DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION AS RANDOM DYNAMICAL SYSTEMS: QUALITATIVE PROPERTIES

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The focused research group on Stochastic Differential Equations driven by Fractional Brownian Motion as Random Dynamical Systems met from around 9:30am to around 5pm from Monday September 29 to Saturday October 4, 2008. It included 8 participants and one observer.

The goal of the group was to exchange ideas between two largely distinct aspects of differential systems driven by self-similar stochastic processes: the stochastic analysis angle and the theory of random dynamical systems. Each of the 9 people gave talks on various topics in each of these aspects. These talks were not aimed at presenting individual research results, rather they were meant to introduce the audience to the general theory, and to present the most current tools being used. Thereafter, the 9 met in smaller groups to discuss ways of exploiting synergies within the collective expertise, defining strategies for solving major problems in stochastic differential equations with fractional Brownian motion.

The expository talks covered the following topics on fractional Brownian motion (fBm) and random dynamical systems:

- **Ciprian A. Tudor** (U. Paris 1 Pantheon-Sorbonne, France): fBm as a Gaussian process, Malliavin calculus for fBm, including the divergence (Skorohod) integral.

- **Fabrice Baudoin** (Purdue University, USA): Rough path theory for integration with respect to fBm, limits of the integration theory for small Hurst parameter.

- **David Nualart** (University of Kansas, USA): Fractional calculus and fBm, Stochastic differential equations driven by fBm, solutions in the rough path sense, estimates of the solutions using a fractional calculus reinterpretation of the rough path theory.

- **Ivan Nourdin** (U. Paris 6 Jussieu, France): Gubinelli's version of rough path theory; integration against fBm via regularization and via Riemann-sum approximations, limits of this integration theory.

- **Maria-Jose Garrido-Atienza** (U. Sevilla, Spain): random dynamical system property for stochastic differential equations driven by fBm, finite-dimensional cases.

- **Björn Schmalfuß** (U. Paderborn, Germany): random dynamical system property for stochastic differential equations driven by fBm, infinite-dimensional cases: results and questions.

- **Jinqiao Duan** (Illinois Institute of Technology, USA): application of fBm-driven stochastic partial differential systems to climate modeling and other physical systems with colored noise, long memory, or self-similarity.

- **Frederi Viens** (Purdue University): Wiener chaos calculus, characterization of normal convergence via Malliavin derivatives, and application to Hurst (self-similarity) parameter estimation.
- **Kening Lu** (Brigham Young University): Formulation of linearized or linear-multiplicative random dynamical systems as products of random matrices, and infinite-dimensional version of the Oseledets theorem for Lyapunov exponents.

All participants used the expository talks to ask many questions of the expositors, in order to enhance their understanding of areas with which they were less familiar. One main topic of investigation that came out of these discussions early on was to seek to prove that the solution of a nonlinear stochastic differential equation driven by fBm in the rough-path sense, with Hurst parameter between 1/4 and 1/2, is in fact a random dynamical system in the sense that the solution satisfies a cocycle property that holds for all starting points simultanously, almost surely (and not merely almost surely for a fixed starting point). This can be done for finite-dimensional systems with Hurst parameter larger than 1/2, working "omega-wise" via standard estimates from the pathwise (Young-type) integration theory; a similar effect should exist when using rough paths. At the moment, the consensus appears to be that the new explicit estimates for rough-path integrals discovered and used by David Nualart and Yaozhong Hu, may provide the best hope for completing this initial problem. Using rough path estimates from Gubinell's theory may also be useful, although this was less clear in our minds. There was a general agreement that the divergence-integral interpretation of fBm-driven stochastic differential equations would not lead to new developments in the study of random dynamical systems.

Other discussions pertained to more specific questions on random dynamical systems for fBm, including existence of stable manifolds, and random attractors, for non-trivial systems, such as those infinite-dimensional ones driven by fBm. We speculate that many fBm-driven systems in infinite dimensions should have finite-dimensional random attractors, a very desirable property from the standpoint of quantitative analysis. Some of us also discussed extensions of the ergodic property to non-Gaussian self-similar and long-memory processes, as well as the question of how to determine statistically the long-memory parameter for such processes, in the non-Gaussian contexts of Wiener chaos or of non-linear time series.

FBm, long-memory processes, self-similar processes, and other colored noises are becoming very popular in the applied sciences. We have had several discussions along these lines on several models. In climate modeling, the atmospheric advection-diffusion-condensation equation is shown empirically to contain long memory; we have discussed estimating the humidity parameter or function via variations methods similar to those that can yield the long-memory parameter itself. Joint estimation of these two parameters should also be possible. Other real-world problems we discussed addressed long memory and self-similarity in financial econometrics, internet traffic, DNA sequencing, and polymers.

References

[1] Arnold, L. (1998). *Random Dynamical Systems*. Springer Monographs in Mathematics, Springer-Verlag, Berlin.

[2] Arnold, L. (2005). Hasselmann's program visited: The analysis of stochasticity in deterministic climate models. In *J.-S. von Storch and P. Imkeller, editors, Stochastic climate models,* 141–158, Birkhäuser.

[3] Baudoin, F.; Coutin, L. (2007). Operators associated with a stochastic differential equation driven by fractional Brownian motions. *Stochastic Process. Appl.* **117** (5), 550-574.

[4] Baudoin, F.; Hairer, M. (2007). A version of Hörmander's theorem for the fractional Brownian motion. *Probab. Theory Related Fields* **139** 373–395.

[5] Bender, C. (2003). An Ito formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter. *Stoch. Proc. Appl.* **104**, 81-106.

[6] Biagini, F.; Hu, Y.; Oksendal, B.; Zhang, T. (2008). *Stochastic calculus for fractional Brownian motion and applications*. Springer.

[7] Caraballo, T. (1990). Asymptotic exponential stability of stochastic partial differential equations with delay. *Stochastics Stochastics Rep.* **33** (1-2), 27-47.

[8] Caraballo, T.; Garrido-Atienza, M.J.; Real, J. (2003). Asymptotic stability of nonlinear stochastic evolution equations. *Stochastic Anal. Appl.*, **21** (2), 310-327.

[9] Caraballo, T.; Garrido-Atienza, M.J.; Schmalfuss, B. (2007). Existence of exponentially attracting stationary solutions for delay evolution equations. *Discrete. Cont. Dyn. Syst. A* **18** (2-3), 271-293.

[10] Chang, S.; Li, S.; Chiang, M.; Hu, S.; Hsyu, M. (2007). Fractal dimension estimation via spectral distribution function and its application to physiological signals. *IEEE Trans. Biol. Engineering* **54** (10), 1895-1898.

[11] Cheridito, P.; Nualart, D. (2005). Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter H in (0,1/2). *Ann. I. H. Poincare* **41**, 1049-1081.

[12] Chueshov, I.; Schmalfuß, B. (2007). Qualitative behavior of a class of stochastic parabolic PDEs with dynamical boundary conditions. *Discrete Contin. Dyn. Syst.*, **18** (2-3): 315–338.

[13] Ciotir, O.; Rascanu, A. (2008). *Viability for stochastic differential equations driven by fractional Brownian motion*, preprint on arxiv arXiv:0808.3997.

[14] Coeurjolly, J.-P. (2001). Estimating the parameters of a fractional Brownian motion by discrete variations of its sample paths. *Stat. Inference for Stoch. Proc.* **4** , 199-227.

[15] Craigmile, P.F. (2003). Simulating a class of stationary Gaussian processes using the Davies-Harte algorithm, with application to long memory processes. *J. Time Series Anal.* **24**, 505-511.

[16] Decreusefond, L.; Üstünel A.-S. (1998). Stochastic analysis of the fractional Brownian motion. *Potential Analysis*, **10**, 177-214.

[17] Duan, J.; Nadiga, B. (2007). Stochastic parameterization of large eddy simulation of geophysical flows. *Proc. American Math. Soc.* **135**, 1187-1196.

[18] Duncan, T. E.; Hu, Y. Z.; Pasik-Duncan, B. (2000). Stochastic Calculus for Fractional Brownian Motion. I: Theory. *SIAM Journal on Control and Optimization* **38**, 582-612.

[19] Garrido-Atienza, M. J.; Maslowski, B.; Schmalfuß, B. (2008). Random attractors for ordinary stochastic equations driven by a fractional Brownian motion with Hurst parameter greater than 1/2. Submitted.

[20] Garrido-Atienza, M. J.; Lu, K.; Schmalfuß, B. Unstable invariant manifolds for stochastic PDEs driven by a fractional Brownian motion. In preparation.

[21] Giraitis, L., Leipus, R., Robinson, P.M., and Surgailis, D. (2004). LARCH, leverage, and long memory. *Journal of Financial Econometrics*, **2** (2), 177-210.

[22] Grigoriu, M. (2007). Linear systems with fractional Brownian motion and Gaussian noise. *Probabilistic Engineering Mechanics*, **22** (3), 276-284.

[23] Gubinelli, M. (2008). Abstract integration, Combinatorics of Trees and Differential Equations. Preprint.

[24] Hairer, M. (2005). Ergodicity of stochastic differential equations driven by fractional Brownian motion. *Ann. Probab.* **33** (2), 703-758.

[25] Hairer, M.; Ohashi, A. (2007). Ergodic theory for SDEs with extrinsic memory. *Ann. Probab.* **35** (5), 1950-1977.

[26] Hasselmann, K. (1976). Stochastic climate models: Part I. Theory. *Tellus*, **28** (1976), 473-485.

[27] Hu, Y. (2005). Integral transformations and anticipative calculus for fractional Brownian motions. *Mem. Amer. Math. Soc.* **175**, no. 825.

[28] Hu, Y. (2007). and D. Nualart: Differential equations driven by Hölder continuous functions of order greater than $1/2$. in *Stochastic analysis and applications*, 399–413, Abel Symp., 2, Springer.

[29] Hu, Y.-Z., Nualart, D. (2008). Rough path analysis via fractional calculus. To appear in *Transaction of American Mathematical Society*.

[30] Jumarie, G. (2002). Would dynamic systems involving human factors be necessarily of fractal nature? *Kybernetes*, **31** (7-8) 1050-1058.

[31] Kou, S.; Sunney-Xie, X. (2004). Generalized Langevin equation with fractional Gaussian noise: subdiffusion within a single protein molecule. *Phys. Rev. Lett.* **93** (18).

[32] Langa, J. A.; Robinson, J. C. (2006).Fractal dimension of a random invariant set. *J. Math. Pures Appl.*, **9** (2), 269–294.

[33] Langa, J. A.; Schmalfuss, B. (2004). Finite dimensionality of attractors for non-autonomous dynamical systems given by partial differential equations. *Stoch. Dyn.*, **4** (3), 385-404.

[34] Ledoux, M.; Qian, Z.; Zhang, T. (2002). Large deviations and support theorem for diffusion processes via rough paths. *Stochastic Process. Appl.* **102** (2), 265-283.

[35] Lian, Z; Lu, K. (2008). Lyapunov exponents and invariant manifolds for randon dynamical systems in Banach spaces, to appear in Memoirs of the American Mathematical Society

[36] Lisei, H. (2001).Conjugation of flows for stochastic and random functional differential equations. *Stochastics and Dynamics* **1** (2), 283-298.

[37] Lu, K.; Schmalfuß, B. (2007).Invariant manifolds for stochastic wave equations. *J. Differential Equations*, **236** (2), 460-492.

[38] Lyons, T. J. (1998). Differential equations driven by rough signals. *Rev. Mat. Iberoamericana* **14** (2), 215-310.

[39] Lyons, T.; Qian, Z. Flow of diffeomorphisms induced by a geometric multiplicative functional. *Probability Theory and Related Fields* **112** (1), 91-119 (1998)

[40] Lyons, T.; Qian, Z. (2002). *System control and rough paths*. Oxford Mathematical Monographs. Oxford University Press, Oxford.

[41] Mandelbrot, B. (1963). The variation of certain speculative prices. *Journal of Business* **XXXVI**, 392-417.

[42] Maslowski, B.; Schmalfuss, B. (2004). Random dynamical systems and stationary solutions of differential equations driven by the fractional Brownian motion. *Stochastic Anal. Appl.* **22** (6), 1577–1607.

[43] Mishura, Y. (2008). *Stochastic calculus for fractional Brownian motion and related processes*. Lecture Notes in Mathematics 1929.

[44] Mocioalca, O.; Viens, F. (2004). Skorohod integration and stochastic calculus beyond the fractional Brownian scale. *Journal of Functional Analysis*, **222** (2), 385-434.

[45] Maslowski, B.; Nualart, D. (2003). Evolution equations driven by a fractional Brownian motion. *Journal of Functional Analysis*, **202**, 277-305.

[46] Majda, A.M.; Timofeyev, I.; Vanden Eijnden, E. (1999). Models for stochastic climate prediction. *PNAS*, **96**, 14687-14691.

[47] Mehrabi, A.R.; Rassamdana, H.; Sahimi, M. (1997). Characterization of long-range correlation in complex distributions and profiles. *Physical Review E* **56**, 712.

[48] Millet, A.; Sanz-Solé, M. (2006). Large deviations for rough paths of the fractional Brownian motion. *Ann. Inst. H. Poincaré Probab. Statist.* **42** (2), 245-271.

[49] Neuenkirch, A.; Nourdin, I.; Rößler, A.; Tindel, S. (2006). Trees and asymptotic developments for fractional diffusion processes. *Ann. Inst. H. Poincaré Probab. Statist.*, to appear.

[50] Nourdin, I.; Nualart, D.; Tudor, C.A. (2007). Central and non-central limit theorems for weighted power variations of fractional Brownian motion. *Submitted*, 30 pages.

[51] Nualart, D. (2003). Stochastic integration with respect to fractional Brownian motion and applications. *Contemp. Math.* **336,** 3–39.

[52] Nualart, D. (2006). *Malliavin Calculus and Related Topics, 2nd ed.* Springer Verlag, Berlin.

[53] Nualart, D.; Rascanu, A. (2002). Differential equations driven by fractional Brownian motion. *Collectanea Mathematica* **53**, 55-81.

[54] Odde, D.; Tanaka, E.; Hawkins, S.; Buettner, H. (1996). Stochastic dynamics of the nerve growth cone and its microtubules during neurite outgrowth. *Biotechnology and Bioengineering* **50** (4), 452-461.

[55] Palmer, T. N.; Shutts, G. J.; Hagedorn, R.; Doblas-Reyes, F. J.; Jung, T.; Leutbecher, M. (2005). Representing model uncertainty in weather and climate prediction. *Annu. Rev. Earth Planet. Sci.* **33**, 163-193.

[56] Palmer, T. N. (2001). A nonlinear dynamical perspective on model error: A proposal for non-local stochastic-dynamic parameterization in weather and climate prediction models. *Q. J. Meteorological Soc.* **127** B, 279-304.

[57] Pipiras, V.; Taqqu, M.S. (2000). Integration questions related to fractional Brownian motion. *Probab. Theory Rel. Fields* **118,** 121–291.

[58] Pipiras, V.; Taqqu, M.S. (2001). Are classes of deterministic integrands for fractional Brownian motion on a interval complete? *Bernoulli* **7,** 873–897.

[59] Skorohod, A. V. (1975). On a generalization of a stochastic integral. *Theory Probab. Appl.* **20,** 219–233.

[60] Taniguchi, T.; Liu, K.; Truman. (2002). Existence, uniqueness, and asymptotic behavior of mild solutions to stochastic functional differential equations in Hilbert spaces. *Journal of Differential Equations* **181** (1), 72-91.

[61] Tindel, S.; Tudor, C.A.; Viens, F. (2003). Stochastic Evolution Equations with Fractional Brownian Motion. *Probability Theory and Related Fields* **127** (2), 186-204.

[62] Tudor, C.A.; Viens, F. (2008). Variations and estimators for the selfsimilarity order through Malliavin calculus. *Submitted*, 37 pages.

[63] Tudor, C.A.; Viens, F. (2007). Statistical aspects of the fractional stochastic calculus. *Annals of Statistics*, **35** (3), 1183-1212.

[64] Viens, F.; Zhang, T. (2008). Sharp Estimation of the Almost Sure Asymptotic Behavior for a Brownian Polymer in a Fractional Brownian Environment. To appear in *J. Functional Analysis*. 47 pages.

[65] Waymire, E.; Duan, J. (2005). *Probability and Partial Differential Equations in Modern Applied Mathematics.* Springer-Verlag.

[66] Willinger, W.; Taqqu, M.; Teverovsky, V. (1999). Long range dependence and stock returns. *Finance and Stochastics* **3**, 1-13.

[67] Willinger, W.; Taqqu, M.; Leland, W.E.; Wilson, D.V. (1995). Selfsimilarity in high speed packet traffic: analysis and modelisation of ethernet traffic measurements. *Statist. Sci.* **10**, 67-85.

[68] Zähle, M. (1999). On the link between fractional and stochastic calculus. *Stochastic dynamics (Bremen, 1997)*, 305–325, Springer.

[69] Zähle, M. (1998). Integration with respect to fractal functions and stochastic calculus. I *Probab. Theory Related Fields*, **111** (3), 333–374.