Self-Similarity and Branching in Group Theory

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1 Overview of the Field and Workshop Objectives

The idea of self-similarity is one of the most basic and fruitful ideas in mathematics of all times. In the last few decades it established itself as the central notion in areas such as fractal geometry, dynamical systems, and statistical physics. Recently, self-similarity started playing a role in algebra as well, first of all in group theory.

Regular rooted trees are well known self-similar objects (the subtree of the regular rooted tree hanging below any vertex looks exactly like the whole tree). The self-similarity of the tree induces the self-similarity of its group of automorphisms and this is the context in which we talk about self-similar groups. Of particular interest are the finitely generated examples, which can be constructed by using finite automata. Groups of this type are extremely interesting and usually difficult to study as there are no general means to handle all situations. The difficulty of study is more than fairly compensated by the beauty of these examples and the wealth of areas and problems where they can be applied.

One of the earliest examples of a self-similar group, is the famous Grigorchuk 2-group, introduced in [4]. This group was the first example of a group of intermediate growth, solving the celebrated Milnor problem. It was also the first example of an amenable group that is not elementary amenable.

The idea of branching entered Algebra via the so-called branch groups that were introduced by Grigorchuk at St. Andrews Group Theory Conference in Bath 1997. Branch groups are groups that have actions of 'branch type' on spherically homogeneous rooted trees. The phrase "of branch type" means that the dynamics of the action (related to the subnormal subgroup structure) mimics the structure of the tree. Spherically homogenous trees appear naturally in this context, both because they are the universal models for homogeneous ultra-metric spaces and because a group is residually finite if and only if it has a faithful action on a spherically homogeneous tree.

The importance of the choice of the 'branch type' action is reflected in the fact that it is the naturally opposite to the so-called diagonal type. While any residually finite group can act faithfully on a rooted homogeneous tree in a diagonal fashion, the actions of branch type are more restrictive and come with some structural implications. The Grigorchuk 2-group is the prototypical example of a finitely generated branch group.

Actions of branch type give rise to many examples of just-infinite groups (thus answering a question implicitly raised in [3] on existence of exotic examples of just infinite groups) and to a number of examples of 'small groups' (or atomic groups) in the sense of S. Pride [7]. Branch groups also played a role in the construction of groups with non-uniform exponential growth, answering a question of Gromov [9].

The ideas of self-similarity and branching interact extremely well in group theory. There is a large intersection between these two classes of groups and this workshop was devoted to some important features and examples of this interaction.

The subject of self-similarity and branching in group theory is quite young and the number of different directions, open questions, and applications is growing rather quickly.

Of particular importance, is the relationship of self-similar groups to Julia sets and fractals from holomorphic dynamics via iterated monodromy groups, introduced by Nekrashevych [5]. This led to the solution of the longstanding 'Twisted Rabbit Problem' by Bartholdi and Nekrashevych [2].

In addition to the standard objectives of a mathematical workshop, this meeting was intended to serve as a forum for

- Formal exchange of information and ideas through formal presentations.

We strongly emphasized the secondary (but nevertheless absolutely crucial) aspects of such a meeting as a forum for

- Informal exchange of information and ideas through informal conversations, chance meetings, learning by "osmosis", and so on.
- Furthering of the existing collaborative efforts between the participants, as well as development of new professional relations.

To accomplish these objectives the number of formal plenary presentations were kept to 3–4 per day in order to save time for the following activities:

- Meetings in smaller groups focused on specific aspects and goals, such as profinite aspects, holomorphic dynamics and iterated monodromy groups, amenability and probabilistic aspects, algebraic and algorithmic aspects, including connections to automata. All participants were included in such activities according to their own inclination. The goal of these focused teams was to make progress toward resolving some of the more challenging problems in the area, at least at the level of establishing lasting and directed collaborative efforts, based on sound working ideas and strategies.
- Demonstration of GAP packages for working with self-similar groups, developed by graduate students D. Savchuk and Y. Muntyan at Texas A & M University. The work on the packages was supported by an NSF grant of R. Grigorchuk and Z. Sunic and the packages are freely accessible to anyone interested in using them.
- Problem session at the beginning and end of the Workshop.

2 Press Release

The Banff International Research Station hosted top researchers in its workshop on "Self-Similarity and Branching in Group Theory", October 12 – October 17, 2008. The importance of self-similarity and branching phenomena in group theory has recently come to the forefront. Self-similar groups are the algebraic counterparts to fractals. Fractals quite often arise as Julia sets of certain rational functions, say polynomials. For instance, the Basilica of Saint Mark fractal is the Julia set of the polynomial $z^2 + 1$. The famous Sierpinski gasket is also the Julia set of a rational function. To each such rational function, there is associated a self-similar group, which encodes algebraically the Julia set and the dynamics of the rational function on the Julia set.

The study of self-similar groups has led to new insights and a better understanding of fractals and their related dynamics. A longstanding-problem concerning the rabbit fractal and the airplane fractal was solved via the method of self-similar groups. Self-similar groups also have interactions with Computer Science, since much of their structure can be encoded by finite state machines. These machines can be used in turn to produce the fractals.

3 Basic notions

3.1 Definition of a self-similar group

For an alphabet X on k letters, the set of all finite words X^* over X has the structure of a rooted regular k-ary tree in which the empty word is the root and each vertex u has k children ux, $x \in X$. Denote the group of automorphisms of the tree X^* by $Aut(X^*)$.

For a tree automorphism $g \in Aut(X^*)$ and a vertex $u \in X^*$, define the **section** of g at u to be the unique tree automorphism g_u such that the equality

$$g(uw) = g(u)g_u(w)$$

holds for all words $w \in X^*$.

A group of tree automoprhisms $G \leq Aut(X^*)$ is **self-similar** if all sections of all elements in G are elements in G.

3.2 Geometric definition of a branch group

Let T be a spherically homogeneous tree. For a group of tree automorphisms $G \leq Aut(T)$ define the rigid stabilizer at the vertex $u \in T$ by

$$RiSt_G(u) = \{g \in G \mid Supp(g) \in T_u\},$$

where T_u is the subtree of T hanging below the vertex u. The rigid stabilizer of the n-th level L_n in T is defined by

$$RiSt_G(L_n) = \langle RiSt_G(u) \mid u \in L_n \rangle = \prod_{u \in L_n} RiSt_G(u)$$

A group G is **geometrically branch group** is it is a spherically transitive subgroup of $Aut(T)$, for some spherically homogeneous tree T, such that all rigid level stabilizers $RiSt_G(L_n)$ have finite index in G.

3.3 Algebraic definition of a branch group

A group G is **algebraically branch group** if there exists a sequence of integers $\overline{k} = \{k_n\}_{n=0}^{\infty}$ and two decreasing sequences of subgroups $\{R_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ of G such that

(1) $G = R_0 = V_0$

(2) $k_n \geq 2$, for all $n > 0$, $k_0 = 1$

(3) for all n,

$$R_n = V_n^{(1)} \times V_n^{(2)} \times \cdots \times V_n^{(k_0 k_1 k_2 \ldots k_n)}, \tag{1}$$

where each $V_n^{(j)}$ is an isomorphic copy of V_n,

(4) for all n, the product decomposition (1) of R_{n+1} is a refinement of the corresponding decomposition of R_n in the sense that the j-th factor $V_n^{(j)}$ of R_n, $j = 1, \ldots, k_0 k_1 \ldots k_n$ contains the j-th block of k_{n+1} consecutive factors

$$V_{n+1}^{((j-1)k_{n+1}+1)} \times \cdots \times V_{n+1}^{(jk_{n+1})}$$

of R_{n+1},

(5) for all n, the groups R_n are normal in G and

$$\bigcap_{n=0}^{\infty} R_n = 1,$$

(6) for all n, the conjugation action of G on R_n permutes transitively the factors in (1), and

(7) for all n, the index $[G : R_n]$ is finite.

4 Open Problems

A number of open problems about self-similar groups and branch groups were raised.

1. Are all contracting self-similar groups amenable? Contracting groups do not have free subgroups (Nekrashevych [6]).
2. Are all automaton groups of polynomial growth amenable? Groups of bounded growth (Bartholdi, Kaimanovich, Nekrasheyvch [1]) and linear growth (Amir, Angel, Virag) are amenable. Groups of polynomial growth do not have free subgroups (Sidki [8]).
3. Is there a residually finite non-amenable group without free subgroups?
4. Is the word problem decidable for finitely generated self-similar groups? How about the uniform problem where the groups are given by functional recursion?
5. Can one decide whether an initial automaton has finite order?
6. Can one decide whether an initial automaton is spherically transitive? (This can be done for n-adic transformations).
7. Can one decide whether an automaton group is infinite?
8. Can one decide whether an automaton group is spherically transitive? (This is decidable for groups of n-adic transformations).
9. Is the word problem for automaton groups PSPACE complete? (It is easy to see that the problem is in PSPACE).
10. Are the products of closed subgroups in the Grigorchuk group closed?
11. Is solvability of equations decidable for the Grigorchuk group?
12. Do contracting groups have decidable conjugacy problem?
13. Do automaton groups have decidable conjugacy problem?
14. Does every hyperbolic group have a faithful self-similar action?
15. Construct self-similar actions of free pro-p groups.
16. What are the kernels of the natural action of fiitely generated algebraically branch groups on rooted trees. In particular, does the center always have finite index in the kernel?
17. Are there finitely generated nonamenable branch groups without free subgroups?
18. Is every maximal subgroup of a finitely generated branch group necessarily of finite index?
19. Do there exist finitely presented branch groups?
20. Is the conjugacy problem decidable in all finitely generated branch groups in which the word problem is decidable?
21. Are all finitely generated hereditarily just infinite groups linear? Do there exist finitely generated, hereditarily just infinite, torsion groups?

5 Participants

Name	Affiliation
Abert, Miklos	University of Chicago
Amir, Gideon	University of Toronto
Benli, Mustafa G.	Texas A& M University
Bumagin, Inna	Carleton University
Glasner, Yair	Ben Gurion University of the Negev
Grigorchuk, Rostislav	Texas A& M University
Kharlampovich, Olga	McGill University
Mccune, David	University of Lincoln at Nebraska
Miasnikov, Alexei	McGill University
Morris, Dave	University of Lethbridge
Nekrashevych, Volodymyr	Texas A& M University
Sapir, Mark	Vanderbilt University
Savchuk, Dmytro	Texas A& M University
Steinberg, Benjamin	Carleton University
Sunic, Zoran	Texas A & M University
Vorobets, Yaroslav	Texas A& M University
Vorobets, Mariya	Texas A& M University

6 The Workshop Program

Monday

7:00–8:45	Breakfast
8:45–9:00	Introduction and Welcome to BIRS by BIRS Station Manager, Max Bell 159
9:00–10:00	Zoran Šunić, *Branching in group theory I*
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	Volodymyr Nekrashevych, *Self-similar groups, limit spaces and tilings*
11:30–13:00	Lunch
13:00–14:00	Yair Glasner, *A zero-one law for finitely generated subgroups of* $SL(2, Q_p)$.
14:10–15:00	Lorenzo Sadun *Introduction to aperiodic tilings, talk from the aperiodic tilings section*
15:00–15:30	Coffee Break, 2nd floor lounge, Corbett Hall.
15:30–16:10	Dmytro Savchuk, *GAP package AutomGrp for computations in self-similar groups and semigroups: functionality, examples and applications*
16:15–17:15	Problem/Discussion Session
17:30–19:30	Dinner

Tuesday

7:00–9:00	Breakfast
9:00–10:00	Zoran Šunić, *Branching in group theory II*
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	Lecture
11:30–13:00	Lunch
13:00–14:00	Lecture
14:00–15:00	Gideon Amir, *Amenability of automata groups with linear growth automorphisms*
15:00–15:30	Coffee Break, 2nd floor lounge, Corbett Hall.
15:30–16:30	Miklos Abert, *On weak containment of measure preserving actions*
17:30–19:30	Dinner
19:30–21:00	Mark Sapir, *Residual finiteness of 1-related groups*

Wednesday

7:00–9:00	Breakfast
9:00–10:00	Olga Kharlampovich, *Undecidability of Markov Properties*
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	Alexei Miasnikov, *The conjugacy problem for the Grigorchuk group has polynomial time complexity*
11:30	Group Photo; meet on the front steps of Corbett Hall
11:30–13:30	Lunch
13:30–	Free Afternoon
17:30–19:30	Dinner
19:30–20:30	Benjamin Steinberg, *The Ribes-Zalesskii Product Theorem and rational subsets of groups*

Thursday

9:00–10:00	Yaroslav Vorobets, *Automata generating free groups and free products of cyclic groups*
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	Volodymyr Nekrashevych, *Free selfsimilar groups going back to Gauss*
11:30–13:00	Lunch
13:00–13:50	Yair Glasner
14:00–15:00	Problem/Discussion Session
15:00–15:30	Coffee Break, 2nd floor lounge, Corbett Hall.
17:30–19:30	Dinner

Friday

7:00–9:00	Breakfast
9:00–10:00	Zoran Šunić
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	Discussion/Problem Session
11:30–13:30	Lunch

7 Abstracts

Speaker: **Miklos Abert** (University of Chicago)
Title: *On weak containment of measure preserving actions*
Abstract: We study asymptotic properties of chains of subgroups in residually finite groups using the dynamics of boundary representations and the structure of periodic invariant measures on Bernoulli actions. This allows us to analyze when the Schreier graphs coming from a chain of subgroups can approximate another action of the group. For chains with property tau, we exhibit a strong rigidity result, while for amenable groups, we prove that every chain approximates every action. As a byproduct, we show that covering towers of regular graphs admit a new kind of spectral restriction which is related to the independence ratio. This leads us to solve a problem of Lubotzky and Zuk. In another direction, we relate the cost of a boundary representation to the growth of rank and the first L^2 Betti number of the group. This allows us to relate the 'fixed price problem' of Gaboriau to the 'rank vs Heegaard genus' conjecture in 3-manifold theory and show that they contradict each other.

Speaker: **Gideon Amir** (University of Toronto)
Title: *Amenability of automata groups with linear growth automorphisms*
Abstract: We prove using random walks that automata of linear growth generate amenable groups, generalizing previous work of Bartholdi, Kaimanovich and Nekrashevych. This is joint work with O. Angel and B. Virag.

Speaker: **Yair Glasner** (Ben Gurion)
Title: *A zero-one law for finitely generated subgroups of $SL(2, Q_p)$.*

Abstract: Let $G = SL(2, Q_p)$. Let $k > 2$ and consider the space $Hom(F_k, G)$ where F_k is the free group on k generators. This space can be thought of as the space of all marked k-generated subgroups of G, i.e., subgroups with a given set of k generators.

There is a natural action of the group $Aut(F_k)$ on $Hom(F_k, G)$ by pre-composition. I will prove that this action is ergodic on the subset of dense subgroups. This means that every measurable property either holds or fails to hold for almost all k-generated subgroups of G together.

Speaker: **Volodymyr Nekrashevych** (Texas A&M)
Title: *Self-similar groups, limit spaces and tilings*
Abstract: We explore the connections between automata, groups, limit spaces of self-similar actions, and tilings. In particular, we show how a group acting “nicely” on a tree gives rise to a self-covering of a topological groupoid, and how the group can be reconstructed from the groupoid and its covering. The connection is via finite-state automata. These define decomposition rules, or self-similar tilings, on leaves of the solenoid associated with the covering.

Speaker: **Olga Kharlampovich** (McGill)
Title: *Undecidability of Markov Properties*
Abstract: A group-theoretic property P is said to be a Markov property if it is preserved under isomorphism and if it satisfies:

1. There is a finitely presented group which has property P.

2. There is a finitely presented group which cannot be embedded in any finitely presented group with property P.

Adyan and Rabin showed that any Markov property cannot be decided from a finite presentation. We give a survey of how this is proved.

Speaker: **Alexei Miasnikov** (McGill)
Title: *The conjugacy problem for the Grigorchuk group has polynomial time complexity*
Abstract: We discuss algorithmic complexity of the conjugacy problem in the original Grigorchuk group. Recently this group was proposed as a possible platform for cryptographic schemes (see [4, 15, 14]), where the algorithmic security of the schemes is based on the computational hardness of certain variations of the word and conjugacy problems. We show that the conjugacy problem in the Grigorchuk group can be solved in polynomial time. To prove it we replace the standard length by a new, weighted length, called the *norm*, and show that the standard splitting of elements from $St(1)$ has very nice metric properties relative to the norm.

Speaker: **Mark Sapir** (Vanderbilt)
Title: *Residual finiteness of* 1*-related groups*
Abstract: We prove that with probability tending to 1, a 1-relator group with at least 3 generators and the relator of length n is residually finite, virtually residually (finite p)-group for all sufficiently large p, and coherent. The proof uses both combinatorial group theory, non-trivial results about Brownian motions, and non-trivial algebraic geometry (and Galois theory). This is a joint work with A. Borisov and I. Kozakova.

Speaker: **Dmytro Savchuk** (Texas A&M)
Title: *GAP package AutomGrp for computations in self-similar groups and semigroups: functionality, examples and applications*
Abstract: Self-similar groups and semigroups are very interesting from the computational point of view because computations related to these groups are often cumbersome to be performed by hand. Many algorithms related to these groups were implemented in AutomGrp package developed by the authors (available at http://www.gap-system.org/Packages/automgrp.html). We describe the functionality of the package, give some examples and provide several applications. This is joint with Yevgen Muntyan

Speaker: **Benjamin Steinberg** (Carleton)
Title: *The Ribes-Zalesskii Product Theorem and rational subsets of groups*

Abstract: Motivated by a conjecture of Rhodes on finite semigroups and automata, Ribes and Zalesskii proved that a product of finitely many finitely generated subgroups of a free group is closed in the profinite topology. We discuss a proof of this result due to the speaker and Auinger, as well as generalizations to other groups by various authors. Applications are given to computing membership in rational subsets of groups. In particular, for a torsion group, like the Grigorchuk group, every rational subset is a finite union

$$\bigcup gH_1 \cdots H_n$$

of translates of products of finitely generated subgroups and so such a separability result would give decidability of membership in rational subsets.

Speaker: **Zoran Šunić** (Texas A&M)
Title: *Branching in group theory*
Abstract: We provide an introduction to the notion of a branch group. We cover the definition, motivation, examples, and some basic properties. In addition, we mention some applications that are based on the branch structure of the given branch groups.

Speaker: **Yaroslav Vorobets** (Texas A&M)
Title: *Automata generating free groups and free products of cyclic groups*
Abstract: An invertible finite automaton canonically defines a finitely generated group of automorphisms of a regular rooted tree. We will describe a class of finite automata that define free nonabelian groups. Freeness is established via the dual automaton approach, which provides a new techniques to solve the word problem for automaton groups.

8 The GAP package AutomGrp

One of the highlights of the meeting was a tutorial by Savchuk on his program with Muntyan implementing self-similar groups in GAP. We include here some screenshots from the tutorial.

Automaton groups and semigroups can be defined in "AutomGrp" are as follows.

Automaton groups:

```
gap> GrigorchukGroup :=
     AutomatonGroup("a=(1,1)(1,2),b=(a,c),c=(a,d),d=(1,b)");
< a, b, c, d >

gap> Basilica := AutomatonGroup( "u=(v,1)(1,2), v=(u,1)" );
< u, v >
```

Automaton semigroups:

```
gap> SG := AutomatonSemigroup( "f0=(f0,f0)(1,2), f1=(f1,f0)[2,2]" );
< f0, f1 >
```

Self-similar groups:

```
gap> WRG := SelfSimilarGroup("x=(1,y)(1,2),y=(z^-1,1)(1,2),z=(1,x*y)");
< x, y, z >
```

The package computes basic properties of groups/semigroups generated by automata.

```
gap> IsSphericallyTransitive(GrigorchukGroup);
true

gap> IsAbelian(GrigorchukGroup);
false
```

```
gap> IsFractal(GrigorchukGroup);
true
```

```
gap> IsAmenable(GrigorchukGroup);
true
```

Basic operations: sections

```
gap> Section(p*q*p^2, [1,2,2,1,2,1]);
p^2*q^2

gap> Decompose(p*q^2);
(p*q^2, q*p^2)(1,2)

gap> Decompose(p*q^2,3);
(p*q^2, q*p^2, p^2*q, q^2*p, p*q*p, q*p*q, p^3, q^3)(1,8,3,5)(2,7,4,6)
```

Finding relations in groups and semigroups.
The following command finds all relations in Grigorchuk group up to length 16

```
gap> FindGroupRelations(GrigorchukGroup,8);
a^2
b^2
c^2
d^2
b*c*d
d*a*d*a*d*a*d*a
c*a*c*a*c*a*c*a*c*a*c*a*c*a*c*a
[ a^2, b^2, c^2, d^2, b*c*d, d*a*d*a*d*a*d*a, c*a*c*a*c*a*c*a*c*a*c*a*c*a*c*a]
```

Or all relations in $\langle ac, ada \rangle$ up to length 10

```
gap> FindGroupRelations([a*c,a*d*a], ["p", "q"], 5);
q^2
q*p*q*p^-1*q*p*q*p^-1
p^-8
[ q^2, q*p*q*p^-1*q*p*q*p^-1, p^-8 ]
```

Find all elements in Grigorchuk group of order 16 up to length 5

```
gap> FindElements(GrigorchukGroup,Order,16,5);
[ a*b, b*a, c*a*d, d*a*c, a*b*a*d, a*c*a*d, a*d*a*b, a*d*a*c, b*a*d*a,
  c*a*d*a, d*a*b*a, d*a*c*a, a*c*a*d*a, a*d*a*c*a, b*a*b*a*c, b*a*c*a*c,
  c*a*b*a*b, c*a*c*a*b ]
```

Order of an element
In Basilica group the element $u^{35}v^{-12}u^2v^{-3}$ has infinite order

```
gap> Basilica := AutomatonGroup( "u=(v,1)(1,2), v=(u,1)" );
< u, v >
gap> Order( u^35*v^-12*u^2*v^-3 );
infinity
```

Contracting groups. We can check that `WRG` is contracting and compute the nucleus.

```
gap> IsContracting( WRG );
true
gap> GroupNucleus( WRG );
[ 1, y*z^-1*x*y, z^-1*y^-1*x^-1*y*z^-1, z^-1*y^-1*x^-1, y^-1*x^-1*z*y^-1,
  z*y^-1*x*y*z, x*y*z ]
```

9 Outcome of the Meeting

The meeting successfully served its main purpose of providing a forum for exchange of recent results and new ideas, as well as establishing new collaborative efforts aimed at solving problems that were already known or were introduced during the meeting. In addition, the idea of having a presentation of the freely available GAP package of Muntyan and Savchuk for working with self-similar groups proved to be very well received, as many participants realized the ease with which this package could be used and its efficiency at performing calculations (leaving the user to only worry about the more creative side of his/her research). Also, all participants responded positively to the idea to have representatives of the two Workshops that were present at Banff at the same time give introductory presentations to the participants of the other Workshop.

Rostislav Grigorchuk and Yair Glasner made progress on the question whether all maximal subgroups in finitely generated branch groups have finite index extending their observation that maximal subgroups in branch groups are themselves branch groups.

Volodymyr Nekrashevych defined a notion of a self-similar groupoid (of a self-covering bimodule), which generalizes self-similar groups, self-coverings of topological spaces (such as post-critically finite rational functions restricted to their Julia sets) and adjacency groupoids of self-similar aperiodic tilings.

Zoran Šunić proved results on distance transitivity of the Hanoi Towers group $H^{(k)}$. In addition, he showed that every normal subgroup N in a distance 2-transitive group of tree automoprhisms acts transitively on every subtree T_u such that there exists $n \in N$ that fixes u and acts nontrivially on the children of u. In particular, every normal subgroup that stabilizes level n but does not stabilize level $n+1$ acts transitively on all subtrees hanging at level n.

Graduate students present at the meeting obtained several ideas for questions they could try to work on and include in their dissertations. For instance, Mccune will try to determine which right-angled Artin groups can be realized as automaton groups, Savchuk will try to determine if the construction of free products of cyclic groups of order 2 suggested by Šunić is correct, and Benli became interested in L-presentations of branch groups.

References

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