Helical solutions for 3D incompressible Euler equations in an infinite cylinder

DAOMIN CAO(Inst. of Applied Math., CAS)

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Collaborators: Jie Wan (Beijing Institute of Technology)

Content of my talk

The motion of an ideal incompressible fluid(with unit density of mass) in a domain D without external force is described by the following Euler equations

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P, & (x,t) \in D \times (0,T), \\ \quad \nabla \cdot \mathbf{v} = 0, & (x,t) \in D \times (0,T), \\ \quad \mathbf{v} \cdot \mathbf{n} = 0, & \partial D \times (0,T), \end{cases} \tag{1}$$

where

v: the velocity, P: the pressure,

n: the outward unit normal to ∂D.

Defined vorticity vector of v by $\vec{\omega} = curl\mathbf{v} = \nabla \times \mathbf{v}$.

Taking the curl in the first equation of (1) we have the equation for vorticity

$$\partial_t \vec{\omega} + \mathbf{v} \cdot \nabla \vec{\omega} = (\vec{\omega} \cdot \nabla) v. \tag{2}$$

(2) was first studied by Helmholtz in 1858 and thus is called Helmholtz equation.

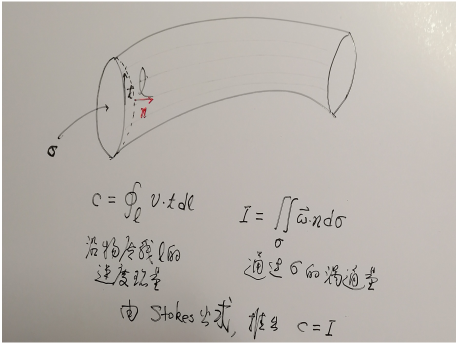
H. Helmholtz, On integrals of the hydrodynamics equations which express vortex motion, J. Reine Angew. Math., 55(1858), 25–55.

Helmholtz considered the vorticity equations of the flow and found that the existence of vortex rings, which are toroidal regions in which the vorticity has small cross-section, translate with a constant speed alone the axis of symmetry. The translating speed of vortex rings was then studied by Kelvin and Hick in 1899.

Define the circulation of a vortex

$$c = \oint_{\ell} \mathbf{v} \cdot \mathbf{t} dl = \iint_{\sigma} \vec{\omega} \cdot \mathbf{n} d\sigma, \tag{3}$$

where ℓ is any oriented curve with tangent vector field t that encircles the vorticity region once and σ is any surface with boundary ℓ.



Lamb showed that if the vortex ring has radius R, circulation c and its cross-section a is small, then the vortex ring moves at the velocity

$$\frac{c}{4\pi R}\left(\ln\frac{8R}{a}-\frac{1}{4}\right). \tag{4}$$

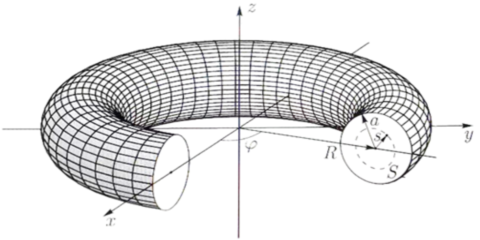
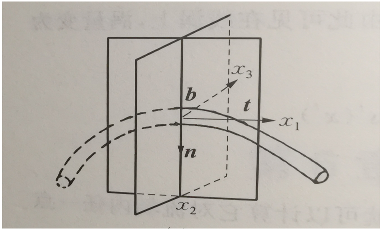


Fig. 1. A vortex ring of radius R with cross-section radius a.

Then L.S. Da Rios in 1906, in his doctoral thesis, showed that if one somehow knows that at some time the vorticity concentrates smoothly and symmetrically in a small tube around a smooth curve, then one can compute the instantaneous velocity of the curve to leading order. These computations suggest that the curve should evolve, after a possible rescaling in time, by an equation, known by various names, including the binormal curvature flow, the vortex filament equation, and the local induction approximation.

For the vortex filament with a small section of radius ε and a fixed circulation, uniformly distributed around an evolving curve $\Gamma(t)$, suppose that $\Gamma(t)$ is parameterized as $\gamma(s,t)$, where s is the parameter of arclength,



then $\gamma(s,t)$ asymptotically obeys a law of the form

$$\partial_t\gamma = \frac{c}{4\pi}|\ln\varepsilon|(\partial_s\gamma\times\partial_{ss}\gamma) = \frac{c\bar{K}}{4\pi}|\ln\varepsilon|\mathbf{b}_{\gamma(t)}, \tag{5}$$

c : **the circulation on the boundary of sections to the filament,**

$\mathbf{b}_{\gamma(t)}$: **the unit binormal,**

$\bar{K}$: **local curvature.**

If we rescale the time $t = |\ln \varepsilon|^{-1}\tau$, then

$$\partial_\tau \gamma = \frac{c\bar{K}}{4\pi} \mathbf{b}_{\gamma(\tau)}. \tag{6}$$

Therefore, the vortex filaments move simply in the binormal direction with speed proportional to the local curvature and the circulation.

When Γ is a circular filament, the leading term of (4) coincides with the coefficient of right hand side of (5) since in this case the local curvature $\bar{K} = \frac{1}{r^*}$.

R.L. Jerrard and C. Seis, On the vortex filament conjecture for Euler flows, Arch. Ration. Mech. Anal., 224(2017), 135–172.

Jerrard and Seis first gave a precise form to da Rios' computation with much weaker conditions.

Their result shows that under some conditions of a solution $\vec{\omega}_\varepsilon$ of (2), there holds in the sense of distribution,

$$\vec{\omega}_\varepsilon(\cdot, |\ln \varepsilon|^{-1}\tau) \to c\delta_{\gamma(\tau)}\mathbf{t}_{\gamma(\tau)}, \quad \textbf{as } \varepsilon \to 0, \tag{7}$$

where $\gamma(\tau)$ satisfies (6), $\mathbf{t}_{\gamma(\tau)}$ is the tangent unit vector of γ and $\delta_{\gamma(\tau)}$ is the uniform Dirac measure on the curve.

Up to now the existence of a family of solutions to (2) satisfying (7), where $\gamma(\tau)$ is a given curve evolved by the binormal flow (6), is still an open problem, called the vortex filament conjecture except for the several type of curves with special forms:

the straight lines, the traveling circles and the traveling-rotating helices.

The problem of vortex concentrating near straight lines, corresponds to the planar Euler equations concentrating near a collection of given points governed by the 2D point vortex model.

Two examples of curves of the binormal flow (6):

Example one

$$\gamma(s,\tau)=\left(r^*\cos\left(\frac{s}{r^*}\right),r^*\sin\left(\frac{s}{r^*}\right),\frac{c}{4\pi r^*}\tau\right)^T, \tag{8}$$

where $\mathbf{v}^T$ denotes the transposition of a vector $\mathbf{v}$.

It is a circle with radius r^* traveling along the x_3 axis with speed $\frac{c}{4\pi r^*}$

The second one of the binormal flow (6) that does not change its form in time is the rotating-translating helix, parameterized as

$$\gamma(s,\tau)=\left(r_*\cos\left(\frac{-s-a_1\tau}{\sqrt{k^2+r_*^2}}\right), r_*\sin\left(\frac{-s-a_1\tau}{\sqrt{k^2+r_*^2}}\right), \frac{ks-b_1\tau}{\sqrt{k^2+r_*^2}}\right)^T, \tag{9}$$
$$a_1=\frac{ck}{4\pi(k^2+r_*^2)},\ \ b_1=\frac{cr_*^2}{4\pi(k^2+r_*^2)}.$$

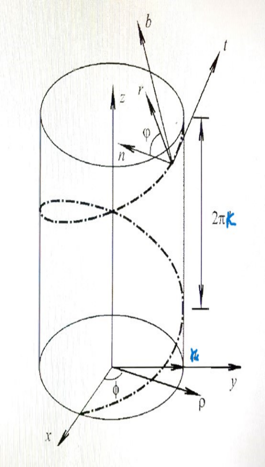
where

$r_*>0$ **is the distance between a point in** $\gamma(\tau)$ **and the** x_3**-axis,**

$2\pi k$ **is the pitch of the helix,**

local curvature: $\frac{r_*}{k^2+r_*^2}$

local torsion: $\frac{k}{k^2+r_*^2}$


b
t
z
r
φ
n
2πk
y
ρ
ϕ
x

It should be noted that the curve parameterized by (9) **is a rotating-traveling helix.**

This helix degenerates into the traveling circle if $k \to 0$ **and to a straight line when** $|k| \to \infty$.

Define for any $\theta \in [0, 2\pi]$

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad Q_\theta = \begin{pmatrix} R_\theta & 0 \\ 0 & 1 \end{pmatrix}.$$

Computing directly we can find

$$\gamma(s,\tau) = Q_{\frac{a_1\tau}{\sqrt{k^2+r_*^2}}}\gamma(s,0) + \left(0, 0, -\frac{b_1\tau}{\sqrt{k^2+r_*^2}}\right)^T.$$

When $k > 0$, (9) corresponds to the left-handed helix, and if $k < 0$ then (9) corresponds to the right-handed helix.

We will consider the case $k > 0$ only, the case $k < 0$ can been dealt with similarly.

Axi-symmetric Case – The vortex ring

L. E. Fraenkel, On steady vortex rings of small cross-section in an ideal fluid, Proc. R. Soc. Lond. A., 316(1970), 29–62.

Vortex rings with small cross-section without change of form concentrating near a traveling circle satisfying (8) in the sense of (7)

Elliptic equations for 2D and 3D axi-symmetric flows:

In terms of the Stokes stream function Ψ, the problem can be reduced to a free boundary problem on the half plane $\Pi = \{(r,z) \mid r > 0\}$ of the form:

$$
(P) \quad \begin{cases}
\mathcal{L}\Psi = 0 & \textbf{in } \Pi \backslash A, \\
\mathcal{L}\Psi = \lambda f(\Psi - q) & \textbf{in } A, \\
\Psi(0,z) = -\mu \leq 0, & \\
\Psi = 0 & \textbf{on } \partial A, \\
\frac{1}{r}\frac{\partial \Psi}{\partial r} \to -W \textbf{ and } \frac{1}{r}\frac{\partial \Psi}{\partial z} \to 0 \ \textbf{ as } \ r^2 + z^2 \to \infty, &
\end{cases}
$$

where

$$\mathcal{L} := -\frac{1}{r}\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big) - \frac{1}{r^2}\frac{\partial^2}{\partial z^2}, \quad \textbf{3D with axi-symmetry}$$

$$\mathcal{L} := -\Delta, \quad \textbf{in the case of 2D}$$

After then many articles on desingularization results: constructing vortex rings

under different conditions, on different kinds of domains, with different vortex profiles.

J.Norbury, [1972, Proc. Cambridge Philos.Soc.], A steady vortex ring close to Hill's spherical vortex .

J.Norbury,[1973,J. Fluid Mech.,], A family of steady vortex rings.

L.E. Fraenkel and M.S. Berger, A global theory of steady vortex rings in an ideal fluid, Acta Math., 132(1974), 13–51.

W-M.Ni [1980, J.Anal.Math.], Using Mountain Pass lemma, more general f

$$\begin{cases} -\Delta u = g(r^2 u - \frac{1}{2} W r^2 - k), \quad in\ \mathbb{R}^5 \\ u \to 0 \quad as\ |x| \to \infty, \end{cases}$$

where $\Delta = \sum_{i=1}^{5} \frac{\partial^2}{\partial^2 x_i}$**,** $r = \sqrt{x_1^2 + x_2^2 + x_3^3 + x_4^4}, z = x_5$**.**

A.Ambrosetti and Mancini [1981,Nonli.Anal.],

G.R. Burton, Rearrangements of functions, maximization of convex functionals, and vortex rings, Math. Ann., 276 (2)(1987), 225–253.

M.Struwe [1988, Acta. Math.],

A. Ambrosetti and M.Struwe [1989,ARMA],

A.Ambrosetti and J.Yang [1990, MMMAS]: f super-linear.

J.Yang, Global vortex rings and asymptotic behavior, Nonlinear Analysis,25(1995),531-546.

S. de Valeriola and J. Van Schaftingen, Desingularization of vortex rings and shallow water vortices by semilinear elliptic problem, Arch. Ration. Mech. Anal., 210(2)(2013), 409–450.

C–, J.Wan and W.Zhan, Desingularization of vortex rings in 3 dimensional Euler flow, JDE, 270(2021), 1258–1297.

C–, J.Wan, G.Wang and W.Zhan, Asymptotic behavior of global vortex rings, Nonlinearity, 35(2022),368–3705.

C–, G.Qin, W.Zhan and C.Zou, Remarks on orbital stability of steady vortex rings, Trans. Amer. Math. Soc., 376(2023), 3377–3395.

D.Chae and O.Imanuvilov, Existence of axisymmetric weak solutions of the 3D Euler equations, E. JDE,1998,

J.G.Liu and Z.Xin[CPAM(1995)],

Q.S.Jiu and Z.Xin[Acta Math.Sinica(2004);JDE(2006, 2007)],

V.V.Melshko, A.A.Gourjii and T.S.Krasnopolskaya[Vortex rings: History and state of the art, J.Math.Sciences, 187 (2012), 772 - 808].

Global well-posedness of solutions to the vorticity equation (2) with helical symmetry was studied in several papers.

A. Dutrifoy, Existence globale en temps de solutions hélicoïdales des équations d'Euler, C. R. Acad. Sci. Paris Sér. I Math., 329(1999), no. 7, 653–656.

B. Ettinger and E.S. Titi, Global existence and uniqueness of weak solutions of three-dimensional Euler equations with helical symmetry in the absence of vorticity stretching, SIAM J. Math. Anal., 41(2009), no. 1, 269–296.

Q. Jiu, J. Li and D. Niu, Global existence of weak solutions to the three-dimensional Euler equations with helical symmetry, J. Differential Equations, 262 (2017), no. 10, 5179–5205.

H. Abidi and S. Sakrani, Global well-posedness of helicoidal Euler equations, J. Funct. Anal., 271 (2016), no. 8, 2177–2214.

A.C. Bronzi, M.C. Lopes Filho and H.J. Nussenzveig Lopes, Global existence of a weak solution of the incompressible Euler equations with helical symmetry and L^p vorticity, Indiana Univ. Math. J., 64(2015), no. 1, 309–341.

For a helix $\gamma(\tau)$ satisfying (9), there are a few results of existence of true solutions of (2) concentrating on this curve in the sense of (7).

J. Dávila, M. del Pino, M. Musso and J. Wei, Travelling helices and the vortex filament conjecture in the incompressible Euler equations, Calc. Var. Partial Differential Equations. 61 (2022), art. 119.

the vorticity maybe not compactly supported.

Let $k > 0$. Define a one-parameter group of isometries of $\mathbb{R}^3$

$$\mathcal{G}_k = \{H_\theta \ \mid \ \mathbb{R}^3 \to \mathbb{R}^3,\ \theta \in \mathbb{R}\},$$

where the transformation H_θ is defined by

$$H_\theta \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \cos\theta + x_2 \sin\theta \\ -x_1 \sin\theta + x_2 \cos\theta \\ x_3 + k\theta \end{pmatrix}. \tag{10}$$

So H_θ is a superposition of a rotation around the $x_3 - axis$ and a translation along the $x_3 - axis$, that is,

$$H_\theta \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = Q_\theta \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ k\theta \end{pmatrix}.$$

Define a vector field

$$\overrightarrow{\zeta} = \begin{pmatrix} x_2 \\ -x_1 \\ k \end{pmatrix}.$$

Then $\overrightarrow{\zeta}$ is the field of tangents of symmetry lines of $\mathcal{G}_k$.

Helical function: A scalar function f is called a *helical* function if

$$f(H_\theta x) = f(x) \text{ for any } \theta \in \mathbb{R}, x \in \mathbb{R}^3.$$

Namely, f is invariant under the action of $\mathcal{G}_k$.

An equivalent definition:

A scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ is a helical function if and only if

$$f(x', x_3) = f(R_\theta x', x_3 + k\theta) \text{ for all } \theta \in \mathbb{R}, x = (x', x_3) \in \mathbb{R}^3. \tag{11}$$

For a scalar function f **satisfying** (11)**,**

$$f(x) = f(R_{-\frac{x_3}{k}} x', 0),$$

that is, f **is determined by its values on the horizontal plane**

$$\{x = (x', x_3) \ \mid x_3 = 0\}.$$

By direct computations it is easy to see that a C^1 **function** f **is helical if and only if**

$$\overrightarrow{\zeta} \cdot \nabla f = 0.$$

Helical vector field: A vector $\mathbf{h} = (h_1, h_2, h_3)$ is called helical if

$$\mathbf{h}(H_\theta x) = Q_\theta \mathbf{h}(x), \quad \text{for any } \theta \in \mathbb{R}, x \in \mathbb{R}^3.$$

Equivalently, $\mathbf{h} = (h_1, h_2, h_3)$ is helical if and only if

$$\mathbf{h}(x', x_3) = Q_{-\theta}\mathbf{h}(R_\theta x', x_3 + k\theta), \quad \text{for any } \theta \in \mathbb{R}, x \in \mathbb{R}^3. \tag{12}$$

If $\mathbf{h}$ satisfies (12), then

$$\mathbf{h}(x) = Q_{-\frac{x_3}{k}}\mathbf{h}(R_{-\frac{x_3}{k}}x', 0).$$

Therefore **h is determined by the values on the horizontal plane**

$$\{x = (x', x_3) \ \mid x_3 = 0\}.$$

Direct computation shows that a C^1 vector field h is helical if and only if

$$\overrightarrow{\zeta} \cdot \nabla \mathbf{h} = \mathcal{R}\mathbf{h}, \quad \textbf{where } \mathcal{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Equivalently by components it satisfies (12) **if and only if**

$$\nabla h_1 \cdot \overrightarrow{\zeta} = -h_2, \ \nabla h_2 \cdot \overrightarrow{\zeta} = h_1, \ \nabla h_3 \cdot \overrightarrow{\zeta} = 0.$$

Helical solutions: **A function pair $(\mathbf{v}, P)$ is called a** *helical solution* **of (1) in $B_{R^*}(0) \times \mathbb{R}$, if $(\mathbf{v}, P)$ satisfies (1) and both vector field $\mathbf{v}$ and scalar function P are helical.**

We will always assume that the helical solutions satisfy *Orthogonality condition :*

$$\mathbf{v} \cdot \overrightarrow{\zeta} = 0. \tag{13}$$

The role of orthogonality condition is similar to the no swirling condition for 3D axi-symmetric case(It is said that v has vanishing helical swirl).

Orthogonality + Helicity $\Rightarrow$ vanishing of vorticity stretching term

If v satisfies condition (13), then the corresponding vorticity field $\vec{\omega}$ of v satisfies (see lemma 2.11 in the paper of Ettinger-Titi)

$$\vec{\omega} = \frac{\omega}{k}\overrightarrow{\zeta}, \tag{14}$$

where $\omega = \omega_3 = \partial_{x_1} v_2 - \partial_{x_2} v_1$, the third component of vorticity field $\vec{\omega}$, is a helical function.

From (14) we know that $\vec{\omega}$ and v are orthogonal.

Moreover, (2) is equivalent to

$$\partial_t \vec{\omega} + (\mathbf{v} \cdot \nabla)\vec{\omega} + \frac{1}{k}\vec{\omega}\mathcal{R}\mathbf{v} = 0.$$

As a consequence, ω satisfies

$$\partial_t \omega + (\mathbf{v} \cdot \nabla)\omega = 0. \tag{15}$$

We will introduce a *stream function* and reduce the system (2) to a 2D vorticity-stream equation.

Since v is a helical vector field, we have $\overrightarrow{\zeta} \cdot \nabla \mathbf{v} = \mathcal{R}\mathbf{v}$**, which implies that**

$$x_2 \partial_{x_1} v_3 - x_1 \partial_{x_2} v_3 + k \partial_{x_3} v_3 = 0. \tag{16}$$

The orthogonal condition shows that

$$v_3 = -\frac{1}{k} x_2 v_1 + \frac{1}{k} x_1 v_2. \tag{17}$$

By $\nabla \cdot \mathbf{v} = 0$ **and (16),(17) we get**

$$\frac{1}{k^2} \partial_{x_1} [(k^2 + x_2^2) v_1 - x_1 x_2 v_2] + \frac{1}{k^2} \partial_{x_2} [(k^2 + x_1^2) v_2 - x_1 x_2 v_1] = 0,$$

that is $\nabla \cdot \hat{\mathbf{v}} = 0$**, where**

$$\hat{\mathbf{v}} = \frac{1}{k^2} \left((k^2 + x_2^2) v_1 - x_1 x_2 v_2, (k^2 + x_1^2) v_2 - x_1 x_2 v_1 \right).$$

Since $B_{R^*}(0)$ is simply-connected, from $\nabla \cdot \hat{\mathbf{v}} = 0$, correspondingly, we can find a stream function $\varphi : B_{R^*}(0) \to \mathbb{R}$

$$\partial_{x_2}\varphi = \frac{1}{k^2}[(k^2 + x_2^2)v_1 - x_1 x_2 v_2], \;\; \partial_{x_1}\varphi = -\frac{1}{k^2}[(k^2 + x_1^2)v_2 - x_1 x_2 v_1],$$

that is,

$$\begin{pmatrix} \partial_{x_1}\varphi \\ \partial_{x_2}\varphi \end{pmatrix} = -\frac{1}{k^2}\begin{pmatrix} -x_1 x_2 & k^2 + x_1^2 \\ -(k^2 + x_2^2) & x_1 x_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

or equivalently,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\frac{1}{k^2 + x_1^2 + x_2^2}\begin{pmatrix} x_1 x_2 & -k^2 - x_1^2 \\ k^2 + x_2^2 & -x_1 x_2 \end{pmatrix}\begin{pmatrix} \partial_{x_1}\varphi \\ \partial_{x_2}\varphi \end{pmatrix}. \qquad (18)$$

By the definition of ω and (18), we get

$$\mathcal{L}_H\varphi = \partial_{x_1} v_2 - \partial_{x_2} v_1 = \omega, \tag{19}$$

where

$$\mathcal{L}_{K_H}\varphi = -\textbf{div}(K_H(x_1, x_2)\nabla\varphi)$$

is a divergence type operator with the coefficient matrix

$$K_H(x_1, x_2) = \frac{1}{k^2 + x_1^2 + x_2^2}\begin{pmatrix} k^2 + x_2^2 & -x_1x_2 \\ -x_1x_2 & k^2 + x_1^2 \end{pmatrix}. \tag{20}$$

(1). K_H **is a positive definite matrix and** $(K_H(x))_{ij} \in C^\infty(\overline{B_{R^*}(0)})$ **for** $i, j = 1, 2$.

(2). $\mathcal{L}_{K_H}$ **is uniformly elliptic, namely,** $\lambda_1 = 1, \lambda_2 = \frac{k^2}{k^2+|x|^2}$ **are two eigenvalues of** K_H **which have positive lower and upper bounds.**

By the elliptic regularity theory, for any $q \in (1, +\infty)$ **one can define a continuous linear operator** $\mathcal{G}_{K_H} : L^q(B_{R^*}(0)) \to W^{2,q} \cap W_0^{1,q}(B_{R^*}(0))$ **such that** $u = \mathcal{G}_{K_H} f$ **satisfies**

$$\mathcal{L}_{K_H} u = f.$$

To sum up, by the notation introduced before, we need to solve the following 2D vorticity-stream equations in $B_{R^*}(0) \times \mathbb{R}$

$$\begin{cases} \partial_t \omega + \nabla^\perp \varphi \cdot \nabla \omega = 0, & \text{in } B_{R^*}(0), \\ \mathcal{L}_{K_H} \varphi = \omega, & \text{in } B_{R^*}(0), \\ \varphi = 0, & \text{on } \partial B_{R^*}(0), \end{cases} \tag{21}$$

where $\perp$ is given by $(a, b)^\perp = (b, -a)$.

For a solution pair (ω, φ) of (21), one can recover helical velocity field v. Indeed, we can use (18), (17) to obtain v_3 from v_1, v_2.

Boundary condition of φ: by the result of Ettinger and Titi(SIAM J.Math.Anal.2009) from $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial B_{R^*}(0) \times \mathbb{R}$, φ is a constant on $\partial B_{R^*}(0)$. Without loss of generality, we set $\varphi|_{\partial B_{R^*}(0)} = 0$.

Using $\mathcal{G}_{K_H}$, the first equation in (21) can be rewritten as the following vorticity equations

$$\partial_t \omega + \nabla^{\perp} \mathcal{G}_{K_H} \omega \cdot \nabla \omega = 0, \quad \textbf{in } B_{R^*}(0). \tag{22}$$

(21) **is still too hard to be dealt with!!**

Let α **be a constant. We look for rotating solutions to** (21)

$$\omega(x',t)=W(R_{-\alpha|\ln\varepsilon|t}(x')),\quad \varphi(x',t)=\Phi(R_{-\alpha|\ln\varepsilon|t}(x')), \tag{23}$$

where $x'=(x_1,x_2)\in B_{R^*}(0)$.

To solve (21) **we only need to find a pair** (W,Φ) **satisfying**

$$\begin{cases}\nabla W\cdot\nabla^{\perp}\left(\Phi-\frac{\alpha}{2}|x'|^2|\ln\varepsilon|\right)=0, & \textbf{in } B_{R^*}(0),\\ \mathcal{L}_{K_H}\Phi = W, & \textbf{in } B_{R^*}(0),\\ \Phi|_{\partial B_{R^*}(0)}=0. \end{cases} \tag{24}$$

So formally if for some function f_ε,

$$W = f_\varepsilon \left(\Phi - \frac{\alpha}{2} |x'|^2 |\ln \varepsilon| \right) \quad \textbf{in } B_{R^*}(0), \tag{25}$$

then the first equation in (24) automatically holds.

We will consider two different types of f_ε:

$$f_\varepsilon(t) = \frac{1}{\varepsilon^2}(t - \mu_\varepsilon)_+^p, \quad p > 1,$$

and

$$f_\varepsilon(t) = \frac{1}{\varepsilon^2}\mathbf{1}_{\{t-\mu_\varepsilon>0\}},$$

for some μ_ε.

Thus we only need to solve the second equation satisfying the boundary condition (the third equation).

In the sequel we write x' as $x = (x_1, x_2)$.

The stream function method : To look for solutions (stream function) Φ of a semilinear elliptic equations

$$\begin{cases} -\mathbf{div}\cdot(K_H(x)\nabla\Phi) = \frac{1}{\varepsilon^2}\left(\Phi - \left(\frac{\alpha}{2}|x|^2 + \beta\right)|\ln\varepsilon|\right)_+^p, & x \in B_{R^*}(0), \\ \Phi(x) = 0, & x \in \partial B_{R^*}(0), \end{cases} \tag{26}$$

where $p > 1$, α, β are chosen in the following way

$$\alpha = \frac{c}{4\pi k\sqrt{k^2 + r_*^2}}, \quad \beta = \frac{\alpha}{2}(3r_*^2 + 4k^2).$$

The vortex method : **To find solution W of**

$$\begin{cases} \nabla W \cdot \nabla^{\perp}\left(\mathcal{G}_{K_H} W - \frac{\alpha}{2}|x'|^2|\ln\varepsilon|\right) = 0, \\ W = f_{\varepsilon}\left(\mathcal{G}_{K_H} W - \frac{\alpha}{2}|x'|^2|\ln\varepsilon|\right), \end{cases} \tag{27}$$

where

$$f_{\varepsilon}(t) = \frac{1}{\varepsilon^2}\mathbf{1}_{\{t-\mu_{\varepsilon}>0\}},$$

for some μ_{ε}.

If we obtain a solution W of (27), then letting $\Phi = \mathcal{G}_{K_H} W$, we get a pair (W, Φ) satisfies (24).

We will consider the case that D is an infinite pipe in $\mathbb{R}^3$ whose section is a disc with radius R^*,

$$D = B_{R^*}(0) \times \mathbb{R} = \{(x_1, x_2, x_3) \mid (x_2, x_2) \in B_{R^*}(0), x_3 \in \mathbb{R}\}.$$

For two sets A, B, define $dist(A, B) = \min_{x \in A, y \in B} |x - y|$ the distance between sets A and B and $diam(A)$ the diameter of the set A.

Our first result is concerned with the desingularization of traveling-rotating helical vortices in D, whose support set has small cross-section ε and concentrates near a single left-handed helix (9) in the sense of (7).

C–,& Jie Wan, *Helical vortices with small cross-section for 3D incompressible Euler equation, J.Funct.Anal., 284(2023) 109836*

Theorem 3.1

Let $k > 0$, $c > 0$ and $r_* \in (0, R^*)$ be any given numbers. Let $\gamma(\tau)$ be the helix parameterized by equation (9). Then for any $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$, there exists a classical solution pair $(\mathbf{v}_\varepsilon, P_\varepsilon)(x, t) \in C^1(D \times \mathbb{R}^+)$ of (1) such that the support set of $\vec{\omega}_\varepsilon$ is a topological traveling-rotating helical tube that does not change form and for all τ, in the sense of distribution

$$\vec{\omega}_\varepsilon(\cdot, |\ln \varepsilon|^{-1}\tau) \to c\delta_{\gamma(\tau)}\mathbf{t}_{\gamma(\tau)}, \quad as\ \varepsilon \to 0.$$

Moreover, there are $R_1, R_2 > 0$ such that

$$R_1\varepsilon \leq diam(supp(\vec{\omega}_\varepsilon) \cap (\mathbb{R}^2 \times \{0\})) \leq R_2\varepsilon.$$

We can also construct multiple traveling-rotating helical vortices in $B_{R^*}(0) \times \mathbb{R}$ with polygonal symmetry.

Let us consider the curve $\gamma(\tau)$ parameterized by (9). For any integer m, define for $i = 1 \cdots, m$ the curves $\gamma_i(\tau)$ parameterized by

$$\gamma_i(s,\tau) = Q_{\frac{2\pi(i-1)}{m}}\gamma(s,\tau). \tag{28}$$

Theorem 3.1 can be generalized to the helical vortices concentrating near multiple helices with polygonal symmetry.

Theorem 3.2

Let $k > 0$, $c > 0$ and $r_* \in (0, R^*)$ be any given numbers and $m \geq 2$ be an integer. Let $\gamma_i(\tau)$ be the helix parameterized by (28). Then for any $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$, there exists a classical solution pair $(\mathbf{v}_\varepsilon, P_\varepsilon)(x,t) \in C^1(D \times \mathbb{R}^+)$ of (1) such that the support set of $\vec{\omega}_\varepsilon$ is a collection of m topological traveling-rotating helical tubes that does not change form and for all τ,

$$\vec{\omega}_\varepsilon(\cdot, |\ln \varepsilon|^{-1}\tau) \to c \sum_{i=1}^{m} \delta_{\gamma_i(\tau)} \mathbf{t}_{\gamma_i(\tau)}, \quad as\ \varepsilon \to 0.$$

Moreover, there are $R_1, R_2 > 0$ such that

$$R_1 \varepsilon \leq diam\left(supp(\vec{\omega}_\varepsilon) \cap B_{\bar{\rho}}\left(Q_{\frac{2\pi(i-1)}{m}}(r_*, 0)\right) \times \{0\}\right) \leq R_2 \varepsilon.$$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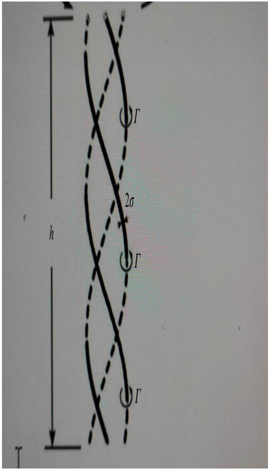
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Our third result is on the vortex patch type solutions. Take the nonlinearity $f_\varepsilon(t) = \frac{1}{\varepsilon^2}\mathbf{1}_{\{t-\mu_\varepsilon>0\}}$ **for some** μ_ε.

C–,& Jie Wan, *Structure of Green's function of elliptic equations and helical vortex patches for 3D incompressible Euler equation, Math Ann.,https://doi.org/10.1007/s00208-023-02589-8*

Theorem 3.3 (Existence of vortex patch type solutions)

Let $k > 0$, $c > 0$ *and* $r_* \in (0, R^*)$ *be three given numbers. Let* $\gamma(\tau)$ *be the helix parameterized by equation* (28). *Then for any* $\varepsilon \in (0, \min\{1, \sqrt{2\pi R^*/c}\})$, *there exists a solution pair* $(\mathbf{v}_\varepsilon, P_\varepsilon)(x, t)$ *of* (1) *such that the support set of* $\vec{\omega}_\varepsilon$ *is a topological traveling-rotating helical tube that does not change form and concentrates near* $\gamma(\tau)$,

(Theorem 3.3 Continued)

that is for all τ,

$$\vec{\omega}_\varepsilon(\cdot, |\ln \varepsilon|^{-1}\tau) \to c\delta_{\gamma(\tau)}\mathbf{t}_{\gamma(\tau)}, \quad \text{as } \varepsilon \to 0.$$

Moreover, the following properties hold:

i). Let $\omega_\varepsilon(x_1, x_2, t)$ *be the third component of* $\vec{\omega}_\varepsilon(x_1, x_2, 0, t)$. *Then*

$$\omega_\varepsilon = \frac{1}{\varepsilon^2}\mathbf{1}_{\{\mathcal{G}_{K_H}\omega_\varepsilon - \frac{\alpha|x|^2}{2}\ln\frac{1}{\varepsilon} - \mu_\varepsilon > 0\}},$$

where μ_ε *is a Lagrange multiplier.*

ii). Define $\bar{A}_\varepsilon = supp(\omega_\varepsilon)$ *the cross-section of* $\vec{\omega}_\varepsilon$. *Then there are* $r_1, r_2 > 0$ *such that*

$$r_1\varepsilon \leq diam(\bar{A}_\varepsilon) \leq r_2\varepsilon.$$

Before giving the orbital stability, we need to introduce some notation. Let $\mathcal{E}(\omega)$ and $\mathcal{I}(\omega)$ be the kinetic energy and the moment of inertia defined respectively by

$$\mathcal{E}(\omega) = \frac{1}{2}\int_{B_{R^*}(0)} \omega \mathcal{G}_{K_H}\omega dx, \tag{29}$$

$$\mathcal{I}(\omega) = \frac{1}{2}\int_{B_{R^*}(0)} |x|^2 \omega dx. \tag{30}$$

Define

$$\mathcal{E}_\varepsilon(\omega) = \mathcal{E}(\omega) - \alpha \ln\frac{1}{\varepsilon}\mathcal{I}(\omega) = \frac{1}{2}\int_{B_{R^*}(0)} \omega \mathcal{G}_{K_H}\omega dx - \frac{\alpha}{2}\ln\frac{1}{\varepsilon}\int_{B_{R^*}(0)} |x|^2 \omega dx.$$

Consider the maximization of $\mathcal{E}_\varepsilon(\omega)$ over the constraint set

$$\mathcal{M}_\varepsilon = \left\{ \omega \in L^\infty(B_{R^*}(0)) \mid \int_{B_{R^*}(0)} \omega dx = c, \ \textbf{and } 0 \le \omega \le \frac{1}{\varepsilon^2} \right\}.$$

Let us define the set of maximizers

$$\mathcal{S}_\varepsilon := \{\omega \in \mathcal{M}_\varepsilon \mid \mathcal{E}_\varepsilon(\omega) = \sup_{\mathcal{M}_\varepsilon} \mathcal{E}_\varepsilon\}. \tag{31}$$

$\mathcal{S}_\varepsilon$ **is not empty. Each element in** $\mathcal{S}_\varepsilon$ **is a rotation-invariant vortex patch to** (22) **and leads to a 3D Euler flow with helical symmetry.**

Consider the following initial problem

$$\begin{cases} \partial_t \omega + \nabla^\perp \mathcal{G}_{K_H}\omega \cdot \nabla\omega = 0, & B_{R^*}(0) \times (0, T), \\ \omega(\cdot, 0) = \omega_0(\cdot), & B_{R^*}(0). \end{cases} \tag{32}$$

Theorem 3.4 (Orbital stability)

Let $2 \leq q < +\infty$, $\varepsilon \in (0, \min\{1, \sqrt{|B_{R^*}(0)|/c}\})$, *and* $\mathcal{S}_\varepsilon$ *be defined by* (31). *Then* $\mathcal{S}_\varepsilon$ *is orbitally stable in* L^q *norm, or equivalently, for any* $\rho > 0$, *there exists a* $\delta > 0$, *such that for any* $\omega_0 \in L^q(B_{R^*}(0))$ *satisfying*

$$\inf_{\omega \in \mathcal{S}_\varepsilon} \|\omega_0 - \omega\|_{L^q(B_{R^*}(0))} < \delta,$$

we have

$$\inf_{\omega \in \mathcal{S}_\varepsilon} \|\omega_t - \omega\|_{L^q(B_{R^*}(0))} < \rho$$

for all $t > 0$, *where* ω_t *is a weak solution to the vorticity equation* (32) *with initial vorticity* ω_0.

Remark: **M. Benvenutti [NoDEA(2020)] obtained the nonlinear stability of smooth steady solutions to vorticity equation** (22) **under the assumption that**

$$0 \leq -\frac{\nabla \mathcal{G}_{K_H}\omega(x)}{\nabla \omega(x)} \leq C, \quad \forall\ x \in \Omega. \tag{33}$$

However, for many weak solutions to (22) **like vortex patches,** (33) **does not hold.**

In contrast to M. Benvenutti, we use the characterization of energy maximizers to get the orbital stability of vortex patches constructed in Theorem 3.3.

Whether these solutions are stability is still unknown.

J. Dávila, M. del Pino, M. Musso and J. Wei, Travelling helices and the vortex filament conjecture in the incompressible Euler equations, Calc. Var. &PDEs.,61(2022).art.119.

Constructed a family of Euler flows with helical symmetry in the whole $\mathbb{R}^3$ by reducing to the problem

$$-\mathbf{div}(K_H(x)\nabla u) = f_\varepsilon\left(u - \frac{\alpha}{2}|\ln\varepsilon||x|^2\right) \quad \textbf{in} \quad \mathbb{R}^2,$$

where $f_\varepsilon(t) = \varepsilon^2 e^t$ and α is chosen properly.

The vorticity concentrates near a helix and multiple in the distributional sense.

Note that by the choice of f_ε, the support set of vorticity maybe the whole $\mathbb{R}^3$, namely the support may not be near the given curve.

The main tool in analysis of aymptotic behaviour of vorticities is the Green's function.

For general positive definite matrix $K=(K_{i,j})_{2\times 2}$ and is a simply-connected bounded domain with smooth boundary $U\subset\mathbb{R}^2$, we first study the following Dirichlet problem:

$$\begin{cases}\mathcal{L}_K u := -\mathbf{div}(K(x)\nabla u) = f, & x\in U,\\ u=0, & x\in\partial U,\end{cases} \tag{34}$$

where $K=(K_{i,j})_{2\times 2}$ satisfies

(C1). $K_{i,j}(x)\in C^\infty(\overline{U})$ **for** $1\le i,j\le 2$.

(C2). $-\mathbf{div}(K(x)\nabla\cdot)$ **is uniformly elliptic, that is, there exist** $\Lambda_1,\Lambda_2>0$ **such that**

$$\Lambda_1|\zeta|^2\le (K(x)\zeta|\zeta)\le\Lambda_2|\zeta|^2,\qquad \forall\ x\in U,\ \zeta\in\mathbb{R}^2.$$

Since the coefficient matrix K satisfies $(C1)-(C2)$, one has the classical result:

Proposition 4.1

For every $q \in (1, +\infty)$, there exists a linear continuous operator $\mathcal{G}_K : L^q(U) \to W^{2,q}(U)$ such that for every $f \in L^q(U)$, the function $u = \mathcal{G}_K f$ is a weak solution of the problem (34).

By Cholsky decomposition there is C^∞ positive-definite matrix-valued function T_x determined by K satisfying

$$T_x^{-1}(T_x^{-1})^t = K(x) \quad \forall\ x \in U, \tag{35}$$

and such T_x exists and is unique.

Let $\Gamma(x) = -\frac{1}{2\pi}\ln|x|$ be the fundamental solution of the Laplacian $-\Delta$ in $\mathbb{R}^2$.

Proposition 4.2

Let $q > 2$. There exists a function $S_K \in C^{0,\gamma}_{loc}(U \times U)$ for some $\gamma \in (0,1)$ such that for every $f \in L^q(U)$ and every $x \in U$,

$$\mathcal{G}_K f(x) = \int_U \left[\frac{\left(\sqrt{\det K(x)}^{-1} + \sqrt{\det K(y)}^{-1}\right)}{2} \Gamma\left(\frac{T_x + T_y}{2}(x-y)\right) + S_K(x,y)\right] f(y)dy. \tag{36}$$

Moreover,

$$S_K(x,y) = S_K(y,x),\ \ S_K(x,y) \le C,\ \ \textit{for all}\, x, y \in U.$$

In particular, Proposition 4.2 implies that the Dirichlet problem of elliptic equation in divergence form (34) **has a Green's function** $G_K : U \times U \to \mathbb{R}$ **defined for each** $x, y \in U$ **with** $x \neq y$ **by**

$$G_K(x,y) = \frac{\sqrt{\det K(x)}^{-1} + \sqrt{\det K(y)}^{-1}}{2} \Gamma\left(\frac{T_x + T_y}{2}(x-y)\right) + S_K(x,y). \tag{37}$$

Main steps for the proof of Proposition 4.2.

Define

$$G_0(x,y) := \frac{\sqrt{\det K(x)}^{-1} + \sqrt{\det K(y)}^{-1}}{2} \Gamma\left(\frac{T_x + T_y}{2}(x-y)\right).$$

Denote $T_x = \begin{pmatrix} T_{11}(x) & T_{12}(x) \\ T_{21}(x) & T_{22}(x) \end{pmatrix}$. **and** $z = \frac{T_x+T_y}{2}(x-y)$.

Step 1: **We conclude that**

$$-\nabla_x \cdot (K(x)\nabla_x G_0(x,y)) = -\frac{\sqrt{\det K(x)}^{-1} + \sqrt{\det K(y)}^{-1}}{2} \Delta_z \Gamma(z) + F(x,y), \tag{38}$$

for some $F(\cdot,y) \in L^q(U)(1<q<2)$.

Step 2: (38) **implies that** $-\nabla_x \cdot (K(\cdot)\nabla_x G_0(\cdot, y)) = F(\cdot, y)$ **in any subdomain of** $U \setminus \{y\}$.

For fixed $y \in U$, **let** $S_K(\cdot, y) \in W^{1,2}(U)$ **be the unique weak solution to the following Dirichlet problem**

$$\begin{cases} -\nabla_x \cdot (K(x)\nabla_x S_K(x, y)) = -F(x, y) & \text{in } U, \\ S_K(x, y) = -G_0(x, y) & \text{on } \partial U. \end{cases} \tag{39}$$

Since K **is smooth and positive definite, by classical elliptic regularity estimates, we have** $S_K(\cdot, y) \in W^{2,q}(U)$ **for every** $1 < q < 2$.

Below we give examples of the matrix K in (34) to explain the expansion of Green's function in Proposition 4.2.

Example **1. If $K(x) = Id$, then (34) is the standard Laplacian problem, which corresponds to the vorticity-stream formulation to 2D Euler equations. By Proposition 4.2, the Green's function becomes**

$$G_1(x, y) = \Gamma(x - y) + S_1(x, y), \qquad \forall\ x, y \in U.$$

So in this case, $S_1(x, y) = -H(x, y)$, where $H(x, y)$ is the regular part of Green's function of $-\Delta$ in U with zero-Dirichlet data.

Example **2.** **If** $K(x) = \frac{1}{b(x)} Id$**, where** $b \in C^1(U)$ **and** $\inf_U b > 0$**, then** (34) **corresponds to vorticity-stream formulation to the 2D lake equations. It is not hard to get that** $\det K = \frac{1}{b^2}$ **and** $T = \sqrt{b} Id$**. From Proposition 4.2, the Green's function becomes**

$$G_b(x,y) = \frac{b(x)+b(y)}{2} \Gamma\left(\frac{\sqrt{b(x)}+\sqrt{b(y)}}{2}(x-y)\right) + S_b(x,y) \quad \forall\, x,y \in U,$$

which coincides with previous results.

In the sequel we will always take $U = B_{R^*}(0)$, $K = K_H$ and choose $\alpha > 0$ such that

$$\alpha = \frac{c}{4\pi k\sqrt{k^2 + r_*^2}}. \tag{40}$$

To prove Theorem 3.3, we use the vorticity method.

Consider the maximization of functional

$$\mathcal{E}_\varepsilon(\omega) := \frac{1}{2}\int_{B_{R^*}(0)} \omega \mathcal{G}_{K_H}\omega dx - \frac{\alpha}{2}\ln\frac{1}{\varepsilon}\int_{B_{R^*}(0)} |x|^2\omega dx. \tag{41}$$

over $\mathcal{M}_\varepsilon$ defined by

$$\mathcal{M}_\varepsilon = \left\{\omega \in L^\infty(B_{R^*}(0)) \ \Big|\ \int_{B_{R^*}(0)} \omega dx = c, \text{ and } 0 \le \omega \le \frac{1}{\varepsilon^2}\right\}.$$

Note: For any $\omega \in \mathcal{M}_\varepsilon$, by the classical elliptic estimate, we have $\mathcal{G}_{K_H}\omega \in W^{2,q}(B_{R^*}(0))$ for any $1 < q < +\infty$. Thus $\mathcal{E}_\varepsilon(\omega)$ is a well defined functional on $\mathcal{M}_\varepsilon$.

Lemma 4.3

There exists $\omega = \omega_\varepsilon \in \mathcal{M}_\varepsilon$ *such that*

$$\mathcal{E}_\varepsilon(\omega_\varepsilon) = \max_{\tilde{\omega}\in\mathcal{M}_\varepsilon} \mathcal{E}_\varepsilon(\tilde{\omega}) < +\infty.$$

Moreover,

$$\omega_\varepsilon = \frac{1}{\varepsilon^2}\mathbf{1}_{\{\psi^\varepsilon>0\}} \quad a.e.\ in\ \Omega, \tag{42}$$

where

$$\psi^\varepsilon = \mathcal{G}_{K_H}\omega_\varepsilon - \frac{\alpha|x|^2}{2}\ln\frac{1}{\varepsilon} - \mu^\varepsilon, \tag{43}$$

and the Lagrange multiplier $\mu^\varepsilon \geq -\frac{\alpha|R^*|^2}{2}\ln\frac{1}{\varepsilon}$ *is determined by* ω_ε*. Consequently,* ω_ε *is a weak solution to* (24) *with* $f_\varepsilon(t) = \frac{1}{\varepsilon^2}\mathbf{1}_{\{t>\mu^\varepsilon\}}$.

To show Theorem 3.3 we need to obtain

The limiting behavior of ω_ε as ε tends to 0.

Use C to denote generic positive constants independent of ε.

Define

$$Y(x) := \frac{c}{2\pi\sqrt{\det K_H(x)}} - \alpha|x|^2 = \frac{c\sqrt{k^2+|x|^2}}{2\pi k} - \alpha|x|^2, \qquad (44)$$

where α is chosen by (40). Clearly, Y is radially symmetric. Then one computes directly that

Lemma 4.4

Under the choice of α in (40), the maximizers set of Y in $B_{R^*}(0)$ is $\{x \mid |x| = r_*\}$. That is, $Y|_{\partial B_{r_*}(0)} = \max_{B_{R^*}(0)} Y$. Moreover, up to a rotation the maximizer is unique.

We will prove that, to maximize the energy $\mathcal{E}_\varepsilon$, the support set of ω_ε must shrink to a single point which is a maximizer of Y in $B_{R^*}(0)$ as ε tends to 0. Let

$$\bar{P}_\varepsilon = \inf\{|x| \mid x \in supp(\omega_\varepsilon)\}, \quad \text{and} \quad \bar{Q}_\varepsilon = \sup\{|x| \mid x \in supp(\omega_\varepsilon)\}. \quad (45)$$

$\bar{P}_\varepsilon$ and $\bar{Q}_\varepsilon$ describe the lower bound and upper bound of the distance between the origin and $supp(\omega_\varepsilon)$, respectively.

Lemma 4.5

$$\lim_{\varepsilon\to 0^+} \bar{P}_\varepsilon = r_*, \quad \lim_{\varepsilon\to 0^+} \bar{Q}_\varepsilon = r_*.$$

We can obtained the asymptotic behavior of ω_ε as follows.

Proposition 4.6

[Diameter and location of *supp*(ω_ε)] *For any $\gamma \in (0,1)$, there holds*

$$diam[supp(\omega_\varepsilon)] \leq 2\varepsilon^\gamma$$

provided ε is small enough. Moreover,

$$\lim_{\varepsilon \to 0^+} dist\left(supp(\omega_\varepsilon), \partial B_{r_*}(0)\right) = 0,$$

$$\lim_{\varepsilon \to 0^+} \frac{\ln diam\big(supp(\omega_\varepsilon)\big)}{\ln \varepsilon} = 1.$$

We can get the following optimal asymptotic expansions of the energy $\mathcal{E}_\varepsilon(\omega_\varepsilon)$ and Lagrange multiplier μ^ε.

Lemma 4.7

As $\varepsilon \to 0^+$, there holds

$$\mathcal{E}_\varepsilon(\omega_\varepsilon) = \left(\frac{c^2}{4\pi\sqrt{\det K_H((r_*,0))}} - \frac{c\alpha r_*^2}{2} \right) \ln\frac{1}{\varepsilon} + O(1), \tag{46}$$

$$\mu^\varepsilon = \left(\frac{c}{2\pi\sqrt{\det K_H((r_*,0))}} - \frac{\alpha r_*^2}{2} \right) \ln\frac{1}{\varepsilon} + O(1). \tag{47}$$

Using Lemma 4.3–4.7 we can prove Theorem 3.3.

To prove Theorem 3.4, we need three preliminary lemmas first.

Using the energy characterization that any element in S_ε is a maximizer of $\mathcal{E}_\varepsilon$ in $\mathcal{M}_\varepsilon$, we can obtain the following compactness result.

Lemma 4.8

[Compactness] *Let $\{\omega_n\}$ be a maximizing sequence for $\mathcal{E}_\varepsilon$ in $\mathcal{M}_\varepsilon$, then up to a subsequence there exists $\omega^\varepsilon \in \mathcal{S}_\varepsilon$ such that as $n \to +\infty$, $\omega_n \to \omega^\varepsilon$ in $L^q(\Omega)$ for any $q \in [1, +\infty)$.*

Following Burton's idea, one can get the linear transport theory of 3D Euler flows with helical symmetry as follows.

Lemma 4.9

Let $\omega(x,t) \in L^\infty_{loc}(\mathbb{R}; L^q(\Omega))$ *with* $2 \le q < +\infty$. $\zeta_0 \in L^q(\Omega)$. *Then there exists a weak solution* $\zeta(x,t) \in L^\infty_{loc}(\mathbb{R}; L^q(\Omega)) \cap C(\mathbb{R}; L^q(\Omega))$ *to the following linear transport equation*

$$\begin{cases} \partial_t \zeta + \nabla^\perp \mathcal{G}_{K_H}\omega \cdot \nabla \zeta = 0, \quad t \in \mathbb{R}, \\ \zeta(\cdot, 0) = \zeta_0. \end{cases}$$

Here by weak solution we mean for all $\phi \in C_c^\infty(D \times \mathbb{R})$,

$$\int_{\mathbb{R}} \int_D \partial_t \phi(x,t)\zeta(x,t) + \zeta(x,t)(\nabla^\perp \mathcal{G}_{K_H}\omega \cdot \nabla \phi)(x,t)dxdt = 0.$$

(Lemma 4.9 continued)

$$\lim_{t\to 0}\|\zeta(\cdot,t)-\zeta_0\|_{L^q(D)}=0.$$

Moreover, we have for any $t\in\mathbb{R}$

$$|\{x\in D\mid \zeta(x,t)>a\}|=|\{x\in D\mid \zeta_0(x)>a\}|,\ \forall\ a\in\mathbb{R}.$$

As a consequence, we have for any $t\in\mathbb{R}$

$$\|\zeta(\cdot,t)\|_{L^q(D)}=\|\zeta_0\|_{L^q(D)}.$$

Using the idea of M. Benvenutti[Nonlinear stability for stationary helical vortices, NoDEA Nonl. Diff. Equat. Appl., 27 (2020), no. 2, Paper No. 15, 20 pp.], we can get the energy and angular momentum conservation of solutions ω to the vorticity equation (21).

Lemma 4.10

Let $2 \leq q < \infty$. Let $\omega(t,x) \in L^\infty(\mathbb{R}; L^q(\Omega))$ be a solution of the vorticity equation (21). Then the kinetic energy $\mathcal{E}$ defined by (29) and the angular momentum $\mathcal{I}$ defined by (30) are conserved along the time.

Outline for the proof of Theorem 3.1.

We use the so-called stream function method.

To prove Theorem 3.1 by finding solutions of the equation satisfied by stream function Φ

$$\begin{cases} -\mathbf{div}\cdot(K_H(x)\nabla\Phi) = \frac{1}{\varepsilon^2}\left(\Phi - \left(\frac{\alpha}{2}|x|^2+\beta\right)|\ln\varepsilon|\right)_+^p, & x\in B_{R^*}(0), \\ \Phi(x)=0, & x\in\partial B_{R^*}(0), \end{cases}$$

where $p>1$, α,β are given by

$$\alpha = \frac{c}{4\pi k\sqrt{k^2+r_*^2}}, \quad \beta = \frac{\alpha}{2}(3r_*^2+4k^2).$$

We change the parameter to simplify notation.

Let $v = \frac{\Phi}{|\ln \varepsilon|}$ **and** $\delta = \varepsilon |\ln \varepsilon|^{-\frac{p-1}{2}}$, $q(x) = \frac{\alpha |x|^2}{2} + \beta$, **then**

$$\begin{cases} -\delta^2 \mathbf{div}(K_H(x)\nabla v) = (v - q)_+^p, & x \in B_{R^*}(0), \\ v = 0, & x \in \partial B_{R^*}(0). \end{cases} \tag{48}$$

We will choose α, β **so that**

$$\min_{x \in B_{R^*}(0)} \frac{\alpha |x|^2}{2} + \beta > 0.$$

Let $h(r) = h(|x|) = q^2 \sqrt{det(K_H)}(x)$ **for any** $x \in B_{R^*}(0)$. **We call** z^* **is a** *strict local maximum (minimum) point* **of** $q^2 \sqrt{det(K_H)}$ *up to ratation* **in** $B_{R^*}(0)$, **if** $|z^*|$ **is a strict local maximum (minimum) point of** h **in** $(0, R^*)$.

By choosing α, β properly Theorem 3.1 can be deduced from:

Theorem 4.11

Let α, β be two constants satisfying $\min_{x \in B_{R^*}(0)} \left(\frac{\alpha|x|^2}{2} + \beta\right) > 0$ and $z_1 \in B_{R^*}(0)$ be a strict local maximum (minimum) point of $\left(\frac{\alpha|x|^2}{2} + \beta\right)^2 \cdot \frac{k}{\sqrt{k^2+|x|^2}}$ up to rotation. Then there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0]$, (48) has a solution u_ε satisfying the following properties:

1. *Define $A_\varepsilon = \left\{u_\varepsilon > \left(\frac{\alpha|x|^2}{2} + \beta\right) \ln \frac{1}{\varepsilon}\right\}$. Then there exist $R_1, R_2 > 0$ such that $R_1\varepsilon \leq diam(A_\varepsilon) \leq R_2\varepsilon$, and*

$$\lim_{\varepsilon \to 0} dist(A_\varepsilon, z_1) = 0.$$

2. $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left(u_\varepsilon - \left(\frac{\alpha|x|^2}{2} + \beta\right) \ln \frac{1}{\varepsilon}\right)_+^p dx = \frac{k\pi(\alpha|z_1|^2+2\beta)}{\sqrt{k^2+|z_1|^2}}.$

To prove Theorem 3.1, we define for every $r_* \in (0, R^*)$, $c > 0$ **and**

$$\alpha = \frac{c}{4\pi k\sqrt{k^2 + r_*^2}}, \quad \beta = \frac{\alpha}{2}(3r_*^2 + 4k^2). \tag{49}$$

One computes directly that $(r_*, 0)$ **is a strict minimum point of** $q^2\sqrt{det(K_H)}(x) = \left(\frac{\alpha|x|^2}{2} + \beta\right)^2 \cdot \frac{k}{\sqrt{k^2+|x|^2}}$ **up to rotation and that**

$$2\pi q\sqrt{det(K_H)}((r_*, 0)) = \frac{k\pi(\alpha r_*^2 + 2\beta)}{\sqrt{k^2 + r_*^2}} = c.$$

Main idea of proof for Theorem 4.11.

Fix $R \geq 3R^*$. **For any** $a > 0$, **consider**

$$\begin{cases} -\delta^2 \Delta w = (w-a)_+^p, & \textbf{in } B_R(0), \\ w = 0, & \textbf{on } \partial B_R(0). \end{cases} \tag{50}$$

(50) **has a unique** C^1 **positive solution**

$$W_{\delta,a}(x) = \begin{cases} a + \delta^{\frac{2}{p-1}} s_\delta^{-\frac{2}{p-1}} \phi\left(\frac{|x|}{s_\delta}\right), & |x| \leq s_\delta, \\ a \ln \frac{|x|}{R} / \ln \frac{s_\delta}{R}, & s_\delta \leq |x| \leq R, \end{cases}$$

where $\phi \in H_0^1(B_1(0))$ **satisfies**

$$-\Delta \phi = \phi^p, \quad \phi > 0 \ \ \textbf{in } B_1(0),$$

if s_δ **satisfies the relation**

$$\delta^{\frac{2}{p-1}} s_\delta^{-\frac{2}{p-1}} \phi'(1) = a / \ln \frac{s_\delta}{R}. \tag{51}$$

Indeed (51) **is uniquely solvable if** $\delta > 0$ **is sufficiently small.**

Furthermore

$$\frac{s_\delta}{\delta|\ln\delta|^{\frac{p-1}{2}}} \to \left(\frac{|\phi'(1)|}{a}\right)^{\frac{p-1}{2}} \quad \textbf{as } \delta \to 0.$$

The Pohozaev identity implies

$$\int_{B_1(0)} \phi^{p+1} = \frac{\pi(p+1)}{2}|\phi'(1)|^2, \quad \int_{B_1(0)} \phi^p = 2\pi|\phi'(1)|. \qquad (52)$$

since K is a C^∞ positive definite matrix with all eigenvalues having uniformly positive lower and upper bounds, by the Cholesky decomposition one can find a matrix-valued function $F \in C^\infty(\overline{B_{R^*}(0)})$ such that for any $x \in B_{R^*}(0)$, $F(x)$ is invertible and

$$(F(x)^{-1})(F(x)^{-1})^t = K(x). \tag{53}$$

For simplicity, we denote $F_x = F(x)$. Since $R \geq 3R^*$ large enough, $B_{R^*}(0) \subseteq F_x^{-1}(B_R(0)) + x$ for any $x \in B_{R^*}(0)$. Clearly by the positive definiteness of K, such R exists.

Now for any $\hat{x} \in B_{R^*}(0), \hat{q} > 0$, let $V_{\delta,\hat{x},\hat{q}}$ be a C^1 positive solution of the following equations

$$\begin{cases} -\delta^2 \mathbf{div}(K(\hat{x})\nabla v) = (v - \hat{q})_+^p, & \text{in } F_{\hat{x}}^{-1}(B_R(0)), \\ v = 0, & \text{on } \partial F_{\hat{x}}^{-1}(B_R(0)). \end{cases} \tag{54}$$

Thus one has $V_{\delta,\hat{x},\hat{q}}(x) = W_{\delta,\hat{q}}(F_{\hat{x}}x)$.

Clearly $V_{\delta,\hat{x},\hat{q}}$ has an explicit profile

$$V_{\delta,\hat{x},\hat{q}}(x) = \begin{cases} \hat{q} + \delta^{\frac{2}{p-1}} s_\delta^{-\frac{2}{p-1}} \phi\left(\frac{|F_{\hat{x}}x|}{s_\delta}\right), & |F_{\hat{x}}x| \le s_\delta, \\ \hat{q} \ln \frac{|F_{\hat{x}}x|}{R} / \ln \frac{s_\delta}{R}, & s_\delta \le |F_{\hat{x}}x| \le R. \end{cases}$$

For any $z \in B_{R^*}(0)$ define

$$V_{\delta,\hat{x},\hat{q},z}(x) := V_{\delta,\hat{x},\hat{q}}(x-z), \quad \forall x \in B_{R^*}(0).$$

Since $V_{\delta,\hat{x},\hat{q},z}$ **is not 0 on** $\partial B_{R^*}(0)$**, we need to make a projection of** $V_{\delta,\hat{x},\hat{q},z}$ **on** $H_0^1(B_{R^*}(0))$**. Let** $PV_{\delta,\hat{x},\hat{q},z}$ **be a solution of**

$$\begin{cases} -\delta^2 \mathbf{div}(K(\hat{x})\nabla v) = (V_{\delta,\hat{x},\hat{q},z} - \hat{q})_+^p, & \textbf{in } B_{R^*}(0), \\ v = 0, & \textbf{on } \partial B_{R^*}(0). \end{cases} \tag{55}$$

We claim that for δ **sufficiently small,**

$$PV_{\delta,\hat{x},\hat{q},z}(x) = V_{\delta,\hat{x},\hat{q},z}(x) - \frac{\hat{q}}{\ln \frac{R}{s_\delta}} g_{\hat{x}}(F_{\hat{x}}x, F_{\hat{x}}z), \quad \forall x \in B_{R^*}(0), \tag{56}$$

where $g_{\hat{x}}(x,y) = 2\pi h_{\hat{x}}(x,y) + \ln R$ **for any** $x, y \in F_{\hat{x}}(B_{R^*}(0))$**, and** $h_{\hat{x}}(x,y)$ **is the regular part of Green's function of** $-\Delta$ **on** $F_{\hat{x}}(B_{R^*}(0))$**,**

In the following, we will construct solutions of the form

$$PV_{\delta,\hat{x},\hat{q},z} + r_{\delta,z}$$

where $PV_{\delta,\hat{x},\hat{q},z}$ is the main part and $r_{\delta,z}$ is the small perturbation.

Suppose that z^* is a strict local maximum (minimum) point of $q^2\sqrt{det(K_H)}$, we choose z near z^*. We will let $\hat{x} = z$ and choose $\hat{q} = \hat{q}_{z,\delta}$ properly (depending on z) such that $PV_{\delta,z,\hat{q}_{z,\delta},z}$ is a better approximation of solution. By such choice, we can find $r_{\delta,z}$ such that $PV_{\delta,z,\hat{q}_{z,\delta},z} + r_{\delta,z}$ is a solution.

Note that the associated functional of (48) is

$$I_\delta(u) = \frac{\delta^2}{2}\int_{B_{R^*}(0)} (K(x)\nabla u|\nabla u) - \frac{1}{p+1}\int_{B_{R^*}(0)} (u-q)_+^{p+1}. \qquad (57)$$

Denote

$$P_\delta(Z) = I_\delta(V_{\delta,Z} + \omega_{\delta,Z}),$$

$P_\delta(Z)$ **is a** C^1 **function.**

There holds

$$I_\delta(V_{\delta,Z}) = \sum_{j=1}^{m} \frac{\pi\delta^2}{\ln\frac{R}{\varepsilon}} q^2(z_j)\sqrt{det(K(z_j))} + O\left(\frac{\delta^2 \ln|\ln\varepsilon|}{|\ln\varepsilon|^2}\right).$$

Equations on general domains

Consider

$$\begin{cases} -\varepsilon^2 \mathbf{div}(K(x)\nabla u) = (u - q|\ln\varepsilon|)_+^p, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{58}$$

where $\Omega \subset \mathbb{R}^2$ **is a simply-connected bounded domain with smooth boundary,** $\varepsilon \in (0,1)$ **and** $p > 1$. $K = (K_{i,j})_{2\times 2}$ **is a matrix satisfying**

($\mathcal{K}1$). $K = (K_{i,j})_{2\times 2}$ **is a positive definite and** $K_{i,j}(x) \in C^\infty(\overline{\Omega})$ **for** $1 \le i,j \le 2$.

($\mathcal{K}2$). $-\mathbf{div}(K(x)\nabla\cdot)$ **is a uniformly elliptic operator, that is, there exist** $\Lambda_1, \Lambda_2 > 0$ **such that**

$$\Lambda_1|\zeta|^2 \le (K(x)\zeta|\zeta) \le \Lambda_2|\zeta|^2, \qquad \forall\ x \in \Omega,\ \zeta \in \mathbb{R}^2.$$

$q(x)$ **is a function defined in** $\overline{\Omega}$ **satisfying**

(Q1). $q(x) \in C^\infty(\overline{\Omega})$ **and** $q(x) > 0$ **for any** $x \in \overline{\Omega}$.

Denote $det(K)$ the determinant of K.

Theorem 4.12

Let K satisfy $(\mathcal{K}1)$-$(\mathcal{K}2)$ and q satisfy $(Q1)$. Then, for any given m distinct strict local minimum (maximum) points $x_{0,j}(j = 1, \cdots, m)$ of $q^2\sqrt{det(K)}$ in Ω, there exists $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0]$, (58) has a solution u_ε. Moreover, the following properties hold

(i) Let $\bar{A}_{\varepsilon,i} = \left\{u_\varepsilon > q \ln \frac{1}{\varepsilon}\right\} \cap B_{\bar{\rho}}(x_{0,i})$, where $\bar{\rho}$ is small. Then there exist $(z_{1,\varepsilon}, \cdots, z_{m,\varepsilon})$ and $R_1, R_2 > 0$ independent of ε satisfying

$$\lim_{\varepsilon \to 0}(z_{1,\varepsilon}, \cdots, z_{m,\varepsilon}) = (x_{0,1}, \cdots, x_{0,m}), \;\; B_{R_1\varepsilon}(z_{i,\varepsilon}) \subseteq \bar{A}_{\varepsilon,i} \subseteq B_{R_2\varepsilon}(z_{i,\varepsilon}).$$

(ii) Denote $\kappa_i(u_\varepsilon) = \frac{1}{\varepsilon^2}\int_{B_{\bar{\rho}}(x_{0,i})}\left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)_+^p dx$. Then

$$\lim_{\varepsilon \to 0} \kappa_i(u_\varepsilon) = 2\pi q\sqrt{det(K)}(x_{0,i}).$$

Many Thanks