Classification of amalgams for non-spherical Kac-Moody groups (08rit130)

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1 Overview of the Field

Dating back to Felix Klein's Ernlangen's program, mathematicians (and more recently physicists) have used the interplay between a geometric structure and its group of automorphisms.

Some of the most interesting objects in group theory are Lie groups/ algebraic groups and the related finite groups of Lie type. A unifying geometric approach for these groups was proposed by J. Tits (see [25]) in the 60's using combinatorial geometric objects called *spherical buildings*. The approach was extremely successful and gave unifying results for all such groups.

Later Ronan and Tits generalized the notion of a spherical building and introduced the idea of a *twin building*. These objects are more general than spherical buildings but not as general as buildings. Moreover many of the abstract results about spherical buildings could be generalized to twin buildings.

In an apparently unrelated development, Victor Kac [19] and Robert Moody[21] independently considered a class of infinite dimensional Lie algebras that closely resemble the finite dimensional semi-simple ones. These came to be known as Kac-Moody Lie algebras. The usual Chevalley game of exponentiation gave rise to a new series of groups that were called Kac-Moody groups. These objects found applications in various areas of theoretical physics.

In [26], Tits proved that the automorphisms groups of twin buildings are the Kac-Moody groups. This provided a combinatorial framework to the newly discovered class of groups.

The last piece of the puzzle came from the classification of finite simple groups. An important step of the classification of finite simple groups, announced in 1981, and of the ongoing Gorenstein-Lyons-Solomon revision of the classification is the identification of the "minimal counterexample" with one of the known simple groups. This step follows the local analysis step, when inside the minimal counterexample G one reconstructs one or more of the proper subgroups using the inductive assumptions and available techniques. Thus the input of the identification step is a set of subgroups of G that resemble certain subgroups of some known simple group $\hat{G}$ referred to as the target group. The output of the identification step is the statement that G is isomorphic to $\hat{G}$. Two of the most widely used identification tools are the Curtis-Tits theorem (see [13]) and Phan's theorem (see [23]).

The Curtis-Tits theorem allows the identification of G with a simple Chevalley group $\hat{G}$ provided that G contains a system of subgroups identical to the system of appropriately chosen rank-two Levi factors from $\hat{G}$. Phan's theorem is very similar and allows one to identify the group G with a unitary group provided that G contains a set of appropriately chosen rank 2 unitary subgroups. The relation between the two theorems was explained in [1].

2 Recent Developments and Open Problems

One important result obtained using this point of view is Abramenko and Mühlherr's generalization of the Curtis-Tits theorem which assesses that both the twin building and the associated groups can be recognized by local data. In the same manner the Phan theorem was generalized using results of Devillers and Mühlherr [10]. Another series of important results is the large number of generalizations of Phan's result to many other groups of Lie type [1, 2, 4, 5, 14, 15, 16]

Several important classification results on Kac-Moody groups and twin-buildings have been obtained recently. The classification of twin buildings as proposed by Mühlherr is under way but not published. Criteria for a building to be twin-able have recently appeared in work by Devillers, Mühlherr and Van Maldeghem [11], and Ronan [24]. Mühlherr and Caprace have described automorphisms of twin buildings [7, 8] and Caprace and Remy proved abstract simplicity of non-affine Kac Moody groups [9].

3 Scientific Progress Made

As put forth in our proposal for this RIT, the main question we have considered is whether the "Curtis-Tits" amalgams determine the groups in the absence of the actual twin building.

A common and elegant way to describe an amalgam $\mathcal{A}$ uses a diagram similar to the Dynkin diagram of a Lie algebra. Nodes represent the "rank 1" groups in $\mathcal{A}$ and edges represent the "rank 2" groups containing the corresponding rank 1 groups in some prescribed way. In the case of spherical and tree-shaped diagrams, this diagram uniquely determines the rank-2 Curtis-Tits amalgam and hence its universal completion. The Curtis-Tits theorem can be interpreted to say that in fact this diagram is equal to the Dynkin diagram of the group, where nodes now represent Levi components rather than the full parabolic subgroups. Similar results are obtained for certain Phan-type amalgams mentioned above. These results motivate the following fundamental question:

> To what extent do diagrams for Curtis-Tits amalgams determine the amalgam?

We study diagrams for which non-isomorphic amalgams exist. This phenomenon can already be studied in the following setting. As a diagram we consider an arbitrary simple graph without triangles. The "rank 1" groups, corresponding to the vertices, are all isomorphic to $SL_2(k)$, where k is a field. The "rank 2" groups are given by the graph as follows: two $SL_2(k)$'s will generate an $SL_3(k)$ (in the "standard" way) if the corresponding vertices are on an edge and an $SL_2(k) \circ SL_2(k)$ if they are not. The first question is to classify such amalgams. That is, for any amalgam with a given diagram, one needs to describe the universal completion. We do this by describing this completion algebraically and, wherever possible, by describing a geometric object on which this completion acts flag-transitively.

The RIT at BIRS was extremely successful. We managed to classify all amalgams in the case of a loop graph and have devised a method to deal with the general case. It turns out that in the case of a loop diagram the set of isomorphism classes of amalgams is in bijection with $Aut(k) \times \mathbb{Z}$. To our surprise we realized that the classes of amalgams corresponding to twin buildings are just those in $Aut(k)$. The others correspond to a new class of groups that generalize Kac-Moody groups that we call *Curtis-Tits groups*.

We intend to publish three papers as a direct result of our collaboration in Banff. I will briefly describe the results in each.

In the first paper we prove the classification of amalgams and consider the twin building amalgams. These were briefly described by Tits in [26] as groups of invertible matrices over a non-commutative polynomial ring. We correct a small error in that paper and give an equivalent description of these twisted Kac-Moody groups as actual subgroups of untwisted Kac-Moody groups fixed by certain automorphisms. On some abstract level, a result by Mühlherr [22] implies the existence of such groups, but no complete classification like ours is available in the literature.

In the second paper we describe the *non-orientable* Curtis-Tits amalgams and the corresponding groups. These groups are not Kac-Moody groups but they appear as fixed subgroups of Kac-moody groups under a certain kind of type-changing automorphism. The automorphism resembles a Phan-type flip as studied for the first time in [5]. The fixed-point result from [22] is not applicable here, and our methods are based on the results of Devillers and Mühlherr [10].

Finally in the third paper we consider a particular example of the non-orientable Curtis-Tits group. It is a unitary group for a non-symmetric sesquilinear form over $k[t, t^{-1}]$. We construct a Clifford-like super-algebra on which this group acts. This is a very interesting algebra. It can be viewed as a generalization of Manin's quantum plane and it is a sort of q-CCR algebra as defined for example in [18]. Moreover if one specializes the group and the algebra at $t = 1$ we get an orthogonal group and its usual Clifford algebra. If we do the same at $t = -1$ we get a symplectic group and its corresponding Heisenberg algebra. In short, our group is related to those algebras that are used by theoretical physicists to study elementary particles and quantum phenomena.

4 Outcome of the Meeting

The very fact that out of the material developed during the meeting we will be able to publish three separate papers, made the meeting very productive. Scientifically speaking the meeting was very productive in that we have been able to achieve the following goals:

(a) We have been able to give a complete answer to the fundamental question to what extent a Curtis-Tits diagram can determine an amalgam in an important instance, namely that of simply laced diagrams.

(b) We have discovered a new family of groups, namely the non-orientable Curtis-Tits groups.

(c) Although our study of non-orientable Curtis-Tits groups was motivated by pure mathematical considerations, we find that there is a surprising connection between these and certain super-algebras that are used by theoretical physicists to study elementary particles.

Encouraged by these results in (a) we are now in a position to extend our work field to amalgams over other non-spherical diagrams as well as amalgams whose groups are different from $SL_2(k)$'s and $SL_3(k)$'s. In particular, with these results in hand, one can begin to obtain Phan-type amalgams for these groups of Kac-Moody and Curtis-Tits type. Our results in (c) encourage us to seek contact and possible collaboration with colleagues interested in super-algebras and their groups.

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