FIELD THEORY AND COHOMOLOGY OF GROUPS (RESEARCH IN TEAMS)

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Abstract. This is a report on the Research in Teams session "Field Theory and Cohomology of Groups" held at the BIRS April 26–May 10, 2003.

1. Final Report

Most of our time at the BIRS was spent investigating a conjecture which suggests that the Galois group of the maximal quadratic extension of a field has a strong regularity property.

Specifically, the conjecture states that if $1 \to R \to S \to G \to 1$ is a minimal presentation of $G = \mathrm{Gal}(F_q/F)$, where F_q is the maximal quadratic extension of F, then the lower-2-central series of S has a regular intersection with R, in the following sense.

Let $R^{(1,S)} = R$, and $R^{(n+1,S)}$ be the subgroup of $R^{(n,S)}$ generated by $[R^{(n,S)}]^2$ and $[S, R^{(n,S)}]$. Let $\langle S^{(n)} | n \geq 1\rangle$ be the lower-2-central series of S. Then we have

Conjecture. $R \cap S^{(n+1)} = R^{(n,S)}$.

It should be noted that this is a very special property, which is not true for just any group. For example, if G is described as the quotient $\mathbb{Z}/4\mathbb{Z}$, with $S = R = \mathbb{Z}$, then $R \cap S^{(3)} \neq R^{(2,S)}$.

This conjecture was motivated by attempts to find concrete interpretations in field theory for the Milnor conjecture [8] (now a theorem of V. Voevodsky [14]). This celebrated result gives (in the context described here) an isomorphism between the mod 2 Milnor K-theory $K_*^M(F, 2)$ and the mod 2 cohomology of G, $H^*(G)$. Many consequences of this isomorphism have since been derived, for example, enormous advances were made in the (Quillen) K-theory of rings of algebraic integers. However, as of this writing, no simple field-theoretic interpretation of the result is known.

The conjecture stated above would provide such an interpretation. Indeed, for small values of n, it has already been shown that the conjecture is equivalent to the isomorphism given by the Milnor conjecture in low degrees. Specifically, the equality $R \cap S^{(3)} = R^{(2,S)}$ is equivalent [9] to surjectivity in Merkurjev's theorem [5, 6] (the Milnor conjecture for $n = 2$). Also, (assuming the equalities for $n < 3$) the equality $R \cap S^{(4)} = R^{(3,S)}$ is equivalent to injectivity in the Milnor conjecture for $n = 3$. (This result was originally proved by Merkurjev, Suslin, and Rost [7, 11, 12].) Thus, there is strong evidence that the conjecture is in fact a group-theoretic interpretation of the Milnor conjecture.

We were able to test the conjecture in small degrees for many Galois groups $G = \mathrm{Gal}(F_q/F)$ using the computer algebra system MAGMA [1]. Needless to say, no counterexamples were found.

In addition, we were able to prove the conjecture in a variety of special cases, including Demuskin groups, the maximal pro-2-quotient of the absolute Galois group of $\mathbb{Q}$, and the pro-2-completions of fundamental groups of Riemann surfaces. The proofs for these cases rely on techniques developed by John Labute [3, 4]. Of course, the fundamental group of a Riemann surface need not be a Galois group of the type we consider. But the conjecture is a purely group-theoretic statement, and the fundamental group of a Riemann surface is sufficiently similar to the groups we consider to be an interesting case.

We also investigated the connection between our conjecture and various strengthenings of the Milnor conjecture. We considered the conjecture of Positselski and Vishik [10] and the conjectures of Carlsson [2]. Although we were unable to establish a concrete connection, these papers do suggest techniques which might be helpful in proving our conjecture.

In the short term, we plan to write a paper presenting the conjecture and giving two partial proofs: one for all our Galois groups in small degrees, and another in all degrees for some class of examples. Already these results provide some interesting consequences of Voevodsky's theorem and give strong restrictions on possible absolute Galois groups. In the long term, we plan to continue our investigation of the conjecture and its connections to other results in field theory.

It is a pleasure to thank the sponsors of the BIRS for the opportunity to advance our work in such a pleasant setting. In addition, we thank the staff of the BIRS (Andrea Lundquist, Brent Kearney, and Robert Moody), for their hospitality and unfailing good humor. We all look forward to returning to the BIRS in the near future.

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