Recent Progress in Rigidity Theory 08w2137

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1 Overview of the Field

Mathematicians have been interested in the rigidity of frameworks since Euler's conjecture in 1776 that 3-dimensional polyhedra are rigid. The conjecture was verified for convex polyhedra by Cauchy in 1813 and for generic polyhedra by Gluck in 1975. Connelly constructed a counterexample to Euler's original conjecture in 1982. Interest and developments in rigidity have increased rapidly since the 1970's, motivated initially by the combinatorial characterization of rigid 2-dimensional generic bar-and-joint frameworks by Laman in 1970, and also by applications in many areas of science, engineering and design. This 2-day workshop was preceded by a 5-day workshop 'Rigidity, Flexibility, and Motion: Theory, Computation and Applications to Biomolecules' which concentrated on one such application. Discussions begun at that workshop, in particular on global rigidity, played an essential role in the momentum carried into this workshop and into the progress reported below. The applications have encouraged mathematicians not only to develop theoretical results but also fast algorithms for determining whether a framework is rigid, and more generally calculating its rigid components and degrees of freedom. See [9] for definitions and a survey of results on the rigidity frameworks.

2 Open Problems and Recent Developments

Two fundamental open problems are to determine when a 3-dimensional generic bar-and-joint framework is either rigid or globally rigid. An important special case of the first problem, which has applications to the rigidity of molecules, is the 'Molecular Conjecture' of Tay and Whiteley [8]. This would characterize when a 3-dimensional generic bar-and-joint realization of the 'square' of a graph is rigid. A conjectured characterization of when an arbitrary 3-dimensional generic bar-and-joint framework is rigid is given in [6].

The 2-dimensional version of the second problem, i.e. characterizing when a 2-dimensional generic bar-and-joint framework is globally rigid, was solved in [5]. Gortler, Healy, and Thurston [4] recently completely the proof that the d-dimensional version of this problem is equivalent to determining the generic rank of a 'stress matrix' associated to the underlying graph. This result verifies a conjecture of Connelly and is analogous to a result of Gluck which connects the rigidity of a generic framework to the rank of the 'rigidity matrix' of its underlying graph. It implies, in particular, that the global rigidity of generic d-dimensional bar-and-joint frameworks is completely determined by the underlying graph.

Other open problems and results concern the rigidity and global rigidity of different types of frameworks, e.g. symmetric frameworks, body-and-bar frameworks, as well as angles and directions as constraints, alone or with added distance constraints. All of the above topics were addressed during the workshop.

3 Scientific Progress Made During the Workshop and Future Outcomes

Sitharam presented an algorithm which she conjectures could determine whether a 3-dimensional generic bar-and-joint framework is rigid. This generated much discussion, particular between Jackson, Jordán and herself, and will lead to future research collaboration.

There were many discussions on various aspects of global rigidity. Connelly and Whiteley [3] began work on using the above mentioned result of Gortler at al to show that a d-dimensional generic framework is globally rigid if and only if its $(d+1)$-dimensional 'cone' is globally rigid, and also derive a random algorithm for determining whether a d-dimensional generic bar-and-joint framework is globally rigid. Jordán and Whiteley formulated a conjectured characterization of d-dimensional globally rigid generic body-and-bar frameworks. Follow up work of Connelly, Jordán and Whiteley has confirmed this conjecture, and has led them to a closely related conjectured characterization of global rigidity in molecular frameworks [2]. Whiteley asked whether the 'X-replacement' operation preserves global rigidity in 3-space and Jordán constructed an example to show that this is not always the case.

Jackson and Jordán [7] completed their work on showing that the '1-extension' operations preserve global rigidity in 2-dimensional generic direction-length frameworks, and began work with Connelly [1] on determining when redundant rigidity is a necessary condition for global rigidity in these frameworks. Sitheram and Whiteley each spoke about angle constraints in 2-D and 3-D. The resulting conversations of Servatius, Sitheram, Whiteley, have continued around these topics and also analogs of direction-length frameworks for the sphere.

Schulze and Whiteley began a collaborative research project with Watson, combining their independent results on the rigidity of bar-and-joint frameworks under symmetry, and dependence which is implied by flatness, which also follows from flatness.

Servatius, Shai, and Whiteley continued their investigation of Assur Graphs and extending the algorithms for decomposition of mechanisms (one-degree of freedom frameworks).

References

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