INDECOMPOSABLE BINARY STRUCTURES

ILLE (PIMS)
HAHN (University of Montreal)

14 June 2009 - 21 June 2009

1 Overview of the Field

The notion of *interval* is well-known for linear orders. The analogue for (undirected) graphs is called *module* [25] or *homogeneous* set [6]. One uses also *autonomous* set [16, 21, 22] for partially ordered sets. It is still called interval for relations and multirelations [14, 15], and for directed graphs [18, 24]. For 2-structures [11, 13], it is called *clan*. In our framework, it is easier and more efficient to consider labelled 2-structures [13], simply called binary structures [19].

Given a binary structure, a quotient is naturally associated with a partition in clans of its vertex set. The notions above were mainly introduced to obtain a simple notion of quotient. A binary structure admitting a non-trivial quotient is *decomposable*, otherwise it is *indecomposable* (or *prime* or *primitive*).

2 Discussion Highlights

2.1 Weakly Partitive Families

Ille recalled the basic decomposition theorem of Gallai [16, 22] and its generalisation to binary structures [19]. To obtain decomposition results, it is sufficient to consider weakly partitive families (i.e. families of subsets with the same set properties as the families of the clans of binary structures) without an underlying binary structure. From a weakly partitive family on a finite set, we can apply several times the clan decomposition to obtain its decomposition tree. Then the problem is to construct a binary structure whose family of clans coincides with the initial weakly partitive family. In the finite case, this result is classic and easy. In the infinite one, Ille and Woodrow [20] showed that such a binary structure of rank 3 (that is a 3-labelled 2-structure) exists. Villemaire presented the main points of [20] and gave a nice and short proof of this theorem when an infinite rank is allowed.

Rao presented a generalisation of weakly partitive families, the weakly bipartitive families. They arise from the bipartitions obtained for instance from the splits [9, 10] or from the bijoins [23].

2.2 Critical binary structures

Ille recalled the first important results on the indecomposable substructures of an indecomposable binary structure (for instance, see [12]). It results that an indecomposable binary structure contains an indecomposable substructure obtained by deleting one or two vertices. Whence the following definition: an indecomposable binary structure is *critical* if all of its substructures obtained by removing one vertex is decomposable. Schmerl and Trotter [24] characterised the critical binary relational structures. Bonizonni [1] extended their characterisation to 2-structures. Boudabbous and Ille [2] use the *indecomposability graph*, introduced by Ille [17], to obtain a much simpler characterization of critical binary sructures. Ille presented their approach which is based on the characterisation of the connected components of the indecomposability graph.

For tournaments, Culus and Jouve [8] consider *linear clans*, that is, clans inducing subtournaments which are linear orders. They obtain a weaker indecomposability for which they characterised the critical tournaments. Jouve presented their arduous proof.

2.3 Duality theorems

Duality theorems are the analogues for specific classes of directed graphs of the classic result of Gallai [16, 22]: given two partially ordered sets with the same comparability graph, if one of them is indecomposable, then they are equal or dual. Boussaïri, Ille, Lopez and Thomassé [4] obtained a similar result for tournaments by considering the family of the 3-cycles instead of the comparability graph. A. Boussaïri and Ille [3] found a very succinct proof of this result by using the minimal indecomposable tournaments [7] and established other duality theorems. Boussaïri presented their work.

3 Scientific Progress Made

Brignall [5] proposed a nice and natural conjecture on the minimal prime extensions of a graph. During the week, Boussaïri and Ille answered positively to the conjecture and extended their answer to 2-structures.

4 Outcome of the Meeting

After the presentation of Rao (see Subsection 2.1), the group discussed the possible relationships between the indecomposability (for the clans) and that for the splits or for the bijoins. This constitutes a new area of research.

By considering constant or linear clans, Ille and Jouve will try to extend to binary structures the characterisation obtained in [8] (see Subsection 2.2).

After the presentation of Boussaïri (see Subsection 2.3), Boussaïri and Ille tried to extend duality theorems to binary structures. It is difficult and they will probably have to begin with the extension to binary structures of the characterisation of minimal indecomposable graphs [7].

The French participants will apply for support provided by the French Research National Agency to pursue their joint work in this area.

References

[1] P. Bonizonni, Primitive 2-structures with the $(n-2)$-property, *Theoret. Comput. Sci.* 132 (1994) 151-178.

[2] Y. Boudabbous and P. Ille, Indecomposability graph and critical vertices of an indecomposable graph, *Discrete Math.* **309** (2009), 2839-2846.

[3] A. Boussaïri and P. Ille, Different duality theorems, to appear in *Ars Combin.*.

[4] A. Boussaïri, P. Ille, G. Lopez and S. Thomassé, The C_3-structure of the tournaments, *Discrete Math.* **277** (2004), 29–43.

[5] R. Brignall, *Simplicity in relational structures and its application to permutation classes*, Ph.D. Thesis, University of St Andrews, 2007.

[6] A. Cournier and M. Habib, An efficient algorithm to recognize prime undirected graphs. In *Graph-Theoritic Concepts in Computer Science, (E.W. Mayr, ed.)*, Lecture Notes in Computer Science, **657**, 212–224, Springer, Berlin, 1993.

[7] A. Cournier and P. Ille, Minimal indecomposable graph, *Discrete Math.* **183** (1998), 61–80.

[8] J-F Culus and B. Jouve, Convex circuit-free coloration of an oriented graph, *European J. Comb.* **30** (2009), 43–52.

[9] W. H. Cunningham, Decomposition of directed graphs, *SIAM Journal on Algebraic and Discrete Methods* **3** (1982), 214–228.

[10] W. H. Cunningham and J. Edmonds, A combinatorial decomposition theory, *Canad. J. Math.* **32** (1980), 734–765.

[11] A. Ehrenfeucht and G. Rozenberg, Theory of 2-structures, Part I: clans, basic subclasses, and morphisms, *Theoret. Comput. Sci.* **70** (1990), 343–358.

[12] A. Ehrenfeucht and G. Rozenberg, Primitivity is hereditary for 2-structures, *Theoret. Comput. Sci.* **70** (1990), 343–358.

[13] A. Ehrenfeucht, T. Harju and G. Rozenberg, *The Theory of 2-Structures, A Framework for Decomposition and Transformation of Graphs*, World Scientific, Singapore, 1999.

[14] R. Fraïssé, L'intervalle en théorie des relations, ses généralisations, filtre intervallaire et clôture d'une relation. In *Order, Description and Roles, (M. Pouzet and D. Richard eds.)*, 313–342, North-Holland, Amsterdam, 1984.

[15] R. Fraïssé, *Theory of Relations, revised edition*, Studies in Logic **145**, North- Holland, Amsterdam, 2000.

[16] T. Gallai, Transitiv orientierbare Graphen, *Acta Math. Acad. Sci. Hungar.* **18** (1967), 25–66.

[17] P. Ille, Recognition problem in reconstruction for decomposable relations. In *Finite and Infinite Combinatorics in Sets and Logic (B. Sands, N. Sauer and R. Woodrow eds.)*, 189–198, Kluwer Academic Publishers, 1993.

[18] P. Ille, Indecomposable graphs, *Discrete Math.* **173** (1997), 71–78.

[19] P. Ille, La décomposition intervallaire des structures binaires, *La Gazette des Mathématiciens* **104** (2005), 39–58.

[20] P. Ille and R. Woodrow, Weakly partitive families on infinite sets, *Contrib. Discrete Math.* **4** (2009), 54–80.

[21] D. Kelly, Comparability graphs. In *(Graphs and Orders, (I. Rival, ed.)*, 3–40, Reidel, Drodrecht, 1985.

[22] F. Maffray and M. Preissmann, A translation of Tibor Gallai's paper: Transitiv orientierbare Graphen. In *Perfect Graphs, J.L. Ramirez-Alfonsin and B.A. Reed eds.)*, 25–66, Wiley, New York, 2001.

[23] F. de Montgolfier, M. Rao, The bi-join decomposition, proceedings of ICGT'05 (7th International Colloquium on Graph Theory), *The Electronic Notes in Discrete Math.* **22** (2005), 173–177.

[24] J.H. Schmerl and W.T. Trotter, Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures, *Discrete Math.* **113** (1993), 191–205.

[25] J. Spinrad, P4-trees and substitution decomposition, *Discrete Appl. Math.* **39** (1992), 263–291.