

# Prime number races and zeros of Dirichlet $L$ -functions

## 09rit148

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### 1 Overview of the Field

This Research in Teams meeting focused on the finer behaviour of the function  $\pi(x; q, a)$ , which denotes the number of prime numbers of the form  $qn + a$  that are less than or equal to  $x$ . Dirichlet's famous theorem on primes in arithmetic progressions asserts that there are infinitely many primes of the form  $qn + a$  when  $a$  is a reduced residue modulo  $q$  (that is, when  $a$  and  $q$  are relatively prime), and so  $\pi(x; q, a)$  is unbounded.

If  $a$  and  $b$  are reduced residues modulo  $q$ , then we may ask whether the inequality

$$\pi(x; q, a) > \pi(x; q, b) \tag{1}$$

is satisfied for arbitrarily large  $x$ . Chebyshev observed that for the triple  $(q; a, b) = (4; 3, 1)$ , the inequality (1) holds for all small  $x$ . In fact, he asked whether this inequality would continue to hold for all  $x$ . However, in 1914 Littlewood proved that for each of the triples  $(4; 1, 3)$  and  $(4; 3, 1)$ , there are arbitrarily large values of  $x$  such that the inequality (1) holds. These inequalities can be thought of as a "prime number race" between two contestants, Team 1 and Team 3. In these terms, Chebyshev observed that Team 3 usually leads Team 1; Littlewood's theorem asserts that each team takes turns leading the prime number race infinitely often.

Over the years, researchers have attempted to prove that there are triples  $(q; a, b)$  such that (1) holds for arbitrarily large  $x$ . However, only a few such results have been established. For many triples  $(q; a, b)$  with values of  $q$  ranging up to 100, it is known that the inequality (1) holds for arbitrarily large  $x$ ; these results, however, depend on lengthy computer calculations of zeros of Dirichlet  $L$ -functions.

One can further generalize to prime number races with more than two contestants. Let  $a_1, \dots, a_r$  be distinct reduced residues modulo  $q$ . A natural question is whether the system of inequalities

$$\pi(x; q, a_1) > \pi(x; q, a_2) > \dots > \pi(x; q, a_r) \tag{2}$$

holds for arbitrarily large  $x$ ; this question can be interpreted as a prime number race among  $r$  teams. This generalized problem has also received considerable attention without many proven results. The primary goal of *comparative prime number theory* is to establish that any set of inequalities of the form (2) has arbitrarily large solutions  $x$ . In particular, if we focus upon a particular  $r$ -way prime number race (that is, a particular  $r$ -tuple  $\{a_1, \dots, a_r\}$  modulo  $q$ ), we can hope to prove that the inequalities (2) will be satisfied for arbitrarily large values of  $x$  no matter what permutation of the  $a_i$  we choose: we call such a prime number race *inclusive*. If for some permutation of the  $a_i$ , the inequalities (2) are never satisfied when  $x$  is sufficiently large, we call the prime number race *exclusive*.

## 2 Recent Developments and Open Problems

Over the years, more and more prime number races were proved to be inclusive, but this progress came at the cost of assuming stronger and stronger hypotheses on the locations of the zeros of Dirichlet  $L$ -functions. Rubinstein and Sarnak established that all prime number races are inclusive, but only by requiring two strong hypotheses: the Generalized Riemann Hypothesis (GRH), the assertion that all nontrivial zeros of Dirichlet  $L$ -functions have real part equal to  $1/2$ ; and a Linear Independence hypothesis (LI), the assertion that the imaginary parts of these nontrivial zeros are linearly independent over the rational numbers.

It is natural to wonder: are such strong hypotheses really necessary to prove these results in comparative prime number theory? Two recent papers of Ford and Konyagin [1, 2] illustrate the difficulty of these problems in the absence of such hypotheses. They show that certain hypothetical GRH-violating configurations of zeros would in fact result in exclusive prime number races. In fact these malicious zeros can be arbitrarily close to the center of the critical strip and arbitrarily far from the real axis; therefore no prohibition on the zeros that was limited to a strip of width less than  $1/2$ , or of finite height, can suffice to prove that a race game is inclusive. In this sense, their work shows that a hypothesis almost as strong as GRH is necessary to establish this type of result.

One can similarly ask if the LI hypothesis is necessary to prove that prime number races are inclusive, even if we assume GRH. Indeed, Rubinstein and Sarnak deemed LI to be a “working hypothesis”. We believe that it should be possible to construct an infinite set of hypothetical violations of LI that forces a prime number race to be exclusive; doing so would show that a hypothesis almost as strong as LI is also necessary to establish exclusivity results. The main goal of our Research in Teams week was exactly this extension of the Ford–Konyagin work: we planned to search for constructions of hypothetical configurations of zeros, satisfying GRH but violating LI, that force prime number races to be exclusive.

## 3 Scientific Progress Made at the Meeting

Our Research in Teams meeting was extremely productive and successful—although none of our successes, as it turns out, involved progress towards the main goal of the week! As it happened, we did prove one result directly related to the main goal, although it actually demonstrates that the goal is harder to achieve than we first imagined. We also established several other results related more fundamentally to the LI hypothesis.

Since our main goal was to show that “enough” linear dependences among the imaginary parts of zeros of Dirichlet  $L$ -functions could hypothetically cause prime number races to become exclusive, it made sense for us to first ask: how much is “enough”? In other words, we wanted to show that the presence of only a small set of linear dependences could not force a prime number race to be exclusive, so that we would have a better idea of how complicated our construction would have to be. One of our stated objectives for the meeting was to show that a finite set of linear dependences would not suffice; we quickly realized that we could show that even a “density zero” set of linear dependences would not impact whether a race was inclusive or exclusive. Very surprisingly, however, we were able during the week to improve this result even further, to show that even certain “density one” sets of linear dependences would not impact a race.

To be more precise, assume the generalized Riemann hypothesis, and let  $\gamma$  be an imaginary part of a zero of a Dirichlet  $L$ -function  $L(s, \chi)$ , where  $\chi$  is a character modulo  $q$ . We say that  $\frac{1}{2} + i\gamma$  is *self-sufficient* if there is no linear combination of other imaginary parts of zeros of Dirichlet  $L$ -functions modulo  $q$  that equals  $\gamma$  (so that the LI hypothesis is the assumption that every such  $\gamma$  is self-sufficient). We proved that if every  $L$ -function modulo  $q$  has a dense enough set of self-sufficient zeros—for example, at least  $\varepsilon T / \log T$  zeros with imaginary part bounded by  $T$  in absolute value when  $T$  is sufficiently large—then every prime number race modulo  $q$ , including the full  $\phi(q)$ -way race, is inclusive. (Note that the total number of zeros of  $L(s, \chi)$  with imaginary part bounded by  $T$  has order of magnitude  $T \log T$ , and so quite a thin set of self-sufficient zeros suffices to make a race inclusive.) In fact, for particular races modulo  $q$ , we don’t even require this property for all  $L$ -functions modulo  $q$ , but only for certain subsets of the  $L$ -functions depending on the  $r$ -tuple  $\{a_1, \dots, a_r\}$  of residue classes. This counterintuitively strong theorem suggests that it could be very difficult to find configurations of zeros of Dirichlet  $L$ -functions, satisfying GRH but violating LI, that force prime number races to be exclusive: any such construction would have to involve almost all of the zeros.

Our other successes involved modest (yet still groundbreaking) progress towards actually establishing

the LI hypothesis. Two consequences of LI would be that all zeros of Dirichlet  $L$ -functions are simple and that  $L(\frac{1}{2}, \chi)$  never equals zero; both consequences have been proved to hold a positive proportion of the time in a suitable sense. However, almost no theoretical progress had been made towards showing that more complicated linear dependences seldom occur. Our work during the Research in Teams week advanced this knowledge in multiple ways.

For example, if we consider a fixed arithmetic progression  $\{s_k\} = \{\frac{1}{2} + i(\alpha + k\beta)\}$  and a particular function  $L(s, \chi)$ , it is not possible for every single  $s_k$  to be a zero of  $L(s, \chi)$ . Lapidus and van Frankenhauser have shown [3, chapter 9] that there are at least  $T^{5/6}$  values of  $k$  between 1 and  $T$  for which  $\zeta(s_k) \neq 0$  (although their method requires that  $\alpha = 0$ ) and a similar result for Dirichlet  $L$ -functions. During the meeting, we were able to show that  $L(s_k, \chi) \neq 0$  for at least  $cT/\log T$  values of  $k$  between 1 and  $T$  for some positive constant  $c$ , with no restriction on  $\alpha$ .

One can also consider a fixed linear form and investigate how often the corresponding linear combination of zeros could be another zero. For concreteness, consider a fixed  $k$ -tuple  $(\alpha_1, \dots, \alpha_k)$  of real numbers strictly between 0 and 1 (the case where the  $\alpha_j$  are rational has a direct bearing on the LI hypothesis, but our result holds for any real numbers). We proved, assuming the Riemann hypothesis, that among all  $k$ -tuples  $(\gamma_1, \dots, \gamma_k)$  in the box  $[T, 2T]^k$  such that  $\zeta(\frac{1}{2} + i\gamma_j) = 0$  for each  $1 \leq j \leq k$ , at least  $T^{k-\varepsilon}$  of them have the property that  $\zeta(\frac{1}{2} + i(\alpha_1\gamma_1 + \dots + \alpha_k\gamma_k))$  is nonzero.

Our method for establishing this last result can accommodate real numbers  $\alpha_j$  that exceed 1, as long as they satisfy a particular bound depending on  $k$ ; however, there are still natural cases to consider that fall outside the scope we could treat. For example, if  $\frac{1}{2} + i\gamma$  is a zero of  $\zeta(s)$ , then can  $\frac{1}{2} + 2i\gamma$  also be a zero of  $\zeta(s)$ ? The LI hypothesis predicts that the answer is always “no”. We were able to reduce the problem of showing that the answer is often “no” to bounding a mysterious exponential sum over primes that was considered by Vinogradov, namely  $\sum_{T < n < 2T} \Lambda(n)n^{-1/4}e^{2\pi im\sqrt{n}}$ . The conjectured upper bound for that exponential sum would provide a lower bound for the number of expressions of the form  $\frac{1}{2} + 2i\gamma$  that are nonzero.

## 4 Future Directions

The accomplishments of our Research in Teams meeting have left a collection of accessible further questions to address. Above we described how we showed that at least  $cT/\log T$  of the first  $T$  elements in an arithmetic progression on the critical line are not zeros of  $L(s, \chi)$ ; we believe that by overcoming certain technicalities, we can actually improve this result to show that a positive proportion of points in any such arithmetic progression are not zeros of  $L(s, \chi)$ . A similar remark applies to our result on the number of values of a fixed linear form that are not zeros of  $\zeta(s)$ : by striving for some technical improvements, we hope to increase the lower bound for the number of such values, ideally to  $(T \log T)^k$  which is the order of magnitude of the sample space itself. We also plan to see if we can extend this latter result from the Riemann zeta function to all Dirichlet  $L$ -functions.

Regarding our last result, which depends upon the exponential sum considered by Vinogradov, one obvious avenue of research is to try to establish the conjectured upper bound for that sum. On the other hand, for our application we need such an upper bound only on average over  $m$ , which makes a successful analysis seem more likely. We also plan to generalize our approach to consider  $\frac{1}{2} + \alpha i\gamma$  rather than simply  $\frac{1}{2} + 2i\gamma$ . Finally, our original goal still remains open: can we construct a configuration of hypothetical zeros of Dirichlet  $L$ -functions whose linear dependences would force a corresponding prime number race to be exclusive?

With this combination of established results and fruitful additional lines of inquiry, all created in one productive (and luxurious) week at BIRS, we certainly deem our Research in Teams meeting an unqualified success.

## References

- [1] Kevin Ford and Sergei Konyagin, *The prime number race and zeros of Dirichlet  $L$ -functions off the critical line*, Duke Math. J 113 (2002), 313–330.

- [2] Kevin Ford and Sergei Konyagin, *The prime number race and zeros of Dirichlet L-functions off the critical line, II*, Bonner Mathematische Schiften (D. R. Heath-Brown, B. Z. Moroz, editors), vol. 360, Bonn, Germany, 2003.
- [3] Michel L. Lapidus and Machiel Van Frankenhuysen, *Fractal Geometry and Number Theory: Complex Dimensions of Fractal Strings and Zeros of Zeta Functions*, Birkhauser Verlag AG.