Topology in and Around Dimension Three

Steven Boyer (Université du Québec à Montréal),
Martin Scharlemann (University of California at Santa Barbara),
Abigail Thompson (University of California at Davis)

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1 Goals of the workshop

The study of 3-dimensional manifolds from a topological and geometrical point of view is a subject rich in technique, application, and connections with other areas of mathematics. Research is driven by various open problems of a foundational nature. The Poincaré conjecture and the more general geometrization conjecture of Thurston are the best known. Many individuals and groups have worked on these problems and their approaches have tended to be highly specific. Distinct sub-fields have arisen within 3-manifold theory distinguished not only by techniques and strategies, but also by the background mathematical culture needed to assimilate their methods. Thus, though at one level the goal of the workshop was to provide a forum for the examination of the current state of the field, at another, it was to bring together researchers from the various subfields to share their particular expertise with the other participants. Also included were a handful of researchers whose work focused on other areas, but had a relevance for 3-manifold topology.

2 An overview of the areas covered in the workshop

One of the main concerns of 3-manifold topology is the development of useful ways to describe them. For instance as geometric objects, as Dehn fillings, or by Heegaard splittings. This was reflected in the topics broached at the workshop.

2.1 Geometric structures on 3-manifolds

For over a quarter century, Thurston's vision of 3-manifolds as geometric objects has pervaded much of the thinking about these spaces. His geometrization conjecture [Th1], [Th2] provides an explicit description of the building blocks for compact 3-manifolds and, more generally, compact 3-orbifolds. Its verification has been a major focus of research and has spurred the enhancement of existing methods as well as the development new ones. Beyond that, the geometric viewpoint has led to the introduction of arithmetic, analytic, algebro-geometric, and other techniques into the subject (e.g. [MR], [P1,2,3], [CS1]).

The geometrization conjecture for manifolds contends that a compact 3-manifold admits a geometric decomposition. This means that it can be cut open along an essentially canonical family of surfaces of non-negative Euler characteristic in such a way that each of the remaining pieces admits a complete, locally homogeneous Riemannian metric. Thus the pieces admit geometric structures based on one of the eight 3-dimensional geometries [Sc] of which the spherical, Euclidean and hyperbolic geometries are the best known. To date, the conjecture has been verified in many cases and we can be quite specific in describing a connected, irreducible manifold for which it is not yet known. It is closed, connected, orientable, contains no essential surfaces, and its fundamental group does not contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. The conjecture contends that such a manifold is hyperbolic if it has an infinite fundamental group with trivial centre, otherwise it is the quotient of the 3-sphere by a finite group of isometries which acts freely.

Quite recently, G. Perelman has made important advances in theory of Ricci flow on 3-manifolds [P1], [P2], [P3] which he claims leads to a proof of Thurston's conjecture. This has yet to be verified, but it is clear that his methods will have a significant impact on 3-dimensional topology. At the workshop, Ian Agol showed how they could be used to improve estimates on the minimal volume problem for hyperbolic 3-manifolds. Many of the other talks were related to geometrization in one way or other, as will be detailed in the following sections.

2.2 Virtual properties of 3-manifolds

A 3-manifold is called *Haken* if it is irreducible and contains a 2-sided incompressible surface. Waldhausen's *virtual Haken conjecture* states that a compact, connected, orientable, irreducible 3-manifold M with infinite fundamental group is finitely covered by a Haken manifold. Stronger versions state that a finite cover of M with positive first Betti number can be found (the *virtually positive first Betti number conjecture*) and in most cases, the cover can be taken to have an arbitrarily large first Betti number (the *virtually infinite first Betti number conjecture*). It is known that a positive solution to Waldhausen's conjecture would imply that the geometrization conjecture holds for irreducible 3-manifolds except, perhaps, for those with a finite fundamental group (note that these possible exceptions include homotopy 3-spheres) [GMT], [Ga1], [Ga2], [CJ]. A positive solution to Waldhausen's conjecture would have other important consequences in 3-manifold topology as well: homotopy equivalences between manifolds to which the conjecture applies are homotopic to homeomorphisms, homotopic homeomorphisms of such manifolds are isotopic, the residual finiteness of the fundamental groups of 3-manifolds, etc.

These conjectures have received an increasing amount of attention in recent years and various approaches have been developed to examine them. This fact was reflected in the four talks at the workshop which dealt with them. Nathan Dunfield described his joint work with W. Thurston on random 3-manifolds. Given a (closed, connected, orientable) 3-manifold M and a finite (simple) group G, they obtain results on the probability that M admits a finite regular cover with group G, and the probability that there is such a cover with positive first Betti number (http://www.its.caltech.edu/ dunfield/). Marc Lackenby discussed the relationship between the virtual Haken conjecture and the growth rates of the Heegaard genus of the finite covers of a closed, connected, orientable 3-manifold [La1], [La2] and in particular has found has found sufficient conditions for a positive solution to the the conjectures. X. Zhang described arithmetic conditions on the trace fields of the representations $\pi_1(M) \to PSL_2(\mathbb{C})$ which are sufficient for a positive solution to the the conjectures [Zh]. Finally, Genevieve Walsh showed that a significant proportion of the manifolds obtained by Dehn surgery on a 2-bridge knot are virtually Haken [Wa].

2.3 Dehn filling

The problem of understanding the geometric decompositions of the Dehn fillings of a hyperbolic manifold M with a torus boundary has attracted a lot of attention over the last twenty years. Thurston developed a theory of hyperbolic Dehn filling [Th1], [Th2] which formed the basis of the proof of his orbifold theorem [BPL], [CHK]. Peter Shalen spoke in the workshop of the programme he and Marc Culler have developed to prove the Poincaré conjecture from a Dehn filling point of view (see e.g. [CS2]). The programme is based on certain Seifert fibred space recognition theorems (using $SL_2(\mathbb{C})$-character variety methods), complemented by a method for producing certain types of knots in homotopy 3-spheres.

Over the years, a fairly accurate picture of the finite set of *exceptional fillings* of M, i.e. those which yield non-hyperbolic manifolds, has emerged [Go], though a precise description has yet to be found, and there is a lot of work to be done on the problem of describing the topology of M when it admits more than one exceptional filling. Ying-Qing Wu spoke in the workshop on his recent work classifying the exceptional Dehn fillings of hyperbolic arborescent knot exteriors.

If the virtual Haken conjecture is true, then existing results [GLu], [BZ2] can be used to show that at most five Dehn fillings of a compact, connected, orientable hyperbolic 3-manifold M with torus boundary can be non virtually Haken. To date, it is not even known whether such a manifold admits only finitely many non virtually Haken Dehn filling, though partial results are known [CL], [BZ1], [DT]. Genevieve Walsh's work mentioned in §2.2 is a contribution to this problem.

2.4 Heegaard structures on 3-manifolds and applications

Heegaard splittings of 3-manifolds are among the most natural decompositions one can hope for, yet they remain difficult to analyze and to work with. Part of the difficulty lies in the construction of interesting examples. "Interesting" is the key word here; examples of manifolds with Heegaard splittings are rife, it is the sorting-out of the examples and the distinguishing among them that is the problem. Despite the hazards, approaching central 3-manifold questions via Heegaard splittings continues to be attractive, first because the the decomposition into handlebodies seems so simple, and second because they arise naturally in many different instances. They also transparently connect the topology of the manifold with the algebra of its fundamental group.

Schultens and Kobayashi both gave talks on particular examples of splittings in which there is a discrepancy between the anticipated and the actual Heegaard genus. Schultens (in joint work with R. Weidmann) described the structure of Heegaard splittings of graph manifolds, and used this to produce examples in which the Heegaard genus is arbitrarily larger than the rank of the fundamental group of the manifold. In the previously known examples of this phenomena the difference was one. In the other direction, Kobayashi examined the asymptotic behavior of the tunnel number of a knot in the 3-sphere under the connect-sum operation, and proved that it "usually" grows more slowly than one would expect, that is, the Heegaard genus of the complement of the summed knots is smaller than one would expect. The simplest example of an application of his theorem is to tunnel number one knots which are not 1-bridge on an unknotted torus. These knots have also appeared crucially as examples to show that the tunnel number is super-additive under connect sum [MSY]; thus they appear to somehow raise the Heegaard genus artificially when a few are added together, and to lower it artificially in the limit.

Schleimer discussed properties related to the distance of a Heegaard splitting as defined by Hempel [He]. Reducible Heegaard splittings have distance zero; weakly reducible splittings have distance at most one, and splittings of toroidal manifolds are known to have distance at most two. Thus examining properties of manifolds with Heegaard splittings with distance at most three is closely related to the Geometrization Conjecture; Schleimer discussed the relation between the distance of a Heegaard splitting for a manifold and its possible geometric structure. A more classical idea is to try to determine whether a 3-manifold is simply-connected by discerning properties of a Heegaard splitting. In this spirit Rolfsen showed that a Heegaard diagram for a homology 3-sphere, interpreted as a collection of curves on the boundary of a handlebody, can be imbedded in the 3-sphere in such a way that each curve of the diagram bounds a surface with interior disjoint from the handlebody. Lackenby and Dunfield's work is discussed elsewhere, but both dealt with general properties of 3-manifolds approached via Heegaard splittings. One could argue that Heegaard splittings can provide a unifying theme for approaching some of the central problems in 3-manifolds, including many of those those discussed at this conference.

In another direction, Rob Kirby described his work with Paul Melvin on finding a combinatorial description of the chain complexes which arise in Oszvath-Szabo Floer homology for 3-manifolds. This theory is based on a method, pioneered by Andrew Casson, which uses a Heegaard splitting to "coordinatize" a 3-manifold. The usefulness and power of this use of Heegaard splittings can be seen by the important applications of Casson's work [AM], and, more recently, that of Kronheimer, Wrowka, Oszvath and Szabo [KMOS], to our understanding of 3-manifolds and surgery theory.

2.5 Knots and braids

A background theme of the workshop was the emerging connections between questions in 3-manifold theory and related questions in dimensions 2 and 4. Several talks on the general subjects of knots and braids were illustrative of this connection.

A central problem of classical knot theory is to develop criteria that determine if two knots in 3-space are isotopic; that is, to determine whether or not there is a smooth level-preserving imbedding $S^1 \times I \to S^3 \times I$ which restricts to the two knots at each end. Expressed in this way, it appears to be almost a 4-dimensional question and, indeed, it becomes one if one simply drops the requirement that the embedding be level-preserving. So, two knots are *concordant* if there is an embedding $S^1 \times I \to S^3 \times I$ that restricts to each knot on each end.

This simple extension of knot isotopy gives rise to a group structure on classical knots, the so called knot concordance group, and understanding the group has been an important theme in geometric topology for some time, in higher dimension as well as dimensions 3 and 4, cf [Le]. Classical work of Casson and Gordon [CG] showed that the mere algebraic criteria that work in higher dimension fail in the classical dimension, and this has given rise to the question of exactly how complicated the classical knot cobordism groups can be.

In coordinated talks, Cochran and Teichner demonstrate that this concordance group is very, very complicated. Indeed, there is a geometric construction, called a *grope*, for producing such concordances, and an associated numerical level of complexity called the height of the grope. This gives a sort of filtration of the concordance group by the integers and they are able to show that each stage is non-trivial. Their methods are wide-ranging and deep, from hands-on Kirby calculus, to homological algebra over noncommutative rings, to von Neumann rho invariants of Cheeger-Gromov. In addition, there is a related version of the work which can be thought of as a completely 3-dimensional program.

Shelley Harvey described a related approach to a different collection of geometric problems, including the problems of determining whether a 3-manifold fibers over S^1, whether it is a Seifert fibered space, and whether its product with S^1 admits a symplectic structure. There are sequences of integral invariants r_n, d_n (the latter related to the size of the successive quotients of the derived series of the fundamental group) which are computable obstructions to the 3-manifold having the geometric structures just noted.

Harvey's question about symplectic structure on certain 4-manifolds points out another topic that interrelates topology in dimensions 3 and 4. A symplectic structure on a bounded 4-manifold gives rise to what is called a contact structure on the 3-manifold boundary. Both Etnyre and Menasco separately discussed connections between the standard contact structure on S^3 and classical knots in S^3. Etnyre described how to associate a Legendrian torus to a knot, which seems to give rise to useful invariants based on the contact homology of the torus, invariants related to earlier combinatorial invariants studied by Lenhard Ng [Ng1, Ng2]. Menasco described the construction of knots with the same transversal invariants in the standard contact structure on S^3 yet the knots are not transversally isotopic. The construction makes use of Menasco's joint work with Joan Birman on the Markov Theorem without stabilization.

Finally, in a wide-ranging talk, Stephen Bigelow discussed connections between braids and the representation theory of Iwahori-Hecke algebras.

2.6 Dimensions 2, 4 and other matters

A general relationship between 3-dimensional hyperbolic geometry, based on the Lie group $PSL_2(\mathbb{C})$, and certain aspects of topological quantum field theory, related to the quantum group $U_q(sl_2(\mathbb{C}))$, seems to be emerging, although still in a very imprecise way. At the workshop, Francis Bonahon discussed his joint work with Xiaobo Liu which gives an example of such a connection, describing quantum hyperbolic invariants of surface diffeomorphisms based on the Chekhov-Fock quantization of Teichmuller space. The main point was that finite dimensional representations of the Chekhov-Fock algebra (a completely algebraic object) are controlled by the same data as pleated surfaces in hyperbolic 3-manifolds.

Danny Ruberman talked about his joint work with Nikolai Saveliev on a topic in "3.5-dimensional" topology: the relation between Rohlin's invariant of 3-manifolds and 4-dimensional gauge theory [RS1], [RS2]. Taubes had shown how the Casson invariant for homology 3-spheres could be defined using gauge theory. For 4-manifolds with the $\mathbb{Z}[\mathbb{Z}]$-homology of $S^1 \times S^3$, a Casson-type invariant can also be defined via gauge theory, as well as a Rohlin invariant. Ruberman discussed the natural conjecture that these are the same, modulo 2, in analogy to the 3-dimensional case, and outlined a proof for the case when the manifold fibers over the circle with finite order monodromy. The conjecture has some interesting implications about the homology cobordism group and other classical problems.

Ian Hambleton discussed his joint work with Mihail Tanase on finite group actions on definite 4-manifolds [HT]. Equivariant Yang-Mills moduli space to investigate the relation between the singular set, isotropy representations at fixed points, and permutation modules realized by the induced action on homology for smooth group actions on certain 4-manifolds.

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