, B_2, \ldots)$.

Resilient pancyclicity of random graphs

Choongbum Lee *joint with M. Krivelevich and B. Sudakov*

The systematic study of resilience, recently initiated by Sudakov and Vu [21], is a fascinating field which provides interesting connection between classical extremal graph theory and random graphs. For example, Sudakov and Vu successfully extended Dirac's theorem, which says that every graph of minimum degree greater than $n/2$ contains a hamilton cycle, to random graphs in a following way.

Theorem 3. *If $p \geq \log^4 n/n$ then, $G = G(n,p)$ a.a.s. has the following property. For any subgraph $H \subset G$ of maximum degree at most $(1/2 + o(1))np$, $G - H$ contains a hamilton cycle.*

A graph G on n vertices is *pancyclic* if it contains cycles of length t for all $3 \leq t \leq n$. A classical theorem of Bondy [5] says if a graph has minimum degree greater than $\lfloor n/2 \rfloor$, then it must be pancyclic. We proved the following theorem which extends this theorem to random graphs.

Theorem 4. *For any fixed integer $l \geq 3$, if $p \gg n^{-1+1/(l-1)}$ then $G = G(n,p)$ asymptotically almost surely has the following property. For any subgraph $H \subset G$ of maximum degree ate most $(1/2 + o(1))np$, $G - H$ contains cycles of length t for all $l \leq t \leq n$.*

These results are tight in two ways. First, the condition on p essentially cannot be relaxed and second, it is impossible to improve the constant $1/2$ in the assumption for the minimum degree.

Anatomy of a young giant component in the random graph

Eyal Lubetzky *joint with J. Ding, J.H. Kim and Y. Peres*

In their seminal papers from the 1960's, Erdős and Rényi established a phenomenon known as *the double jump*. For $p = c/n$ where $c < 1$ is fixed, the largest component $\mathcal{C}_1$ has size $O(\log n)$ with high probability (w.h.p.). When $c > 1$, the size of $\mathcal{C}_1$ is linear in n, and at the critical $c = 1$ it has order $n^{2/3}$ (this latter fact was fully established much later by Bollobás in 1984 and Łuczak in 1990. The critical behavior extends throughout the *critical window*, the regime where $p = (1 \pm \epsilon)/n$ for $\epsilon = O(n^{-1/3})$. Up to the critical point, the structure of $\mathcal{C}_1$ is relatively well understood. For instance, in the fully subcritical regime ($p = (1-\epsilon)/n$ for $\epsilon > 0$ fixed), $\mathcal{C}_1$ is a tree of known (logarithmic) size and diameter. In the critical window ($\epsilon = O(n^{-1/3})$ the distribution of $|\mathcal{C}_1|$ is known. In the supercritical regime ($p = (1+\epsilon)/n$ with $\epsilon^3 n \to \infty$), a variety of methods can determine key features of $\mathcal{C}_1$ up to some continuous functions of ϵ. While these functions remain bounded in the fully supercritical case ($\epsilon > 0$ fixed), the situation becomes much more delicate as ϵ approaches the critical window. For example, one can deduce that the diameter of the fully supercritical $\mathcal{C}_1$ has order $\log n$ merely by analyzing certain (weak) expansion properties of its 2-core). More precise results on the diameter were obtained by Riordan and Wormald and by Łuczak and Seierstad, but until this work they still did not give the asymptotic diameter in the whole supercritical regime. In the fully supercritical case, the giant component consists of an expander "decorated" using paths and trees of at most logarithmic size. However, the existing decompositions of the giant component are not precise enough to handle the case where $\epsilon \to 0$ (e.g., Riordan and Wormald point out that this is the most difficult regime for determining the diameter).

Here we obtain a complete characterization of the supercritical giant component. Rather than merely describing its properties, we present a simple construction whose distribution is contiguous with that of $\mathcal{C}_1$. This construction is particularly elegant when the giant component is "young", namely when $\epsilon = o(n^{-1/4})$ (see [8] for the general case). Let $\mathcal{N}(\mu, \sigma^2)$ denote the normal distribution with mean μ and variance σ^2, and let $\mathrm{Geom}(\epsilon)$ denote the geometric distribution with mean $1/\epsilon$.

Theorem 5 ([8])**.** *Let $\mathcal{C}_1$ be the largest component of the random graph $\mathcal{G}(n,p)$ for $p = \frac{1+\epsilon}{n}$, where $\epsilon^3 n \to \infty$ and $\epsilon = o(n^{-1/4})$. Then $\mathcal{C}_1$ is contiguous to the model $\tilde{\mathcal{C}}_1$, constructed in 3 steps as follows:*

1. *Let $Z \sim \mathcal{N}\left(\frac{2}{3}\epsilon^3 n, \epsilon^3 n\right)$, and select a random 3-regular graph $\mathcal{K}$ on $N = 2\lfloor Z \rfloor$ vertices.*
2. *Replace each edge of $\mathcal{K}$ by a path, where the path lengths are i.i.d.* $\mathrm{Geom}(\epsilon)$.
3. *Attach an independent* $\mathrm{Poisson}(1-\epsilon)$*-Galton-Watson tree to each vertex.*

That is, $\mathbb{P}(\tilde{\mathcal{C}}_1 \in \mathcal{A}) \to 0$ implies $\mathbb{P}(\mathcal{C}_1 \in \mathcal{A}) \to 0$ for any set of graphs $\mathcal{A}$.

In the above, a $\mathrm{Poisson}(\mu)$-Galton-Watson tree is the family tree of a Galton-Watson branching process with offspring distribution $\mathrm{Poisson}(\mu)$. Note that our description of $\tilde{\mathcal{C}}_1$ constructs the kernel in Step 1, the 2-core in Step 2 and the entire component $\tilde{\mathcal{C}}_1$ in Step 3. The above theorem not only states that the kernel of $\mathcal{C}_1$ in this regime is an expander, but it is in fact contiguous to a random 3-regular graph, an object whose expansion properties are well understood. Furthermore, the 2-core is obtained from the kernel by a simple operation ("stretching" the edges into paths of lengths i.i.d. geometric with mean $1/\epsilon$). This allows us to pinpoint the expansion properties of the 2-core and their dependence on ϵ as it tends to 0.

Theorem 5 enables us to interpret distances in the 2-core as passage times in first-passage percolation. This connection gives the asymptotic behavior of the diameter throughout the regime $\epsilon^3 n \to \infty$ and $\epsilon = o(1)$:

Theorem 6 ([9])**.** *Consider the random graph $\mathcal{G}(n,p)$ for $p = \frac{1+\epsilon}{n}$, where $\epsilon^3 n \to \infty$ and $\epsilon = o(1)$. Let $\mathcal{C}_1$ be the largest component G, let $\mathcal{C}_1^{(2)}$ be its 2-core and let $\mathcal{K}$ denote its kernel. Then w.h.p.,*

$$\mathrm{diam}(\mathcal{C}_1) = \big(3 + o(1)\big)(1/\epsilon)\log(\epsilon^3 n)\,, \tag{2}$$

$$\mathrm{diam}(\mathcal{C}_1^{(2)}) = \big(2 + o(1)\big)(1/\epsilon)\log(\epsilon^3 n)\,, \tag{3}$$

$$\max_{u,v\in\mathcal{K}} \mathrm{dist}_{\mathcal{C}_1^{(2)}}(u,v) = \big(\tfrac{5}{3} + o(1)\big)(1/\epsilon)\log(\epsilon^3 n)\,. \tag{4}$$

A deletion method for local subgraph counts

Angelika Steger *joint with R. Spöhel and L. Warnke*

For a given graph H let X_H denote the random variable that counts the number of copies of H in a random graph $G_{n,p}$. Note that $\mathbb{E}[X_H] = \theta(n^{v(H)}p^{e(H)})$, where $v(H)$ and $e(H)$ denote the number of vertices and edges of H. For subgraph counts one can use Janson's inequality to obtain upper bounds on the probability that X_H is smaller than its expectation.

For the corresponding upper tail, however, such bounds are not obtained easily. Over the last years such research has been devoted to proving results of the form $\Pr[X_H \geq (1+\epsilon)\mathbb{E}[X_H]] \leq \exp(-f(n,p))$. As it turned out, this probability is simply not as small as the lower tail. Roughly speaking this is due to the fact that a reasonably small number of edges that cluster in an appropriate way can give rise to lots of subgraphs H.

In order to better control the upper tail of X_H, Rödl and Ruciński (*Threshold functions for Ramsey properties*, J. AMS, 1995) showed that with probability similar to lower bound the number of copies of H can be reduced to at most $(1+\epsilon)\mathbb{E}[X_H]$ by deleting some small fraction of all edges (or of all triangles, whatever is smaller).

In this talk we are interested in the following strengthening of this result. We want to find a subgraph that on the one hand still contains at least $(1-\epsilon)\mathbb{E}[X_H]$ many H-subgraphs, and on the other hand has the property that every vertex (and more generally every small subset of vertices) is contained in 'not too many' H-subgraphs. First we show how the FKG-inequality can be used to link the probability of existence of a 'nice' collection of H-subgraphs to the probability that a certain number of H-subgraphs exists at all. From this we then deduce the following result for the case that $H = \triangle$ is a triangle. (We defer the corresponding statement for general graphs H to the full paper.)

Theorem 7. *For every $\epsilon > 0$ and $0 < p < 1$ there exists $C > 0$ such that with probability at least $1 - \exp(-\Theta(\min\{\mathbb{E}[X_\triangle, n^2p\})$, there exists a set $E_0 \subseteq E(G_{n,p})$ of size at most $\epsilon \cdot \min\{\mathbb{E}[X_\triangle], n^2p\}$ such that $G_{n,p} \setminus E_0$ contains at least $(1-\epsilon)\mathbb{E}[X_\triangle]$ triangles and such that (in $G_{n,p} \setminus E_0$) each vertex $v \in V$ is contained in at most $\max\{C, (1+\epsilon)\mathbb{E}[X_v]\}$ triangles and every edge $e \in E(G_{n,p}) \setminus E_0$ is contained in at most $\max\{C, (1+\epsilon)\mathbb{E}[X_e]\}$ triangles. (Here $\mathbb{E}[X_v]$ and $\mathbb{E}[X_v]$ denote the expected number of triangles that a vertex resp. edge is contained in.)*

On the Density of a Graph and its Blowup

Raphael Yuster *joint with A. Shapira*

It is well known that of all graphs with edge-density p, the random graph $G(n,p)$ contains the smallest density of copies of $K_{t,t}$, the complete bipartite graph of size $2t$. Since $K_{t,t}$ is a t-blowup of an edge, the following intriguing open question arises: Is it true that of all graphs with triangle density p^3, the random graph $G(n,p)$ contains close to the smallest density of $K_{t,t,t}$, which is the t-blowup of a triangle?

Our main result gives an indication that the answer to the above question is positive by showing that for *some* blowup, the answer must be positive. More formally we prove that if G has triangle density p^3, then there is some $2 \leq t \leq T(p)$ for which the density of $K_{t,t,t}$ in G is at least $p^{(3+o(1))t^2}$, which (up to the $o(1)$ term) equals the density of $K_{t,t,t}$ in $G(n,p)$.

This result is best possible in the sense that we can only guarantee that t is bounded by the constant $T(p)$. We cannot have t universally fixed and applied to all sufficiently large graphs with triangle density p^3. Indeed, we prove that for every fixed t there are graphs whose $K_{t,t,t}$-density is far from the corresponding density in a random graph with the same triangle-density.

The result extends to blowups of other complete graphs, other than the triangle.

3 General random graphs

Hamilton Cycles in Random Geometric Graphs

József Balogh

We consider one of the frequently studied models for random geometric graphs, namely the Gilbert Model. Suppose that S_n is a $\sqrt{n} \times \sqrt{n}$ box and that $\mathcal{P}$ is a Poisson process in it with density 1. The points of the process form the vertex set of our graph. There is a parameter r governing the edges: two points are joined if their (Euclidean) distance is at most r.

Having formed this graph we can ask whether it has any of the standard graph properties, such as connectedness. As usual, we shall only consider these for large values of n. More formally, we say that $G = G_{n,r}$ has a property *with high probability* (abbreviated to *w.h.p.*) if the probability that G has this property tends to one as n tends to infinity.

Penrose proved in the nineties that the threshold for connectivity is $\pi r^2 = \log n$. In fact he proved the following very sharp result: suppose $\pi r^2 = \log n + \alpha$ for some constant α. Then the probability that $G_{n,r}$ is connected tends to $e^{-e^{-\alpha}}$.

He also generalised this result to find the threshold for κ-connectivity: namely $\pi r^2 = \log n + (\kappa - 1) \log \log n$. Moreover, he found the "obstruction" to κ-connectivity. Suppose we fix the vertex set (i.e., the point set in S_n) and "grow" r. This gradually adds edges to the graph. For a monotone graph property P let $\mathcal{H}(P)$ denote the smallest r for which the graph on this point set has the property P. Penrose showed that

$$\mathcal{H}(\delta(G) \geq \kappa) = \mathcal{H}(\text{connectivity}(G) \geq \kappa)$$

w.h.p.: that is, as soon as the graph has minimum degree κ it is κ-connected *w.h.p.*.

He also considered the threshold for G to have a Hamilton cycle. Obviously a necessary condition is that the graph is 2-connected. In the normal (Erdős-Rényi) random graph this is also a sufficient condition in the following strong sense. If we add edges to the graph one at a time then the graph becomes Hamiltonian exactly when it becomes 2-connected.

Penrose, conjectured that the same is true for a random geometric graph. We prove the following theorem proving the conjecture.

Theorem 8. *Suppose that $G = G_{n,r}$ the two-dimensional Gilbert Model. Then* w.h.p.*,*

$$\mathcal{H}(G \textit{ is 2-connected}) = \mathcal{H}(G \textit{ has a Hamilton cycle}).$$

Combining this with the earlier result of Penrose we see that, if $\pi r^2 = \log n + \log \log n + \alpha$ then the probability that G has a Hamilton cycle tends to $e^{-e^{-\alpha}}$.

Our proof generalises to higher dimensions, and to other norms.

We also show that in the k-nearest neighbour model, there is a constant κ such that almost every κ-connected graph has a Hamilton cycle.

Explosion

Simon Griffiths *joint with O. Amini, L. Devroye and N. Olver*

Let T be a infinite rooted tree. Let W be a distribution taking values in the non-negative reals. Denote by T^W the random weighted tree obtained by independently giving each edges of T a weight from the distribution W. If there is an infinite path in the tree with finite weight in T^W we say that T^W *explodes.* For a fixed pair (T, W) the event that T^W explodes will have probability zero or one by the Kolmogorov zero-one law. So, for a fixed tree T, we consider the set

$$\mathcal{W}_0(T) = \{W : T^W \text{ explodes almost surely}\},$$

which contains exactly those weight distributions W for which T^W explodes almost surely. For example, if T is a very thin tree, in the sense that the generation sizes of T do not tend to infinity, then $\mathcal{W}_0(T)$ contains only the trivial distribution which takes value 0 almost surely. On the other hand if every vertex in the nth generation of T has 2^{n+1} children then there are very many weight distributions in $\mathcal{W}_0(T)$, including the uniform distribution on $[0, 1]$.

An event related to explosion is the event that there exists some choice of one edge from every level such that the sum of the weights of these edges is finite. Equivalently, this is the event that the sum over levels of the minimum weight of the level is finite; if this occurs we say that T^W is *min-summable*. For a fixed tree T let

$$\mathcal{W}_0(T) = \{W : T^W \text{ min-summable}\}.$$

Trivially, $\mathcal{W}_1(T) \supseteq \mathcal{W}_0(T)$ for all T.

If T is a very thin tree, with generation sizes not tending to infinity then $\mathcal{W}_1(T)$ contains only the trivial distribution which takes value 0 almost surely. In this trivial case we have $\mathcal{W}_0(T) = \mathcal{W}_1(T)$. However, as one moves on to consider less trivial examples T it becomes clear that in a great many cases $\mathcal{W}_1(T)$ strictly contains $\mathcal{W}_0(T)$. It may even appear that, aside from trivial cases, $\mathcal{W}_1(T)$ should always strictly contain $\mathcal{W}_0(T)$. However, somewhat counter-intuitively this is not the case; there are examples of trees with generation sizes growing very fast (double exponentially) for which $\mathcal{W}_0(T) = \mathcal{W}_1(T)$. Our main result is that this phenomenon is in fact quite general in trees obtained by a Galton-Watson process with a heavy tailed offspring distribution.

Theorem 9. *Let Z be a distribution taking values in the positive integers and suppose that there exist constants $\varepsilon > 0$ and $m_0 \in \mathbb{N}$ such that $F_Z^{-1}(1 - \frac{1}{m}) \geq m^{1+\varepsilon}$ for all $m \geq m_0$. Then the random tree T obtained by a Galton-Watson branching process with offspring distribution Z has $\mathcal{W}_0(T) = \mathcal{W}_1(T)$ almost surely.*

Our condition on Z tells one something about the rate of growth of generation sizes and something about the smoothness of this growth. We have examples that make clear that both of these are necessary.

The Bohman-Frieze Process – The Rise of the Young Giant
Joel Spencer *joint with S. Janson, W. Perkins and others*

In the Bohman-Frieze (BF) Process we start with the empty graph. Each round we pick two random vertices. If they are isolated we join them. Otherwise we pick two other random vertices and definitely join them. We use Erdős-Rényi time, at time t, $t\frac{n}{2}$ edges have been selected.

In work with Nick Wormald we found there was a critical time $t_c \sim 1.176$. In the subcritical regime the components were of size $O(\ln n)$ while in the supercritical regime a giant component of size $\Omega(n)$ has been created.

Now we are looking at the barely supercritical regime. At time $t_c + \epsilon$ let the giant have size $f(\epsilon)n$. Our new result is that $f(\epsilon)$ grows linearly as $\epsilon \to 0^+$.

The key is actually in further analysis of the *sub*-critical regime. Let $S(t) = E[|C(v)|]$, v uniformly chosen at time t. With Wormald we had found a differential equation for $S(t)$ and shown that t_c was when $S(t) \to \infty$. (This has natural analogies to classical percolation theory, that the susceptibility goes to infinity exactly when the infinite cluster appears.) Now we looked at $M_2(t) = E[|C(v)|^2]$ and $M_3(t) = E[|C(v)|^3]$. We derived differential equations for them and were able to give power laws for their values at $t_c - \epsilon$ and $\epsilon \to 0^+$. At this stage we were able to "jump" from $t_c - \epsilon$ to $t_c + \epsilon\delta$ by adding random edges (ignoring the ones between isolated vertices) at a certain rate. This led to a Galton-Watson birth process problem, estimating the probability that a birth process would go on forever which was the proportion of vertices in the giant component. It turned out that S, M_2, M_3 allowed us to make good estimates of this probability. Taking limits in ϵ, δ appropriately gave the size of the young giant.

4 Quasi-random graphs

The Quasi-Randomness of Hypergraph Cut Properties
Asaf Shapira *joint with R. Yuster*

We study quasi-random k-uniform hypergraphs, specifically, hypergraphs whose edge distribution is similar to that of random hypergraphs. We consider properties defined by the number of edges of the hypergraph that cross a cut of a given type. For a vector $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\sum \alpha_i = 1$ we define $\mathcal{P}_\alpha$ to be the property of k-uniform hypergraphs that asserts that for any partition of the vertices of the hypergraph into k-sets of sizes $(\alpha_1 n, \ldots, \alpha_k n)$, the number of edges with one vertex in each set is the one we expect to find in a random hypergraph.

Chung and Graham considered the special case of cut properties in graphs and proved that satisfying $\mathcal{P}_\alpha$ guarantees that a graph is quasi-random if and only if $\alpha \neq (1/2, 1/2)$. We obtain the following results:

- We extend the result of Chung-Graham to k-uniform hypergraphs by showing that $\mathcal{P}_\alpha$ guarantees that a k-uniform hypergraph is quasi-random if and only if $\alpha \neq (1/k, \ldots, 1/k)$.
- We strengthen the result of Chung-Graham by showing that the only way a non-quasi random graph can satisfy $\mathcal{P}_{(1/2,1/2)}$ is the the *trivial* one.

5 Extremal combinatorics

DIRECTED GRAPHS WITHOUT SHORT CYCLES
Jacob Fox *joint with P. Keevash and B. Sudakov*

For a directed graph G without loops or parallel edges, let $\beta(G)$ denote the size of the smallest feedback arc set, i.e., the smallest subset $X \subset E(G)$ such that $G - X$ has no directed cycles. Let $\gamma(G)$ be the number of unordered pairs of vertices of G which are not adjacent. We say that a digraph is *r-free* if it does not contain a directed cycle of length at most r. Chudnovsky, Seymour and Sullivan [6] conjectured that if G is a 3-free digraph then $\beta(G)$ is bounded from above by $\gamma(G)/2$, which would be best possible. They proved this holds within a factor of 2. That is, every 3-free digraph G satisfies $\beta(G) \leq \gamma(G)$.

In her thesis, Sullivan [22] posed an open problem to prove that $\beta(G) \leq f(r)\gamma(G)$ for every r-free digraph G, for some function $f(r)$ tending to 0 as $r \to \infty$. We prove that for $r \geq 3$, every r-free digraph G satisfies $\beta(G) \leq 800\gamma(G)/r^2$. This is best possible up to a constant factor, and extends the result of Chudnovsky, Seymour and Sullivan to general r.

This result can also be used to answer a question of Yuster [23] concerning almost given length cycles in digraphs. We show that for any fixed $0 < \theta < 1/2$ and sufficiently large n, if G is a digraph with n vertices and $\beta(G) \geq \theta n^2$, then for any $0 \leq m \leq \theta n - o(n)$ it contains a directed cycle whose length is between m and $m + 6\theta^{-1/2}$. Moreover, there is a constant C such that either G contains directed cycles of every length between C and $\theta n - o(n)$ or it is close to a digraph G' with a simple structure: every strong component of G' is periodic. These results are also tight up to the constant factors.

THE NUMBER OF 3-SAT FUNCTIONS
Jeff Kahn *joint with L. Ilinca*

A *k-SAT function* of (Boolean) variables $x_1, \ldots, x_n$ is one that can be expressed as

$$C_1 \vee \cdots \vee C_t, \tag{5}$$

with each C_i a *k-clause* (that is, an expression $y_1 \wedge \cdots \wedge y_k$, with $y_1 \ldots y_k$ literals corresponding to different variables x_i).

The problem of estimating the number, say $G_3(n)$, of 3-SAT functions of $x_1 \ldots x_n$ was suggested by Bollobás, Brightwell and Leader (2003), who showed

$$G_3(n) \leq \exp[(2\sqrt{\pi})\binom{n}{3}]$$

—as opposed to the easy

$$G_3(n) > 2^n(2^{\binom{n}{3}} - n2^{\binom{n-1}{3}}) \sim 2^{n+\binom{n}{3}} \tag{6}$$

—and conjectured that

$$\log_2 G_3(n) < (1+o(1))\binom{n}{3}.$$

We show that in fact (6) gives the asymptotics not just of $\log G_3(n)$, but of $G_3(n)$ itself; that is,

Theorem $\quad G_3(n) \sim 2^{n+\binom{n}{3}}$.

Actually (an easy but key idea) we show the same asymptotic value for the number of *minimal* formulas (5) (i.e. those for which deletion of any C_i produces a different function). The Frankl-Rödl Regularity Lemma for 3-uniform hypergraphs is one main ingredient in the proof.

Counting Substructures
Dhruv Mubayi

Turán's theorem determines the maximum number of edges in a graph with n vertices and no clique of a fixed size, and extremal graph theory has grown through extensions and generalizations of this result. One such direction is to count the number of copies of a specified clique in a graph with more edges than in the Turán bound. We take this approach further by extending classical results of Rademacher, Erdős, Simonovits, and Lovász-Simonovits to the class of color critical graphs. The techniques are new and quite general, and they yield similar results for hypergraphs. Here is a sample theorem:

Füredi-Simonovits and independently Keevash-Sudakov settled an old conjecture of Sós by proving that the maximum number of triples in an n vertex triple system (for n sufficiently large) that contains no copy of the Fano plane is $p(n) = \binom{\lceil n/2 \rceil}{2}\lfloor n/2 \rfloor + \binom{\lfloor n/2 \rfloor}{2}\lceil n/2 \rceil$.

We prove that there is an absolute constant c such that if n is sufficiently large and $1 \le q \le cn^2$, then every n vertex triple system with $p(n)+q$ edges contains at least

$$6q\left(\binom{\lfloor n/2 \rfloor}{4} + (\lceil n/2 \rceil - 3)\binom{\lfloor n/2 \rfloor}{3}\right)$$

copies of the Fano plane. This is sharp for $q \le n/2 - 2$.

One modern ingredient of our approach is the use of the removal lemma, which is a consequence of the hypergraph regularity lemma. In many cases, our results so far use ad hoc methods for each hypergraph F, and one open problem is to prove general results that apply to large classes of hypergraphs. Another open problem is to count induced copies of graphs or hypergraphs, which is a more challenging problem. A specific case is to consider the enumerative questions for the configurations studied recently by Razborov and Pikhurko, which are closely related to the famous Turán conjecture for hypergraphs.

The Minimum Size of 3-Graphs without a 4-Set Spanning No or Exactly Three Edges
Oleg Pikhurko

For a family of k-graphs $\mathcal{G}$, let $t(n, \mathcal{G})$ be the minimum size of a k-graph on n vertices not containing any member of $\mathcal{G}$ as an induced subgraph. For $0 \le i \le 4$, let G_i be the (unique) 3-graph with 4 vertices and i edges.

One of the most famous open questions in extremal combinatorics is to determine $t(n, G_0)$. It goes back to the fundamental paper by Turán (1941) who conjectured that $t(n, G_0) = |T_n|$, where T_n the 3-graph on $[n]$ whose edges are triples $\{x, y, z\}$ with $x, y \in V_i$ and $z \in V_i \cup V_{i+1}$ for some $i \in \mathbb{Z}_3$, where $V_0 \cup V_1 \cup V_2 = [n]$ is some partition into almost equal parts. Successively better lower bounds on $t(n, G_0)$ were proved by de Caen (1994), Giraud (unpublished), and Chung and Lu (1999) (see also Razborov (2008)).

Note that T_n is also G_3-free; thus $t(n, \{G_0, G_3\}) \le t_n$. Applying his flag algebras technique Razborov (2008) proved the matching asymptotic lower bound $t(n, \{G_0, G_3\}) = \left(\frac{4}{9} + o(1)\right)\binom{n}{3}$.

This result is very interesting because there are very few non-trivial hypergraphs or hypergraph families for which the asymptotic of its Turán function is known. Also, it gives us a better understanding of the original conjecture of Turán. For example, if the conjecture is false, then any G_0-free 3-graph G on n vertices beating t_n has to contain an induced copy of G_3. (In fact, if $|G| \leq (1-\Omega(1))\, t_n$ as $n \to \infty$, then G contains $\Omega(n^4)$ G_3-subgraphs by the super-saturation technique of Erdős and Simonovits(83).)

We proved for all $n \geq n_0$ that $t(n, \{G_0, G_3\}) = t_n$ and the Turán 3-graph T_n is the unique extremal 3-graph. This result is also interesting in the context of the rapidly developing theory of graph and hypergraph limits as it shows a way how to obtain exact results from limit computations. The key ingredient here is the stability property which states, roughly speaking, that all almost extremal hypergraphs have essentially the same unique structure. This approach was pioneered by Simonovits in the late 1960s and has led to exact solutions of numerous extremal problems since then. In recent years it has been actively used to prove exact results for the hypergraph Turán problem (by Füredi, Keevash, Mubayi, Simonovits, Sudakov, and others).

6 Ramsey theory

The Ramsey number of dense graphs

David Conlon

The Ramsey number $r(H)$ of a graph H is the smallest number n such that in every 2-colouring of the edges of the complete graph K_n there exists a monochromatic copy of H. That these numbers exist was proven by Ramsey [19] and independently rediscovered by Erdős and Szekeres [11].

The most famous question in the field is that of estimating the Ramsey number $r(t)$ of the complete graph K_t on t vertices. Despite some small improvements [7, 20], the standard estimates, that $\sqrt{2}^t \leq r(t) \leq 4^t$, have remained largely unchanged for over sixty years. The question we address here is what happens if one takes a slightly less dense graph H?

The density ρ of a graph H with t vertices and m edges is $\rho = m/\binom{t}{2}$. We would like to determine the Ramsey number of a graph H with t vertices and given density ρ. An easy lower bound follows from taking a clique of size $\sqrt{\rho}t/2$ and connecting up the rest of the t vertices with a sparse collection of edges. This gives $r(H) \geq 2^{\sqrt{\rho}t/4}$. We provide a nearly matching upper bound.

Theorem 10. *There exists a constant c such that any graph H on t vertices with density ρ satisfies*

$$r(H) \leq 2^{c\sqrt{\rho}\log(2/\rho)t}.$$

We also look at graphs H with maximum degree at most ρt. A result of Graham, Rödl and Ruciński [13] implies that $r(H) \leq 2^{c\rho t \log^2 t}$. We show how to remove the log factors, replacing them with corresponding terms depending only on ρ.

Theorem 11. *There exists a constant c such that any graph H on t vertices with maximum degree ρt satisfies*

$$r(H) \leq 2^{c\rho \log^2(2/\rho)t}.$$

Finally, we consider random graphs H with density ρ. Such graphs will not only satisfy a condition saying that the maximum degree is at most $2\rho t$, but the edges will be quite nicely spread within any given subset. This allows us, within a certain range of ρ, to improve the above bound as follows.

Theorem 12. *There exist constants c and c' such that, if H is a binomially chosen random graph with probability $\rho \geq c' \frac{\log^{3/2} t}{\sqrt{t}}$, H almost surely satisfies*

$$r(H) \leq 2^{c\rho\log(2/\rho)t}.$$

An easy probabilistic argument shows that any graph H with density ρ satisfies $r(H) \geq 2^{c\rho t}$, so this result is also very close to being sharp.

Sizes of induced subgraphs of ramsey graphs **Alexander Kostochka** *joint work with N. Alon, J. Balogh and W. Samotij*

For a graph $G = (V, E)$, call a set $W \subseteq V$ *homogenous*, if W induces a clique or an independent set. Let $\hom(G)$ denote the maximum size of a homogenous set of vertices of G. For a positive constant $c > 0$, an n-vertex graph G is called *c-Ramsey* if $\hom(G) \leq c \log n$.

Ramsey theory states that every n-vertex graph G satisfies $\hom(G) \geq (\log n)/2$, and for almost all such G, we have $\hom(G) \leq 2 \log n$. In other words, in a random graph G, the value $\hom(G)$ is of logarithmic order. Moreover, the only known examples of graphs with $\hom(G) = O(\log n)$ come from various constructions based on random graphs with edge density bounded away from 0 and 1. Therefore it is natural to ask whether c-Ramsey graphs look "random" in some sense. Erdős, Faudree and Sós stated the following conjecture:

Conjecture 1. *For every positive constant c, there is a positive constant $b = b(c)$, such that if G is a c-Ramsey graph on n vertices, then the number of distinct pairs $(|V(H)|, |E(H)|)$, as H ranges over all induced subgraphs of G, is at least $bn^{5/2}$.*

At the time the conjecture was stated, its authors knew how to prove an $\Omega(n^{3/2})$ lower bound. The same lower bound was also obtained as a corollary of a much stronger result of Bukh and Sudakov. Very recently it has been improved to $\Omega(n^2)$ by Alon and Kostochka. In fact, the following stronger result was proved. Let $\phi(k, G)$ denote the number of distinct sizes of k-vertex induced subgraphs of G.

Theorem 13. *For every $0 < \epsilon < 1/2$ there is an $n_0(\epsilon)$ so that the following holds. Let $n > n_0$ and let G be an n-vertex graph with $\epsilon < |E(G)|/\binom{n}{2} < 1 - \epsilon$. Then, for every k with $k \leq \frac{\epsilon n}{3}$,*

$$\phi(k, G) \geq 10^{-7} k. \tag{7}$$

The bound is sharp up to a constant factor (in the class of graphs G $\epsilon < |E(G)|/\binom{n}{2} < 1 - \epsilon$). For c-Ramsey graphs, we now improve it as follows:

Theorem 14. *For every positive constants c and ϵ, there is a positive constant $b = b(c, \epsilon)$, such that if G is a c-Ramsey graph on n vertices, then the number of distinct pairs $(|V(H)|, |E(H)|)$, as H ranges over all induced subgraphs of G, is at least $bn^{1+\frac{\sqrt{30}}{4}-\epsilon} \approx bn^{2.3693-\epsilon}$.*

7 Additive combinatorics

Inverse Littlewood-Offord theorems
Van Vu

In this talk, we gave a brief survey about Inverse Littlewood-Offord theory, initiated a few years ago by Tao and the speaker.

The classical Littlewood-Offord theorem, in the discrete form, is as follows. Let $a_1, \ldots, a_n$ be non-zero integers, and $\xi_1, \ldots, \xi_n$ be iid Bernoulli random variables (taking values ± 1 with probability 1/2). Let $S := a_1\xi_1 + \cdots + a_n\xi_n$. Then (with $v = (a_1, \ldots, a_n)$)

$$p(v) := \max_{z \in \mathbf{Z}} \mathbf{P}(S = z) = O(\log n/\sqrt{n}).$$

The $\log n$ term was removed by Erdős (1940s), using Sperner lemma. The improved bound is sharp, as can be see by taking $a_1 = \ldots a_n = 1$. The Littlewood-Offord-Erdős theorem is very well-known, and has been extended in various directions. For example, Sárközy and Szemerédi (1960s) proved that if the a_i are different, then one can improve the bound further to $O(1/n^{3/2})$, which is again sharp. In fact Stanley showed that the extremal construction is when the a_i form an arithmetic progression. A general theme of these extensions is that if one forbid more additive relations among the a_i, then the bound gets better, and we gave many examples to illustrate this fact.

The Inverse L-O project, started in 2005, puts the problem in a new perspective. Assume that $p(v)$ is large, say $p(v) \geq 1/n^C$ for some constant C, we try to characterize the set $\{a_1, \ldots, a_n\}$. A weak characterization was given by Tao and the speaker in 2006. We significantly improved it in 2008, and successfully used it to confirm a long standing conjecture in the theory of random matrices (Circular Law Conjecture). However, even the improved version is still not optimal. The technical part of the talk is thus devoted to the discussion of a recent result of Hoi Nguyen and the speaker (2009), which provides a characterization with optimal parameters. As a consequence, we obtain a new, short, proof of many previous quantitative results (such as the Sárközy-Szemerédi theorem mentioned above) and also a stable, quantitative, version of Stanley theorem.

8 Probabilistic methods in other areas of mathematics

Going up in dimension: Probabilistic and combinatorial aspects of simplicial complexes
Nati Linial

The main thesis of my talk is that combinatorics has much to gain by "going up in dimension". We first recall.

Definition 1. *Let V be a finite set of* vertices. *A collection of subsets $X \subseteq 2^V$ is called a* simplicial complex *if it satisfies the following condition: $A \in X$ and $B \subseteq A \Rightarrow B \in X$. A member $A \in X$ is called a* simplex *or a* face *of dimension $|A| - 1$. The dimension of X is the largest dimension of a face in X.*

In theoretical computer science simplicial complexes were used in (i) The study of the evasiveness conjecture, starting with [Kahn, Saks and Sturtevant '83] (ii) Impossibility theorems in distributed asynchronous computation (Starting with [Herlihy, Shavit '93] and [Saks, Zaharoglou '93]).

In combinatorics: (i) Lovász's proof of A. Frank's conjecture on graph connectivity 1977. (ii) Lower bounds on chromatic numbers of Kneser's graphs and hypergraphs. (Starting with [Lovász '78]). (iii) The study of matching in hypergraphs (Starting with [Aharoni Haxell '00]).

The major challenges that we raise are: (i) To start a systematic attack on topology from a combinatorial perspective, using the extremal/asymptotic paradigm. In particular we hope to introduce the probabilistic method into topology. In the other direction we suggest to use ideas from topology to develop new probabilistic models (random lifts of graphs offer a small step in this direction). We also hope to introduce ideas from topology into computational complexity.

Can we develop a theory of random complexes, similar to random graph theory? Specifically we seek a higher-dimensional analogue to $G(n, p)$. To fix ideas we consider the simplest possible case: Two-dimensional complexes with a full one-dimensional skeleton. Namely, we start with a complete graph K_n and add each triple (=simplex) independently with probability p. This probability space of two-dimensional complexes is denoted by $X(n, p)$.

We recall from Erdős and Rényi's work:

Theorem 15 (ER '60)**.** *The threshold for graph connectivity in $G(n, p)$ is*

$$p = \frac{\ln n}{n}$$

We next ask when a simplicial complex should be considered connected. Unlike the situation in graphs, this question has many (in fact infinitely many) meaningful answers, i.e.: (i) The vanishing of the first homology (with any ring of coefficients). (ii) Being simply connected (vanishing of the fundamental group).

Theorem 16 (Linial and Meshulam '06)**.** *The threshold for the vanishing of the first homology in $X(n,p)$ over $GF(2)$ is*

$$p = \frac{2\ln n}{n}$$

This extends to d-dimensional simplicial complexes with a full $(d-1)$-st dimensional skeleton. Also, for other coefficient groups. (Most of this was done by Meshulam and Wallach). We still do not know, however:

Question. *What is the threshold for the vanishing of the homology with integer coefficients?*

On the vanishing of the fundamental group we have:

Theorem 17 (Babson, Hoffman, Kahle '09)**.** *The threshold for the vanishing of the fundamental group in $X(n,p)$ is near $p = n^{-1/2}$.*

We next move on to some extremal problems and recall:

Theorem 18 (Brown, Erdős, Sós '73)**.** *Every n-vertex two-dimensional simplicial complex with $\Omega(n^{5/2})$ simplices contains a (triangulation of the) two-sphere. The bound is tight.*

and state:

Conjecture 2. *Every n-vertex two-dimensional simplicial complex with $\Omega(n^{5/2})$ simplices contains a (triangulation of the) torus.*

We can show that if true this bound is tight. This may be substantially harder than the BES theorem, where one actually finds a bi-pyramid. We suspect that such a "local" triangulation of the torus need not exist. With Ehud Friedgut we showed that $\Omega(n^{8/3})$ simplices suffice.

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