Non-local operators and applications

Cyril Imbert (CEREMADE, université Paris-Dauphine),
Antoine Mellet (University of British Colombia),
Régis Monneau (CERMICS, Ecole Nationale des Ponts et Chaussées)

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1 Introduction

One of the main objectives of this workshop was to present a state of the art of current research on non-local operators. Over the last few years, there has been a lot of interests for such operators, and much progress have been made by mathematicians working in many different areas. The goal of this workshop was thus to bring together those mathematicians and encourage interactions between different areas of mathematics.

Our interest for such models is motivated by the wide range of applications, and the flourishing of new mathematical tools and results, stimulated by the theory of (local) elliptic operators. This workshop has permitted to bring together mathematicians to present the most recent trend on this topic.

2 Scientific activities

We now wish to present the scientific activities that took place during this meeting. In the first subsection, we explain how those activities were organized. In the remaining of the section, we describe the results presented by speakers in their talks. Six main topics were treated: non-local moving fronts, fractal Burgers equations, non-linear stochastic differential equations, mean-field and kinetic equations, non-linear elliptic equations, reaction-diffusion equations. Several talks also discussed problems coming from applications such as oil extraction and genetic evolution.

2.1 Organization

Scientific activities consisted in 21 talks and 10 informal discussion sessions. There were two one hour talks and a 25 minute talk in the morning and two or three 25 minute talks in the afternoon. There were also two informal discussion sessions in the afternoon, one before the talks and another one after them. See the tabular below.

	Monday	Tuesday	Wednesday	Thursday
09:00 - 10:00	Vasseur	Méléard	Souganidis	Perthame
10:30 - 11:30	Cardaliaguet	Woyczynski	Roquejoffre	Dolbeault
11:30 - 12:00	Peirce	Jourdain	Informal discussion	Mouhot
01:30 - 02:30	Informal discussion	Informal discussion	Free afternoon	Informal discussion
02:30 - 03:00	Informal discussion	Silvestre		Informal discussion
03:00 - 03:30	Droniou	Gentil		Alibaud
04:00 - 04:30	Sire	Schwab		Karch
04:30 - 05:00	Informal discussion	Monteillet		Margetis
05:00 - 06:00	Informal discussion	Informal discussion		Informal discussion

One hour talks were given by senior researchers and short talks were given by either senior researchers or younger mathematicians. Two PhD students (Schwab and Monteillet) and two young mathematicians[1] (Sire and Alibaud) gave short talks. We would like to mention that several PhD students had planed to come and finally could not make it because of job interviews in France or difficulties with visa (Coville, El Hajj, Forcadel).

2.2 Non-linear elliptic equations

- A. Vasseur. *Regularity of solutions to drift-diffusion equations with fractional Laplacian.*
- L. Silvestre. *Some regularity results for integro-differential equations.*
- R. Schwab. *Periodic Homogenization of Nonlinear Integro-Differential Equations*

Non linear elliptic and parabolic equations involving nonlocal operators arise naturally in various frameworks. A well known example of such an equation is the so called quasi geostrophic equation which was presented in A. Vasseur's talk. These equations enjoy many properties of the usual elliptic and parabolic equations, though the nonlocal character of the problem introduces new, sometime unexpected difficulties. In his talk, L. Silvestre generalize the notion of fully nonlinear equation to the nonlocal setting. This is a work in collaboration with L. Caffarelli, who has been one of the main actor in the recent trend in studying the properties of nonlocal elliptic equation. They are able to extend the usual definition of Extremal operators and viscosity solutions to integro-differential equations and establish the main properties of these equations. R. Schwab, is then interested in the homogenization of such equations. His result extends the recent result of L. Caffarelli, P. Souganidis and L. Wang to the nonlocal framework.

The quasi geostrophic equation was introduced by Constantin and Wu in 1999 as a toy model for the study of possible blow-up in 3D fluid dynamics. It is a non-local non-linear equation for the temperature $\theta : \mathbb{R}^2 \to \mathbb{R}$:

$$\begin{aligned} &\partial_t \theta + u \cdot \nabla \theta = -\Lambda \theta, \\ &u = R^{\perp} \theta \end{aligned} \tag{1}$$

with the operator $\Lambda = (-\Delta)^{1/2}$ is defined by $\widehat{\Lambda\theta} = |\xi|\widehat{\theta}$ and where $R^{\perp}$ denotes the orthogonal of the Riesz transform. Other nonlocal operators can be consider, the most natural choices being other powers of the Laplace operator: $\Lambda = (-\Delta)^{\alpha}$ with $\alpha \in (0, 1)$. The case $\alpha = 1/2$ is usually refer to as the critical case.

Together with Luis Caffarelli, A. Vasseur proves that the solutions of the drift-diffusion equation

$$\partial_t \theta + u \cdot \nabla \theta = -\Lambda \theta$$

are locally Holder continuous for L^2 initial data and under minimal assumptions on the drift u. As an application they show that solutions of the quasi-geostrophic equation (1) with initial L^2 data and critical diffusion $(-\Delta)^{1/2}$, are locally smooth for any space dimension. The main difficulty is to obtain Hölder regularity. The method is inspired by the celebrated proof of E. De Giorgi for C^{α} regularity of the solutions of elliptic equation with bounded measurable coefficients.

[1]They had a position less than two years ago.

Over recent years, there has been a lot of interest from the mathematical community for non-local operators, and many of the well known properties of standard elliptic and parabolic equations have been extended to non local ones. In that direction, Luis Caffarelli and Luis Silvestre study fully nonlinear integro-differential equations. These are the non local version of fully non-linear elliptic equations of the form $F(D^2u, \nabla u, u, x) = 0$. Typical examples are the ones that arise from stochastic control problems with jump processes.

We first recall that linear integro-differential operators have the form

$$Lu(x) = \int_{\mathbb{R}^n} (u(x+y) - u(x) - \chi_B(y)\nabla u(x) \cdot y) K(y) dy$$

where the most typical case is

$$K(y) = \frac{1}{|y|^{n+\sigma}}$$

corresponding to the fractional Laplacian. In stochastic control problems with jumps processes it is classical to deal with nonlinear equations of the form

$$0 = lu(x) := \sup_{\alpha} L_\alpha u(x).$$

From two-player stochastic games we would get even more complicated equations of the form

$$0 = lu(x) := \inf_{\beta} \sup_{\alpha} L_{\alpha\beta} u(x).$$

These equations form basic examples of fully nonlinear integro-differential equations.

In order to study these equations, we need to generalized of the notion of uniformly ellipticity for fully nonlinear nonlocal equations. This, as explained in L. Silvestre's talk, can be done using the Pucci extremal operators $M_\sigma^\pm$ defined by

$$M_\sigma^+ u(x) = \sup_{\lambda a(y) \le \Lambda \; a(y) = a(-y)} (2-\sigma) \int (u(x+y) - u(x)) \frac{a(y)}{|y|^{n+\sigma}} dy$$

and a similar definition for M_σ^-.

Then, a nonlocal operator l is said to be uniformly elliptic of order σ if

$$M_\sigma^- v(x) \le l(u+v)(x) - lu(x) \le M_\sigma^+ v(x)$$

(σ is always in $(0, 2)$).

L. Caffarelli and L. Silvestre are then able to obtain results analogous to the Alexandroff estimate, (Krylov-Safonov) Harnack inequality and $C^{1,\alpha}$ regularity for uniformly elliptic equations. Interestingly, as the order of the equation approaches two, in the limit the estimates become the usual regularity estimates for second order elliptic partial differential equations.

Since those nonlocal equations have similar properties as the local ones, it seems natural to investigate their behavior under various classical perturbation limits. In his PhD thesis work, R. Schwab investigate the homogenization limit of such equations. The recent work of L. Caffarelli, P. Souganidis and L. Wang for the homogenization of fully non-linear equations of elliptic and parabolic type introduced a new approach to obtain homogenization result in stationary ergodic media. In his talk, R. Schwab shows that the method can be adapted to a somewhat general class of nonlinear, nonlocal uniformly "elliptic" equations. Motivated by the techniques of the homogenization of fully nonlinear uniformly elliptic second order equations by L. Caffarelli, P. Souganidis and L. Wang, he shows how a similar obstacle problem can be used to identify the effective equation in the nonlocal setting.

2.3 Reaction-diffusion equations

- J.-M. Roquejoffre. *Free boundary problems for the fractional Laplacian.*
- Y. Sire. *Rigidity results for elliptic boundary reaction problems.*

Reaction-diffusion equations are important in many area of applied mathematics involving phase transition. In view of the recent progress in understanding the behavior of nonlocal elliptic operators, it seems natural to extend to such operators some of the well known results of the theory of elliptic equations. This is one of the goal of J.-M. Roquejoffre and Y. Sire in their respective talks. Both of them rely heavily on the extension formula of L. Caffarelli and L. Silvestre which allows us to rewrite fractional Laplace operators as boundary operator for degenerate (local) elliptic operators. This formula, which first appeared in 2005 has generated a lot of new development in the field and the works presented below are good examples.

One of the most studied stationary free boundary problem is Bernoulli problem, which consists of a elliptic equation sets in the positivity set of the solution:

$$\Delta u = 0 \text{ in } \{u > 0\}$$

and a free boundary condition

$$|\nabla u|^2 = 1 \text{ on } \partial\{u > 0\}$$

where the unknown function u is non-negative.

This problem arises in the modeling of flame propagation as the limit of the singular reaction problem

$$\Delta u = \beta_\delta(u)$$

where β_δ is an approximation of the Dirac mass. The same free boundary problem also appears in heat flux minimization, as the Euler Lagrange equation for the minimization of the non continuous functional

$$\int |\nabla u|^2 \; + \chi_{\{u>0\}} dx.$$

The study of such problems is very delicate because of the lack of a priori regularity or the free boundary $\partial\{u > 0\}$. The first regularity results go bak to the early 80's with the works of Alt-Caffarelli and Alt-Caffarelli-Friedman.

In his talk, J.-M. Roquejoffre presented some recent work with L. Caffarelli and Y. Sire concerning a non-local version of this famous problem. In this problem, the Laplace equation is replaced by a fractional Laplace equation:

$$(-\Delta)^s u = 0 \text{ in } \{u > 0\}.$$

In that case, the appropriate free boundary condition (which correspond to the natural Neuman condition for fractional Laplace operators) is

$$u(x) \sim A[(x - x_0) \cdot \nu(x_0)]^s$$

for any $x_0 \in \partial\{u > 0\}$, where $\nu(x_0)$ denotes the inward unit normal vector to $\partial\{u > 0\}$. This problem can be seen as the limit for equations of the form

$$(-\Delta)^s u = -\beta_\delta(u)$$

which arise in physic as a first attempt to take into account non local effects in the modeling of reaction-diffusion phenomena.

L. Caffarelli, J.-M. Roquejoffre and Y. Sire investigates the variational formulation of this problem and strongly rely on the extension formula of Caffarelli-Silvestre. They are able to obtain the main result in the theory, namely the optimal regularity of the minimizer ($u \in \mathcal{C}^s$)), the hölder growth away from the free boundary and the positive density of $\{u > 0\}$ and $\{u = 0\}$ along the free boundary. This implies in particular that blow-up limits have non-trivial free boundaries and that free boundaries cannot form cusps. However, it leaves as an open question a very classical property of the free boundary for the usual Bernoulli problem: Does the free boundary have finite $(n-1)-$Hausdorff measure?

Finally, they investigate the behavior of minimizers in the neighborhood of the regular free boundary points (differentiability points) and show that indeed, u satisfies

$$u(x) \sim A[(x-x_0)\cdot\nu(x_0)]^s$$

where A is a universal constant. The set of differentability points can be proved to be dense in the free boundary, but further regularity results, as in the usual Bernoulli problem are still, for the most part, open problems.

De Giorgi's famous conjecture concerns some symmetry properties of the solution of equations of the form

$$(-\Delta)u = f(u) \text{ in } \mathbb{R}^n$$

with $f(u)=u^3-u$. The conjectures claims that if u is an entire solution such that

$$|u| \leq 1$$

and

$$\frac{\partial u}{\partial x_n} > 0$$

(where $x=(x,x_n)\in\mathbb{R}^N$), then, at least for $n\leq 8$, the level sets of u must be hyperplanes.

The problem originates in the theory of phase transition and is so closely connected to the theory of minimal hypersurfaces that it is sometimes referred to as the "version of Bernstein problem for minimal graphs".

Since the work of Nassif Ghoussoub and Changfeng Gui in 1998, which proved the conjecture in dimension 2, there has been a lot of activity trying to establish the conjecture in all dimensions.

E. Valdinocci and Y. Sire attempted to investigate a similar conjecture for boundary phase transition problems, which can easily be reformulated as a nonlocal reaction-diffusion problem:

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^n.$$

Following the extension formula established by Caffarelli and Silvestre, this equation can be rewritten as

$$-\text{ div }(y^a\nabla u) = 0 \text{ in } \mathbb{R}^{n+1}$$

and

$$-y^a\partial_y u = f(u) \text{ at } y=0.$$

Y. Sire describes a technique based on a geometric Poincare-type inequality which allows to get some symmetry results for bounded stable solutions of boundary reaction problems in low dimension.

This technics relies on the use of the second variation and the notion of stable solutions. The idea to use the second variation to get fine control on the level sets of the solution goes back to Steinberg-Zumbrum and was used, in particular, by Farina-Scienzi and Valdinocci for the usual E. Valdinocci and Y. Sire show how to apply this technique to elliptic degenerate quasi-linear equations set in the half-space. As a consequence of Caffarelli-Silvestre formula, one gets some rigidity properties of solutions for the corresponding nonlocal equations involving fractional powers of the laplacian.

2.4 Non-local moving fronts

- P. Cardaliaguet. *Front propagation with non-local terms.*
- A. Monteillet. *Convergence of approximation schemes for non-local front propagation equations.*
- P. E. Souganidis. *Non-local approximations of moving interfaces.*

Three speakers presented results concerning non-local moving interfaces: P. Cardaliaguet, A. Monteillet and P. E. Souganidis. Contributions of P. E. Souganidis to the field of moving interfaces are fundamental. The reader is referred to his survey paper [32] which contains many important references on this topic. P. Cardaliaguet [15] is one of the first mathematician who developed tools for studying in a very general setting interfaces whose geometric law is non-local. A. Monteillet is a PhD student of P. Cardaliaguet.

P. E. Souganidis presented results concerning non-local approximations of moving interfaces. A moving interface can be defined by considering a family $K = \{K(t)\}_{t\in[0,T]}$ of compact subsets of $\mathbb{R}^N$:

$$\forall x \in \partial K(t), \forall t \geq 0, \quad V(x,t) = f(x,t,\nu_{x,t},A_{x,t},K) \tag{2}$$

where

- $V(x,t)$ is the normal velocity of a point x of $\partial K(t)$ at time t
- $\nu_{x,t}$ is the unit exterior normal to $K(t)$ at $x \in \partial K(t)$
- $A_{x,t} = \left[\frac{\partial \nu^i_{(x,t)}}{\partial x_j}\right]_{ij}$ is the curvature matrix of $K(t)$ at $x \in \partial K(t)$
- $K \mapsto f(x,t,\nu_{x,t},A_{x,t},K)$ is a non-local dependence in the whole front K (up to time t).

Equation (2) is referred to as the geometric law of the moving interface. Such evolution equations appear in several areas: cristal growth, elasticity, biology, finance, shape optimization design, image processing... For these problems, existence and uniqueness of classical solutions can be obtained by methods of differential geometry (Huisken, Escher-Simonnet, ...) However the front (often) develops singularities in finite time. Hence, two important problems are: on one hand, to define the front after the onset of singularities and, on the other hand, to study its properties.

P. E. Souganidis explained that when one expects the geometric inclusion principle to hold true, generalized moving interfaces $\{K(t)\}_{t\in[0,T]}$ can be defined in several ways. In particular, it can be defined

- either by using the level set method which consists in representing the set $K(t)$ as the zero level set of a function $u(t,\cdot)$. The geometric law (2) is translated into a geometric partial differential equation and this equation is studied by using viscosity solution theory;
- or by using a geometric formulation such as in [7]; loosely speaking, this approach consists in considering a smooth test front that is contained in (resp. contains) the generalized front. If this test front evolves with a speed that is smaller (resp. greater) than the one of the generalized front, then it has to stay inside (resp. outside) the generalized front as time increases.

For local evolutions of the form

$$\forall x \in \partial K(t), \forall t \geq 0, \quad V_{x,t} = f(x,t,\nu_{x,t},A_{x,t})$$

Evans and Spruck [23], Chen, Giga and Goto [17] have defined notion of generalized solution by using the level set approach and techniques of viscosity solutions. Similar but more geometric approaches have been developed by Soner [31], Barles, Soner and Souganidis [5], Belletini and Novaga [8], Barles and Souganidis [7]...

After recalling these definitions and classical results, P. .E. Souganidis considered two non-local approximations of moving interfaces: Bence-Merriman-Osher (BMO) schemes and rescalings of solutions of reaction-diffusion equations. He presented results corresponding to two working papers, one with L. Caffarelli and one with C. Imbert. As far as BMO schemes are concerned, he explained that if the Gaussian kernel is replaced with kernels which decay slowly at infinity (as a proper power law), then either mean curvature flow or "fractional" mean curvature flow are obtained at the limit, depending on the decay rate of kernels. He explained that classical BMO schemes can be seen as Trotter-Kato approximation of rescaled reaction-diffusion equations in the classical case. Hence, it can be proved that mean field equations associated with stochastic Ising models with long range interactions can be rescaled in order to prove that, on one hand, mean curvature flows can be obtained at the limit [22, 5] and on the other hand, fractional mean curvature flows can be obtained (working paper).

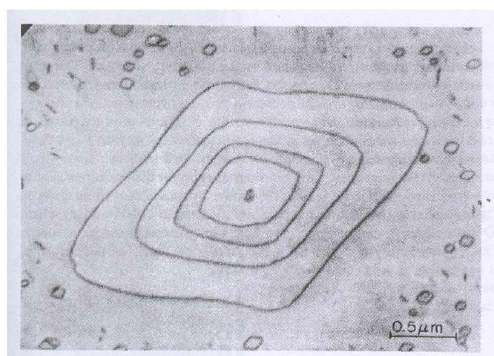
P. Cardaliaguet presented results related to flows with and without inclusion principle. We recall that a geometric flow satisfies the inclusion principle if, at initial time, a front $\mathcal{O}_1$ is included in another front $\mathcal{O}_2$, this inclusion is preserved by the flow. A typical example of inclusion preserving flows is the one associated with the following geometric law

$$V_{x,t} = 1 + \lambda |\nabla u|^2$$

where u is the solution to

$$\begin{cases} -\Delta u = 0 & \text{in } K(t) \setminus S \\ u = 1 & \text{on } \partial S \\ u = 0 & \text{on } \partial K \end{cases}$$

Cardaliaguet explained that this flow can be interpreted as a gradient flow for the Bernoulli problem. In [16], a notion of sub- and super-flow is defined by using (smooth) test fronts. Such an idea first appeared in [7] where a geometric formulation of moving interfaces is developed in order to be able to solve singular perturbation problems arising in the phasefield theory of reaction-diffusion equations. Moreover, an inclusion principle is proved in [16] and a generic uniqueness result is obtained. Moreover, a link with the energy of the problem is presented; it is related to the definition of minimizing movements in the spirit of the seminal paper of De Giorgi, Marino and Tosques [18]. As far as flows without inclusion principle are concerned, Cardaliaguet presented a general existence result obtained with G. Barles, O. Ley and A. Monteillet [4]. Precisely, a generalized moving interface is constructed here by the level set method (see above). He also presented the first uniqueness result for a Fitzhugh-Nagumo type system [4]. Finally, he presented existence result for dislocation dynamics (see Figure 2.4).



A. Monteillet presented results about approximation schemes for computing the weak solution constructed in [4]. More precisely, he considered a general class of stable, monotone and consistent schemes in order to be able to apply the fundamental result of Barles and Souganidis [6] which can be adapted to the non-local geometric equation studied in [4] by using a new stability result of Barles [3].

2.5 Mean-field and kinetic equations

- J. Dolbeault. *Mean field models in gravitation and chemotaxis.*
- I. Gentil. *A Lévy-Fokker-Planck equation: entropies and convergence to equilibrium.*
- C. Mouhot. *Some properties of non-local operators from collisional kinetic theory.*

Dolbeault'stalk was intended to provide an overview of some results of mean field theory, mostly in case of an attractive Poisson law. He first presented some stability results for stationary solutions of the gravitational Vlasov-Poisson model [20]. Connection with drift-diffusion equations were obtained in a diffusion limit. As a side result, he presented some results and conjectures on the two-dimensional Keller-Segel model, which share properties which are similar to gravitational models, but for which the mass is a critical parameter [14]. He then presented some results for a three-dimensional flat model of gravitation, showed the existence of solutions stationary with high Morse index and state some conjectures about their stability [19].

Gentil presented results related to a Lévy-Fokker-Planck equation

$$\begin{cases} \partial_t u = \mathcal{I}[u] + \operatorname{div}(u\nabla V) & x \in \mathbb{R}^d, t > 0, \\ u(0,x) = u_0(x) & x \in \mathbb{R}^d \end{cases}$$

where u_0 is non-negative and in $L^1(\mathbb{R}^d)$ and V is a given proper potential for which there exists a nonnegative steady state. The operator $\mathcal{I}$ is a Lévy operator

$$\mathcal{I}[u](x) = \operatorname{div}(\sigma\nabla u)(x) - b \cdot \nabla u(x) + \int_{\mathbb{R}^d} (u(x+z) - u(x) - \nabla u(x) \cdot zh(z))\nu(dz)$$

with parameters (b, σ, ν) where $b = (b_i) \in \mathbb{R}^d$, σ is a symmetric semi-definite $d \times d$ matrix $\sigma = (\sigma_{i,j})$ and ν denotes a nonnegative singular measure on $\mathbb{R}^d$ that satisfies

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int \min(1, |z|^2)\nu(dz) < +\infty;$$

h is a truncature function and we fix it on this article: for any $z \in \mathbb{R}^d$, $h(z) = 1/(1 + |z|^2)$.

The starting point of this work is a paper by Biler and Karch [10] where an exponential decay towards the equilibrium u_∞ in L^p norm is proved by assuming, loosely speaking, that $\sigma \neq 0$. Moreover, the rate of convergence seemed not to be optimal. In order to get such an exponential decay, they study the following family of entropies: for any nonnegative function f,

$$\operatorname{Ent}^{\Phi}_{u_\infty}(f) := \int \Phi(f)\, u_\infty dx - \Phi\left(\int f u_\infty dx\right).$$

where Φ is a convex function. In particular they prove that the equilibrium state u_∞ is an infinite divisible law and that entropies are Lyapunov functions for the Lévy-Fokker-Planck equation.

Gentil explained that the main contributions of [24] are the following results:

- if $\int_{|z|\geq 1} \ln|z|\nu(dz) < +\infty$, there exists an equilibrium state u_∞ (even if $\sigma = 0$); it is also proved that it is an infinite divisible law;
- the energy associated with the Φ-entropy is explicitely computed; it looks like the Dirichlet form associated with $\mathcal{I}$ with respect to the measure $u_\infty(x)dx$;
- under additional assumptions on Φ and ν, the entropy decays exponentially fast; in particular, an optimal exponential rate is obtained.

C. Mouhot presented results related to collisional kinetic (integro)-differential equation

$$\partial_t f + v\nabla_x f + F \cdot \nabla_v f = Q(f, f) \tag{3}$$

where Q is the collision operator; it is local in t, x.

The operator Q is local for linear Fokker-Planck equations, but it is bilinear and integral for collisional dilute gases (Boltzmann) or plasmas (Landau). There are many interesting issues involving these non-local operators (Cauchy, regularity, asymptotic behavior, derivation, hydrodynamic limit,...). Here the speaker focused on the case of Boltzmann collision operators with singular kernel (long-range interactions) (with Landau operator as a limit) in the linearized setting.

The spatially homogeneous non-linear case has be studied a lot see the papers of Funaki, Goudon, Villani, Lions, Méléard, Desvillettes, Graham, Fournier, Guérin,... about the Cauchy problem, the regularity of solutions, the study of grazing collision limit *etc.* But there are fewer works in the spatially inhomogeneous case and the linearized problem; see the papers of Alexandre, Alexandre-Villani (and Chen-Desvillettes-He for the Landau equation). The linearized study is crucial for stability issues.

Mouhot next recalled the definition of the linearized operator. The normalized Maxwellian equilibrium is the function $M(v) = e - |v|^2$. If now f in (3) is chosen under the form $M + Mh$, the following linearized

Boltzmann operator appears: $L(h) = M^{-1}[Q(Mh, M) + Q(M, Mh)]$ with $h(v) \in L^2(M)$. An explicit formula of L is given now

$$Lh(v) = \frac{1}{4} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} (h(v') + h(v'_*) - h(v) - h(v_*)) B(|v - v_*|, \sigma) M(v_*) dv_* d\sigma$$

where $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma$ and $v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$. The physical important case is the case where $B = \Phi(|v - v_*|) b(\cos\theta)$ with b a power-law and $\cos\theta = \sigma \cdot (v - v_*)/|v - v_*|$.

Here are the important properties of this operator.

- L is symmetric on the Hilbert space $L^2(M)$.
- It is non-positive (linearized H theorem):

$$D(h) = -(h, Lh) = \frac{1}{4} \int_{v, v_*, \sigma} |h' + h'_* - h - h_*| B M M_* \geq 0 \,.$$

- Its null space $N(L)$ is $(d + 2)$-dimensional and spanned by the collisional invariants $1, v_1, ..., v_d, |v|^2$.

Therefore important question of the existence of a spectral gap: positive distance isolating 0 from the remaining part of the spectrum. After recalling a lot of previous results (Hilbert 1912, Carlman 1957, Grad 1962, Wang-Chang and Uhlenbeck 1970, Bobylev 1988, Pao 1974, Caflisch 1980, Degond-Lemou 1997, Lemou 2000, Guo 2002 *etc*), the speaker stated the main theorem of his talk.

Theorem 1 ([30]) *Assume* $B = \Phi(|v - v_*|) b(\cos\theta)$ *with*

$$\Phi(z) \geq C_\Phi z^\gamma, \quad b(\cos\theta) \geq b_0 (\sin\theta/2)^{-(d-1)-\alpha} \textit{ for } \theta \sim 0 \,.$$

Then

- $\forall \epsilon > 0$, *there exists* $C_{B,\epsilon} > 0$ *(constructive proof) such that*

$$D(h) \geq C_{B,\epsilon} \|h - \Pi(h)\|_{L^2_{\gamma+\alpha-\epsilon}}(M)$$

  where Π *denotes the orthogonal projection on* $N(L)$.
- *There exists* $C_{B,0} > 0$ *(non constructive proof) such that*

$$D(h) \geq C_{B,0} \|h - \Pi(h)\|_{L^2_{\gamma+\alpha}}(M) \,.$$

2.6 From non-linear stochastic differential equations to non-linear non-local evolution equations

- B. Jourdain. *Non-linear SDEs driven by Lévy processes and related PDEs.*
- S. Méléard. *Stochastic approach for some non-linear and non-local partial differential equations.*
- W. A. Woyczynski. *Non-linear non-local evolution equations and their physical origins.*

In Woyczynski'stalk, the physical and biological problems leading to non-linear and non-local evolution equations were reviewed and the outstanding problems in this area discussed.

His talk was divided into five parts. In the first one, he presented a model for describing the growth of an interface. This model involves a non-linear non-local diffusion equation. He also presented numerous examples from physical sciences where distributions associated with α-stable Lévy processes appear. For instance, "at the atomic level, there is no reason to assume automatically that surface diffusion is Gaussian"; he illustrated this point by showing some molecular dynamics calculations [29].

The third part of his talk was devoted to fractal conservation laws and fractal Hamilton-Jacobi-KPZ equations. Results concerning fractal conservation laws are presented Subsection 2.7 below. The results for fractal Hamilton-Jacobi-KPZ equations were obtained with G. Karch [28]. The equation at stake in this paper is the following one

$$\partial_t u = (-\Delta)^{\alpha/2} u + \lambda |\nabla u|^q$$

where λ is a real number. As a matter of fact, more general Lévy operators are considered in the paper but we present the results in this framework for the sake of clarity. The case $\lambda > 0$ corresponds to the deposition case; indeed, it is proved that if $\lambda > 0$, then the total mass $M(t) = \int_{\mathbb{R}^d} u(t,x)dx$ increases as time t increases. In the case $\lambda < 0$, $M(t)$ decreases and we say that we are in the evaporation case. In the deposition case, the existence of a limit of $M(t)$ as $t \to +\infty$ is discussed; it depends on the non-linearity exponent q. In any case, as long as $M(t)$ has a finite limit M_∞ as $t \to +\infty$, it is proved that $u(t)$ behaves like the fundamental solution of the fractional heat equation times M_∞.

The case of the strongly non-linear problems analogous to the classical porous medium equation requires further attention here. Some limitations, such as nonexistence of global solutions for general non-local diffusion-convection mean field models will be indicated.

Recent results on the interplay between the strength of the ”anomalous” diffusive part and ”hyperbolic” non-linear terms will be presented in the case of fractal Hamilton-Jacobi-KPZ equations (see [28] and the joint working paper with B. Jourdain, S. Meleard, G. Karch, and P. Biler).

Méléard’s talk was a survey describing various stochastic approaches for non-linear and non-local equations, in terms of interpretation, existence and uniqueness, regularity and particle approximations of the solution of the equation. She explained why it is in a certain sense easier, in a probabilistic point of view, to study non-local non-linearity.

In the first part of the talk, she briefly recalled the link between some non-linear Fokker-Planck partial differential equations and stochastic differential equations which are non-linear in the sense of McKean and are driven by a Brownian motion. Jourdain developed this part in his talk; see below. S. Méléard recalled the particle approximation result deduced from this stochastic interpretation. she generalized this approach to a non-linear partial differential equation with fractional Laplacian and show how it is related with a non-linear jump process.

In a second part, she considered kinetic equations known as (spatially homogeneous) Fokker-Planck-Landau equations [25] (see also J. Fontbona). She showed that the probabilistic interpretation involves a non-linear stochastic differential equation driven by a space-time white noise. She used this interpretation to define an easily simulable stochastic particle system and prove its convergence in a pathwise sense, to the solution of the Landau equation.

B. Jourdain first recalled how existence for a stochastic differential equation (SDE for short) non-linear in the sense of McKean implies existence for the associated non-linear Fokker-Planck partial differential equation. In the case where the driving Lévy process is square integrable and the diffusion coefficient is Lipschitz continuous, B. Jourdain explained how to prove existence and uniqueness for the SDE by a fixed-point approach. He also exhibited strong rates of convergence of approximations by interacting particle systems as the number of particles tends to infinity. When either the integability properties of the Lévy process or the smoothness assumption on the diffusion coefficient are relaxed, weak existence for the SDE is obtained by weak convergence of the particle systems.

2.7 Fractal conservation laws

- N. Alibaud. *Fractional Burgers equation.*
- J. Droniou. *A numerical approximation of the solutions to fractal conservation laws.*
- G. Karch. *Large time asymptotics of solutions to the fractal Burgers equation.*

Four talks (the three previous ones and the one of W. A. Woyczynski, see above) were related to the study of Fractal conservation laws in one space variable

$$\begin{cases} \partial_t u + \partial_x(f(u)) + (-\Delta)^{\alpha/2} u = 0 & t > 0, x \in \mathbb{R}\,, \\ u(0,x) = u_0(x) & x \in \mathbb{R}\,, \end{cases} \tag{4}$$

where f is a non-linear flux function and $(-\Delta)^{\alpha/2}u$ is the fractional Laplacian.

We recall that the fractional Laplacian is defined as follows: for all Schwartz function ϕ,

$$(-\Delta)^{\alpha/2}\phi = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}\phi)$$

where $\mathcal{F}$ denotes the Fourier transform. There also exists an integral representation of the fractional Laplacian: for all $\phi \in C^{1,1} \cap L^\infty$, $x \in \mathbb{R}$,

$$(-\Delta)^{\alpha/2}\phi(x) = -c(\alpha) \int (u(x+z) - u(x) - Du(x) \cdot z \mathbf{1}_B(z)) \frac{dz}{|z|^{1+\alpha}}$$

where $\mathbf{1}_B(z)$ denotes the indicator function of the unit ball of $\mathbb{R}$ and $c(\alpha) > 0$ is a constant which only depends on α.

An important special case of (4) is the fractal Burgers equation which corresponds to the case $f(u) = u^2/2$.

$$u_t + uu_x + (-\Delta)^{\alpha/2}u = 0 \tag{5}$$

with $\alpha \in (0, 2)$.

Before reviewing the results presented during the workshop about (4), we would like to mention that all the important results known for this equation were proved by people attempting the workshop. As far as existence and uniqueness of a solution is concerned, a first study was done in a framework of fractional Sobolev spaces and Morrey spaces [9, 11]. Next, solutions were studied in a L^∞ framework in [21]. In particular, it is proved in the case $\alpha > 1$ that there exists a smooth solution of (4) as soon as f is Lipschitz continuous and u_0 is bounded.

As far as fractal conservation laws are concerned, W. A. Woyczynski focused on results about the asymptotic behavior of solutions are obtained in [12, 28] in the case where $f(u) = |u|^{r-1}u$ and the physical origin of the equation [26]. He explained that $r_C = 1+(\alpha-1)/d$ is a critical non-linearity exponent. When $r > r_C$, then the solution u of (4) for such f's behaves the one where $f \equiv 0$ (with the same initial datum). It is even possible to get second-order asymptotics. When $r = r_C$, there exists a unique source solution U and the long time behaviour of u can be described by using U [28].

Alibaud explained that in the case $\alpha \leq 1$, the analyis developed in [21] in a L^∞ framework do not apply. In particular, shocks can occur even with smooth initial data u_0 [2] and weak solutions are not unique; see the working paper by Alibaud and Andreianov (Besançon, France). This is the reason why it is necessary to define entropy solutions [1] by using the integral representation of the fractional Laplacian.

Karch presented results about the large time behavior of solutions of the Cauchy problem for (5) supplemented with the initial datum of the form

$$u_0(x) = c + \int_{-\infty}^{x} m(dy)$$

with $c \in \mathbb{R}$, m being a finite (signed) measure on $\mathbb{R}$. If $\alpha \in (1, 2)$, the corresponding solution converges toward the rarefaction wave *i.e.* the unique entropy solution of the Riemann problem for the nonviscous Burgers equation [27]. On the other hand, using a standard scaling technique one can show that equation (5) with $\alpha = 1$ has self-similar solutions of the form $u(x, t) = U(x/t)$. These profiles determine the large time asymptotics of solutions to the initial value problem with $\alpha = 1$. If $\alpha \in (0, 1)$, the Duhamel principle allows us to show that the non-linear term is asymptotically negligible and the asymptotics is determined by the linear part of equation (5). These results will be contained in a working paper by C. Imbert and G. Karch.

Droniou presented a method to compute numerical approximations of solutions of (4). The conservation law is discretized using classical monotone upwind fluxes (either 2-points fluxes, or higher order methods such as MUSCL), and the discretization of the fractal operator is based on its integral representation. He gave a few elements on the analysis of this scheme, and he provided numerical results showing behaviors of the solution (such as shock or speed of diffusion) which have been predicted in the literature on theoretical study of fractal conservation laws. See for instance Figure 2.7 for a numerical simulation of the appearence of shocks [2].

Figure 2: Initial condition (in green) and solution (in red) at time $T = 0.5$ and $\alpha = 0.3$

2.8 Applications

- A. Peirce. *Hydraulic Fractures: multiscale phenomena, asymptotic and numerical solutions.*
- B. Perthame. *Adaptive evolution; concentrations in parabolic PDEs and constrained Hamilton-Jacobi equations.*

Some of the talks more directly described direct applications of nonlocal operators. This is in particular the case of A. Peirce's talk, in which he describe a model for studying propagation of hydraulic cracks. After introducing the problem of Hydraulic Fracture and providing examples of situations in which Hydraulic Fractures are used in industrial problems, A. Peirce presented some numerical method for solving the governing equations, for which very few rigorous properties are known.

Hydraulic fractures (HF) are a class of tensile fractures that propagate in brittle materials by the injection of a pressurized viscous fluid. Natural examples of HF include the formation of dykes by the intrusion of pressurized magma from deep chambers. HF are also used in a multiplicity of engineering applications, including: the deliberate formation of fracture surfaces in granite quarries; waste disposal; remediation of contaminated soils; cave inducement in mining; and fracturing of hydrocarbon bearing rocks in order to enhance production of oil and gas wells.

The governing equations in 1-2D as well as 2-3D models of Hydraulic Fractures involve a coupled system of degenerate nonlinear integro-partial differential equations as well as a free boundary Namely, the width of the fracture $w(x, t)$ satisfies

$$\partial_t w = \partial_x (w^3 \partial_x p)$$

where the pressure p is given by

$$p - \sigma_0 = (-\Delta)^{1/2}(w).$$

This equation is satisfied inside the fracture itself, i.e. for $|x| \leq l(t)$. It is a non local diffusion equation (the pressure law is nonlocal) of order 3. At the tip of the fracture ($x = \pm l$), we must have

$$w(\pm l, t) = 0 \qquad w^3 \partial_x p = 0.$$

A. Peirce then demonstrates, via re-scaling the 1-2D model, how the active physical processes manifest themselves in the HF model and show how a balance between the dominant physical processes leads to special solutions.

He then discussed the challenges for efficient and robust numerical modeling of the 2-3D HF problem including: the rapid construction of Greens functions for cracks in layered elastic media, robust iterative techniques to solve the extremely stiff coupled equations, and a novel Implicit Level Set Algorithm (ILSA) to resolve the free boundary problem. The efficacy of these techniques with numerical results can be demonstrated.

Living systems are subject to constant evolution through the two processes of mutations and selection, a principle discovered by Darwin. In a very simple, general and idealized description, their environment can be considered as a nutrient shared by all the population. This alllows certain individuals, characterized by a 'phenotypical trait', to expand faster because they are better adapted to the environment. This leads to select the 'best fited trait' in the population (singular point of the system). On the other hand, the new-born population undergoes small variance on the trait under the effect of genetic mutations. In these circumstances, is it possible to describe the dynamical evolution of the current trait?

In a work based on collaborations with O. Diekmann, P.-E. Jabin, S. Mischler, S. Cuadrado, J. Carrillo, S. Genieys, M. Gauduchon and G. Barles, B.Perthame study the following mathematical model which models such dynamics:

$$\begin{cases} \partial_t n = d\partial_{xx} n + n(1 - \phi \star n), \quad 0 \le x \le 1, \\ n(t,0) = n(t,1), \qquad \partial_x n(t,0) = \partial_x n(t,1) \\ n(0,x) = n_0 \ge 0 \end{cases}$$

where the convolution kernel ϕ satisfies

$$\phi \ge 0, \qquad \int \phi = 1, \quad \phi = 0 \text{ in } \mathbb{R} \setminus [-b, b].$$

Then it can be showm that an asymptotic method allows them to formalize precisely the concepts of monomorphic or polymorphic population. Then, we can describe the evolution of the 'best fitted trait' and eventually to compute various forms of branching points which represent the cohabitation of two different populations.

The regime under investigation correspond to letting the small parameter ϵ go to zero in

$$\begin{cases} \partial_t n = \epsilon\partial_{xx} n + \frac{1}{\epsilon} n(1 - \phi \star n), \quad 0 \le x \le 1, \\ n(t,0) = n(t,1), \qquad \partial_x n(t,0) = \partial_x n(t,1) \\ n(0,x) = n_0 \ge 0 \end{cases}$$

This leads to concentrations of the solutions and the difficulty is to evaluate the weight and position of the moving Dirac masses that desribe the population. It can be shown however, that a new type of Hamilton-Jacobi equation, with constraints, naturally describes this asymptotic.

3 Outcome of the meeting

This meeting brought together mathematicians with a common interest for nonlocal operators to present their latest results on the topic. Besides the great quality of the talks (see above), this meeting has also given the opportunity for people to meet and exchange ideas on this subject. Many participants have taken advantages of the informal discussion sessions to work together. We know that several new collaborations were started during the meeting, and we think that this is an indication that BIRS is extremely important to the mathematical community.

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