Dissipation anomaly and anomalous dissipation

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Navier-Stokes equations

The 3D incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u \\ \operatorname{div} u = 0, \end{cases} \tag{NSE}$$

Here $u(x,t)$, the velocity, and $p(x,t)$, the pressure, are unknowns; $\nu > 0$ is the kinematic viscosity.

The energy balance:

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = -2\nu \int_{t_0}^{t} \|\nabla u(s)\|_{L^2}^2.$$

Conservation of energy for the Euler equations ($\nu = 0$):

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = 0.$$

Anomalous dissipation

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p, \\ \operatorname{div} u = 0. \end{cases} \qquad \text{(Euler equation)}$$

1994 Constantin, E, and Titi:

$$u \in L^\infty(0,1; B^{1/3-}_{3,\infty}) \qquad \Longrightarrow \qquad \text{no Anomalous Dissipation}$$

2008 C, Constantin, Friedlander, and Shvydkoy.

$$\lim_{q \to \infty} \int_0^1 \lambda_q^{\frac{1}{3}} \|\Delta_q u\|_{L^3} \, dt = 0,$$

$\Longrightarrow$ no Anomalous Dissipation:

$$\|u(t)\|_{L^2}^2 = \|u(0)\|_{L^2}^2, \qquad t \in [0,1].$$

Here $\lambda_q = 2^q$.

- '49 Onsager: $\frac{1}{3}$-Hölder is the critical threshold for energy conservation for the 3D Euler equations.
- '93 Scheffer: Wild solutions in $L^2_{t,x}$.
- '94 Eyink: Energy conservation under a stronger assumption.
- '94 Constantin, E, and Titi: Energy conservation in $L^3_t B^{1/3+}_{3,\infty}$.
- '97 Shnirelman: Wild solutions in $L^\infty_t L^2_x$.
- '01 Duchon and Robert: Refinements for the energy conservation.
- '08 C, Constantin, Friedlander, and Shvydkoy: Energy conservation in $L^3_t B^{1/3}_{3,c_0}$
- '09 De Lellis and Śzekelyhidi: Wild solutions in $L^\infty_{t,x}$ - Convex integration I.
- 13,'14 De Lellis and Śzekelyhidi: Wild solutions in $L^\infty_t C^{\frac{1}{10}-}_x$ - Convex integration II.
- '15 Buckmaster (thesis), De Lellis, Isett (thesis), and Śzekelyhidi (independently): Wild solutions in $L^\infty_t C^{\frac{1}{5}-}_x$.
- '15 Buckmaster: Wild solutions in $C^{\frac{1}{3}-}$ for almost all t.
- '16 Buckmaster, De Lellis, and Śzekelyhidi: Wild solutions in $L^1_t C^{\frac{1}{3}-}_x$.
- '18 Isett: Wild solutions in $C^{\frac{1}{3}-}_{t,x}$ - resolution of Onsager's conjecture for the Euler equations.
- '19 Buckmaster and Vicol: Nonuniqueness of NSE solutions in $C_t L^{2+}_x$.

C, Constantin, Friedlander, and Shvydkoy (2008): Energy balance holds for solutions of the Euler equations such that

$$\lim_{q\to\infty}\int_0^1 \lambda_q^{\frac{1}{3}}\|\Delta_q u\|_{L^3}\,dt = 0.$$

Isett (2022): There exists a weak solution of the Euler equation $u(t)$ that does not satisfy the energy balance and

$$|u(x-\Delta x,t)-u(x,t)| \le C|\Delta x|^{\frac{1}{3}-B\sqrt{\frac{\log\log|\Delta x|^{-1}}{\log\Delta|x|^{-1}}}}.$$

Dissipation anomally

The dissipation anomaly, predicted by Kolmogorov's theory of turbulence, can mathematically be stated as

$$\limsup_{\nu \to 0} \nu \langle \|\nabla u^\nu\|_{L^2}^2 \rangle > 0, \tag{1}$$

This phenomenon is related to the anomalous dissipation, the failure of solutions to the Euler equation satisfy the energy balance.

Reynold's number

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}.$$

Characteristic velocity:

$$U := \left(\operatorname*{Lim}_{T \to \infty} \frac{1}{T} \int_0^T \|u(t)\|_{L^2}^2 \, dt \right)^{1/2}.$$

Change variables

$$\mathbf{x}_1 = \frac{\mathbf{x}}{\ell}, \qquad t_1 = \frac{tU}{\ell}, \qquad \mathbf{u}_1 = \frac{\mathbf{u}}{U}, \qquad p_1 = \frac{p}{U^2}, \qquad \mathbf{f}_1 = \frac{\mathbf{f}\ell}{U^2}.$$

Reynolds number:

$$Re = \frac{U\ell}{\nu}.$$

$$\frac{\partial \mathbf{u}_1}{\partial t_1} - \frac{1}{Re} \Delta \mathbf{u}_1 + (\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 + \nabla p_1 = \mathbf{f}_1.$$

Kolmogorov's dissipation anomaly hypothesis

$$\frac{\epsilon \ell}{U^3} = \mathcal{O}(Re^0) \quad \text{as} \quad Re \to \infty$$

ϵ = total energy dissipation rate per unit mass
ℓ = length scale in the flow
U = turbulent velocity scale
$Re = U\ell/\nu$ the Reynolds number

Rigorous estimates:

Square integrable force f:

$$\frac{\epsilon \ell}{U^3} \leq c_1 + c_2 Re^{-1}.$$

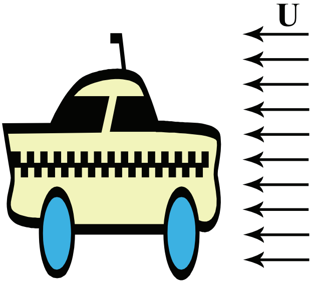
C. Foias (97), C. Doering and C. Foias (02)

Fractal force $f \in H^{-\alpha}$, $\alpha \in [0, 1]$:

$$\frac{\epsilon \ell}{U^3} \leq c_1 Re^{\frac{\alpha}{2-\alpha}} + c_2 Re^{-1}.$$

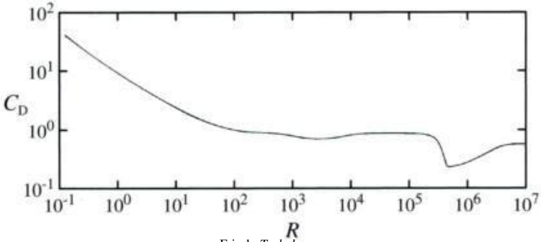
A. C., C. Doering, N. Petrov (06)

$$\epsilon := \operatorname*{Lim}_{T\to\infty} \frac{1}{T} \int_0^T \nu \|\nabla u(t)\|_L^2 \, dt.$$



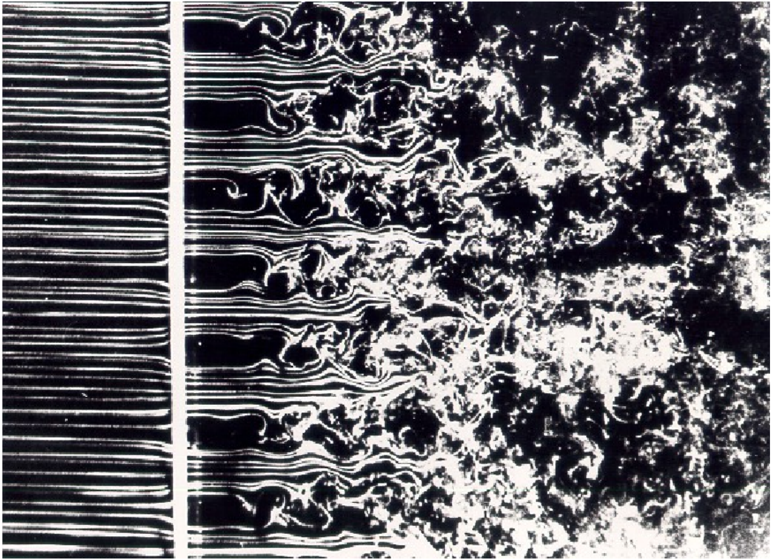
$$C_D := \frac{\epsilon \ell}{U^3} \qquad \text{drag coefficient}$$

Force is proportional to velocity squared for large Reynolds numbers.



Frisch, *Turbulence*.

Fractal forced turbulence



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Particle image velocimetry study of fractal-generated turbulence

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An experimental investigation involving space-filling fractal square grids is presented. The flow is documented using particle image velocimetry (PIV) in a water tunnel as opposed to previous experiments which mostly used hot-wire anemometry in wind tunnels. The experimental facility has non-negligible incoming free-stream turbulence

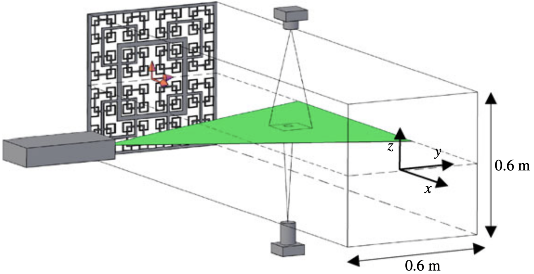

FIGURE 1. (Colour online) Experimental setup.

Upper bounds on the energy dissipation

$$f(x) = F\phi^{Re}(\ell^{-1}x),$$

$F \geq 0$ is the amplitude, ϕ^{Re} is the dimensionless shape.

$\phi^{Re} \to \phi$ in L^2 as $Re \to 0$: $$\frac{\epsilon\ell}{U^3} \leq c_1 + c_2 Re^{-1}.$$ C. Foias (97), C. Doering and C. Foias (02)	$\phi^{Re} \to \phi$ in $H^{-\alpha}$ as $Re \to 0$ ($\alpha \in [0,1]$): $$\frac{\epsilon\ell}{U^3} \leq c_1 Re^{\frac{\alpha}{2-\alpha}} + c_2 Re^{-1}.$$ A. C., C. Doering, N. Petrov (06)

$$\epsilon := \operatorname*{Lim}_{T\to\infty} \frac{1}{T}\int_0^T \nu \|\nabla u\|_{L^2}^2 \, dt,$$

$$U := \left(\operatorname*{Lim}_{T\to\infty} \frac{1}{T}\int_0^T \|u\|_{L^2}^2 \, dt\right)^{1/2}, \qquad Re = \frac{U\ell}{\nu}$$

Dissipation anomally

We consider the vanishing viscosity limit of solutions to the Navier-Stokes equations

$$\begin{cases} \partial_t u^\nu - \nu \Delta u^\nu + \operatorname{div}(u^\nu \otimes u^\nu) + \nabla p^\nu = f^\nu, \\ \operatorname{div} u^\nu = 0, \\ u^\nu(0) = u_{\mathrm{in}}, \end{cases} \tag{2}$$

posed on $\mathbb{T}^d$ or $\mathbb{R}^d$. Here $\nu > 0$ is the viscosity coefficient, the initial data $u_{\mathrm{in}} \in L^2$, and we consider weak solutions on $[0, 2]$ satisfying the energy equality

$$\|u^\nu(t)\|_{L^2}^2 = \|u_{\mathrm{in}}\|_{L^2}^2 - 2\nu \int_0^t \|\nabla u^\nu(\tau)\|_{L^2}^2 \, d\tau + 2 \int_0^t (f^\nu, u^\nu) \, d\tau,$$

for all $t \in [0, 2]$

$$E(t) := \liminf_{\nu \to 0} \|u^\nu(t)\|_{L^2}^2 \qquad \text{Energy limit} \tag{3}$$

$$D(t) := 2 \limsup_{\nu \to 0} \nu \int_0^t \|\nabla u^\nu(\tau)\|_{L^2}^2 \, d\tau \qquad \text{Dissipation limit} \tag{4}$$

$$\|u(t)\|_{L^2}^2 \qquad \text{Energy of the limit} \tag{5}$$

Dissipation anomaly and Anomalous dissipation

$$E(t) := \liminf_{\nu\to 0} \|u^\nu(t)\|_{L^2}^2, \qquad D(t) := 2\limsup_{\nu\to 0} \nu \int_0^t \|\nabla u^\nu(\tau)\|_{L^2}^2 \, d\tau.$$

If $f^\nu \to f$ in $L^1(0,2;L^2)$ and $u^\nu \to u$ in $C_w([0,2];L^2)$, then

$$W(t) := 2\lim_{\nu\to 0} \int_0^t (f^\nu, u^\nu)\, d\tau = 2\int_0^t (f,u)\, d\tau.$$

Hence

$$0 \le \|u(t)\|_{L^2}^2 \le E(t) = \|u_{\text{in}}\|_{L^2}^2 - D(t) + W(t), \tag{6}$$

and, in particular,

$$0 \le D(t) \le \|u_{\text{in}}\|_{L^2}^2 + W(t). \tag{7}$$

Definition

- The family of solutions to (2) u^ν exhibits **dissipation anomaly** on $[0,t]$ if $D(t) > 0$.
- The limiting solution u exhibits **anomalous dissipation** on $[0,t]$ if $\|u_{\text{in}}\|_{L^2}^2 + W(t) - \|u(t)\|_{L^2}^2 > 0$.

Dissipation anomally and Anomalous dissipation

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Definition

- The family of solutions to (2) u^ν exhibits **dissipation anomaly** on $[0,t]$ if $D(t) > 0$.
- The limiting solution u exhibits **anomalous dissipation** on $[0,t]$ if $\|u_{\text{in}}\|_{L^2}^2 + W(t) - \|u(t)\|_{L^2}^2 > 0$.

Dissipation Anomaly $\implies$ Anomalous Dissipation

is the converse true?...

$$\|u^\nu(t)\|_{L^2}^2 = \|u^\nu(0)\|_{L^2}^2 - 2\nu \int_0^t \|\nabla u^\nu(\tau)\|_{L^2}^2 \, d\tau + 2 \int_0^t (f^\nu, u^\nu) \, d\tau,$$

and

$$\lim_{\nu \to 0} \nu \int_0^t \|\nabla u^\nu(\tau)\|_{L^2}^2 \, d\tau > 0 \qquad \text{along some subsequence},$$

implies

$$\|u(t)\|_{L^2}^2 < \|u(0)\|_{L^2}^2 + 2 \int_0^t (f, u) \, d\tau.$$

$$E(t) := \liminf_{\nu\to 0} \|u^\nu(t)\|_{L^2}^2, \qquad D(t) := 2\limsup_{\nu\to 0} \nu \int_0^t \|\nabla u^\nu(\tau)\|_{L^2}^2\, d\tau.$$

$$0 \le \|u(t)\|_{L^2}^2 \le E(t) = \|u_{\text{in}}\|_{L^2}^2 - D(t) + W(t), \tag{8}$$

and, in particular,

$$0 \le D(t) \le \|u_{\text{in}}\|_{L^2}^2 + W(t). \tag{9}$$

Questions:

1. *Can $D(t)$ achieve all the possible values allowed by inequalities in (9)?*
2. *Can $\|u(t)\|_{L^2}^2$ achieve all the possible values allowed by inequalities in (8)?*
3. *Can $E(t)$ and $D(t)$ be continuous with nontrivial $D(t)$?*
4. *Can $\|u(t)\|_{L^2}$ be continuous with nontrivial $D(t)$?*
5. *Does Anomalous Dissipation imply Dissipation Anomaly?*
6. *Can there be infinitely many limiting solutions of the Euler equations in the limit of vanishing viscosity?*

Previous results on Dissipation Anomaly

Theodore D. Drivas, Tarek M. Elgindi, Gautam Iyer, In-Jee Jeong (2022): Dissipation anomaly for the advection-diffusion equation $\partial_t\theta + v\cdot\nabla\theta = \kappa\Delta\theta$.

Scott Armstrong and Vlad Vicol (2023): Dissipation anomaly for the advection-diffusion equation for arbitrary H^1 initial data.

Elia Bruè and Camillo De Lellis (2022):
There is a family of smooth solutions to the 3D NSE $\{u^\nu(t)\}_\nu$ on time interval $[0,1]$ with the force $f^\nu \to f$ in $C([0,2];C^\alpha)$ for all $0<\alpha<1$, initial data $u^\nu(0)=u_{\text{in}}$, such that

$$D(1) = 2\limsup_{\nu\to 0}\nu\int_0^1 \|\nabla u^\nu(\tau)\|_{L^2}^2\,d\tau > 0.$$

Based on a construction by **Alberti, Crippa, and Mazzucato (2019)** of a smooth solution to the transport equation were the density is getting efficiently mixed by a $2D$ velocity v:

$$\partial_t\theta + v\cdot\nabla\theta = 0, \qquad \theta(t)\rightharpoonup 0 \text{ as } t\to 1-, \qquad v\in L_t^\infty C^\alpha.$$

The limiting solution: $2+\frac{1}{2}$ D solution of the forced Euler equation

$$u(x,t) = (v(x_1,x_2,t),\theta(x_1,x_2,t)),$$

Theorem (Discontinuity of $\|u(t)\|_{L^2}$)

Let $u_q^{\nu_m}(t)$ be a sequence of weak solutions to (2) satisfying the energy equality with viscosity $\nu_m \to 0$ and force $f^{\nu_m} \to f$ in $L^1(0,1;L^2)$, converging weakly in L^2 to $u \in L^\infty(0,1;L^2)$

$$u^{\nu_m} \to u \qquad \text{in} \qquad C_w([0,1];L^2),$$

converging strongly at $t=0$

$$u^{\nu_m}(0) \to u(0) \qquad \text{in} \qquad L^2,$$

and exhibiting the dissipation anomaly, i.e.,

$$\limsup_{m\to\infty} \nu_m \int_0^1 \|\nabla u^{\nu_m}\|_{L^2}^2 \, dt > 0. \tag{10}$$

Assume also that there are constants $c>0$, $\alpha>1$ such that for every $m \in \mathbb{N}$ and $t \in [0,1]$ there exists $\tilde{q}(m,t)$ with the following localization property:

$$\|u_q^{\nu_m}(t)\|_{L^2} \leq c\lambda_{|q-\tilde{q}(m,t)|}^{-\alpha}. \tag{11}$$

Then $u(t)$ is discontinuous in L^2 at some $t \in [0,1]$.

Lemma

Dissipation Anomaly on $[t_1, t_2]$, i.e.,

$$\limsup_{m\to\infty} \nu_m \int_{t_1}^{t_2} \|\nabla u^{\nu_m}\|_{L^2}^2 \, dt > 0, \tag{12}$$

and the localization condition imply that there exists $T \in [t_1, t_2]$ with

$$u(T) = 0.$$

Lemma

If $u_q^{\nu_m}(t)$ converges weakly in L^2 to $u(t)$ satisfying the localization condition, such that

$$\limsup_{m\to\infty} \|u^{\nu_m}(t)\|_{L^2} > \|u(t)\|_{L^2}, \tag{13}$$

then

$$u(t) = 0.$$

Theorem

There is a countable family of smooth solutions to the 3D NSE (2) $\{u^\nu(t)\}_\nu$ *on* $[0,2]$ *with force* $f^\nu \to f$ *in* $C([0,2];C^\alpha), \forall 0<\alpha<1$*, initial data* $u^\nu(0)=u_{\rm in}$ *satisfying*

$\exists$ *a solution of Euler eq.* $u \in C_w([0,2];L^2)$*, smooth on* $[0,1)\cup(1,2]$*, with force* f*,* $u(0)=u_{\rm in}$*:*

$$\int_0^1 (f,u)\,dt = 0, \qquad \|u(1)\|_{L^2}^2 = \|u_{\rm in}\|_{L^2}^2 = 1, \qquad u(t)=0 \ \text{for} \ t\in[1,2], \tag{14}$$

and as $\nu\to 0$*, the family of the NSE solutions* u^ν *converges weakly in* L^2 *to* u*,*

$$u^\nu \to u \qquad \text{in} \qquad C_w([0,2];L^2),$$

converges strongly on $[0,1)$*:*

$$u^\nu \to u \qquad \text{in} \qquad C([0,t];L^2), \qquad \forall t\in[0,1).$$

Theorem (part 2)

Moreover, the family $\{u^\nu(t)\}_\nu$ *contains the following sequences.*
First subfamily with total dissipation anomaly on $[0, 1+]$**:** *For any energy level* $e \in [0, 1]$ *there exists a subsequence* $\nu_j^e \to 0$ *as* $j \to \infty$ *such that* $u^{\nu_j^e}$ *dissipates this amount of energy on* $[0, 1]$ *in the limit of vanishing viscosity:*

$$2 \lim_{j\to\infty} \nu_j^e \int_0^1 \|\nabla u^{\nu_j^e}\|_{L^2}^2 \, d\tau = e. \qquad \textit{Partial, total, or no dissipation anomaly} \tag{15}$$

On the other hand, $u^{\nu_m^e}$ *dissipates the total energy on any larger interval:*

$$2 \lim_{j\to\infty} \nu_j^e \int_0^t \|\nabla u^{\nu_j^e}\|_{L^2}^2 \, d\tau = 1, \qquad \forall t \in (1, 2]. \qquad \textit{Total dissipation anomaly} \tag{16}$$

In particular, the limiting energy is discontinuous:

$$E(t) = \lim_{j\to\infty} \|u^{\nu_j^e}(t)\|_{L^2}^2 = \begin{cases} \|u(t)\|_{L^2}^2, & t \in [0, 1), \\ 0, & t \in [1, 2]. \end{cases}$$

Theorem (part 3)

Second subfamily with partial **dissipation anomaly** **on $[0,2]$ and continuous limiting energy $E(t)$:** *For any energy level $e \in [0,1)$ there exists a subsequence $\nu_j^e \to 0$ as $j \to \infty$ such that $u^{\nu_j^e}$ dissipates this amount of energy on $[0,2]$ in the limit of vanishing viscosity:*

$$2 \lim_{j\to\infty} \nu_j^e \int_0^2 \|\nabla u^{\nu_j^e}\|_{L^2}^2 \, dt = e. \qquad \text{Partial or no dissipation anomaly} \quad (17)$$

Moreover, the limiting energy $E(t)$ is positive and continuous on $[0,2]$:

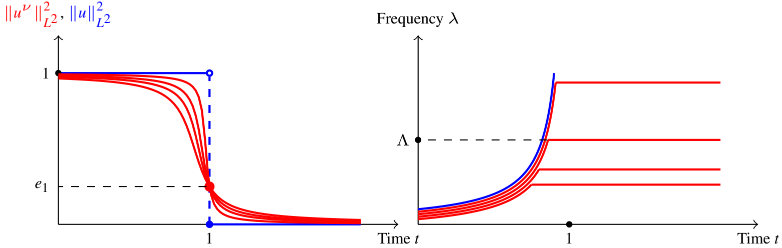
$$E(t) := \lim_{j\to\infty} \|u^{\nu_j^e}(t)\|_{L^2}^2 = \begin{cases} \|u(t)\|_{L^2}^2, & t \in [0,1), \\ \lim_{\tau\to 1-} \|u(\tau)\|_{L^2}^2 = 1, & t = 1, \\ \text{continuous, decreasing}, & t \in [1,2], \\ 1-e, & t = 2. \end{cases}$$

In particular,

$$\lim_{j\to\infty} \|u^{\nu_j^e}(t)\|_{L^2}^2 \geq 1 - e > 0 = \|u(t)\|_{L^2}^2, \qquad t \in [1,2],$$

and hence $u^{\nu_j^e}(t)$ does not converge strongly in L^2 to $u(t)$ for every $t \in [1,2]$.
Also, when $e = 0$, there is no dissipation anomaly by (17), while the limiting solution of the Euler equation looses all of its energy exhibiting anomalous dissipation (14).

Forward energy cascade



(a) Energy profiles $\forall e_1 \in [0, 1]$.

(b) Frequency.

Figure: Convergence of the solutions to the NSE (red) to a solution of the Euler equation (blue).

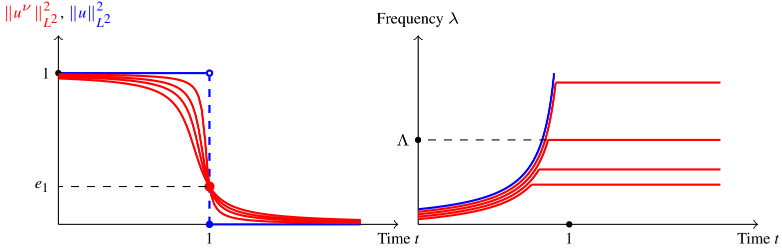
(a) $\Lambda \gg \kappa_d$.

(b) $\Lambda \sim \kappa_d$, $e_2 \in (0, 1]$.

(c) $\Lambda \ll \kappa_d$.

Figure: Energy profiles $E(t)$ for various subsequences of solutions to the NSE.

Forward energy cascade



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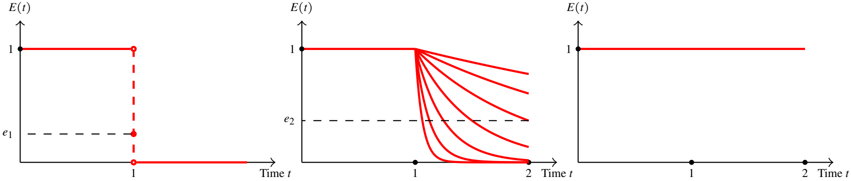

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Figure: Energy profiles $E(t)$ for various subsequences of solutions to the NSE.

Anomalous Dissipation $\not\Rightarrow$ Dissipation Anomaly

$$0 = \|u(2)\|_{L^2}^2 < \|u_{\text{in}}\|_{L^2}^2 = 1, \qquad \int_0^2 (f, u)\, dt = 0.$$

$$\lim_{\nu \to 0} \nu \int_0^2 \|\nabla u^\nu\|_{L^2}^2\, dt = 0.$$

Theorem

There is a (countable) family of smooth solutions to the 3D NSE $\{u^\nu(t)\}_\nu$ *on time interval* $[0,2]$ *with viscosity* $\nu > 0$, *the force* $f^\nu \to f$ *in* $C([0,2];C^\alpha)$,$0<\alpha<1$, $u^\nu(0) = u_{\text{in}}$, *and satisfying:*

There exist two weak solutions of the Euler equation $u_1, u_2 \in C_w([0,2];L^2)$ *smooth on* $[0,1)$ *and* $(1,2]$ *with force* f, *initial data* $u_1(0) = u_2(0) = u_{\text{in}}$, *such that*

$$u_1(t) = u_2(t), \qquad \forall t \in [0,1], \qquad \|u_1(t)\|_{L^2} > \|u_2(t)\|_{L^2}, \qquad \forall t \in (1,2].$$

Two extreme limiting solutions of the Euler equation: *There exist two subsequences*

$$u^{\nu_j^1} \to u_1, \qquad u^{\nu_j^2} \to u_2 \qquad \textit{in} \qquad C_w([0,2];L^2),$$

$u^{\nu_j^1}$ *does not exhibit the dissipation anomaly while* $u^{\nu_j^2}$ *does:*

$$2 \lim_{j\to\infty} \nu_j^1 \int_0^2 \|\nabla u^{\nu_j^1}\|_{L^2}^2 \, dt = 0, \qquad \textit{No dissipation anomaly} \quad (18)$$

$$2 \lim_{j\to\infty} \nu_j^2 \int_0^2 \|\nabla u^{\nu_j^2}\|_{L^2}^2 \, dt = \|u_{\text{in}}\|_{L^2}^2 = 1. \qquad \textit{Total dissipation anomaly} \quad (19)$$

Theorem (part 2)

Arbitrary dissipation anomaly on $[0,2]$ and infinitely many limiting solutions of the Euler equation: *For any $e \in [0,1]$ there exists a subsequence $\nu_j^e \to 0$ as $j \to \infty$ with*

$$2 \lim_{j\to\infty} \nu_j^e \int_0^2 \|\nabla u^{\nu_j^e}\|_{L^2}^2 \, dt = e. \qquad \textit{Partial, total, or no dissipation anomaly} \quad (20)$$

Moreover, there exist infinitely many *solutions of the Euler equation $u_n(t)$, $n = 3, 4, \dots$ with $u_n(0) = u_{\text{in}}$ coinciding with $u_1(t)$ and $u_2(t)$ on $[0,1]$ and satisfying*

$$\|u_n(2)\|_{L^2}^2 < \|u_{\text{in}}\|_{L^2}^2 = 1, \qquad n = 3, 4, \dots,$$

and

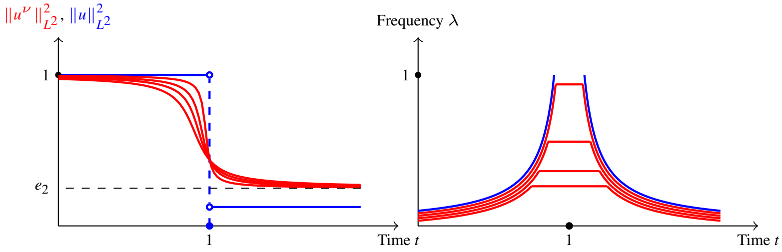
$$\lim_{n\to\infty} \|u_n(2)\|_{L^2}^2 = 1.$$

Finally, each $u_n(t)$ is attained in the limit of vanishing viscosity, i.e., for every $n \in \mathbb{N}$,

$$u^{\nu_j^n} \to u_n, \qquad \textit{in} \qquad C_w([0,2]; L^2),$$

as $j \to \infty$, for some subsequence $\nu_j^n \to 0$.

Forward - backward energy cascades



(a) Energy. $e \in [0, 1]$. (b) Frequency.

Figure: Convergence of the solutions to the NSE (red) to a solution of the Euler equation (blue).

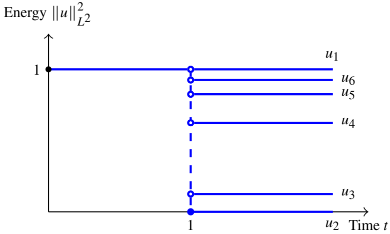
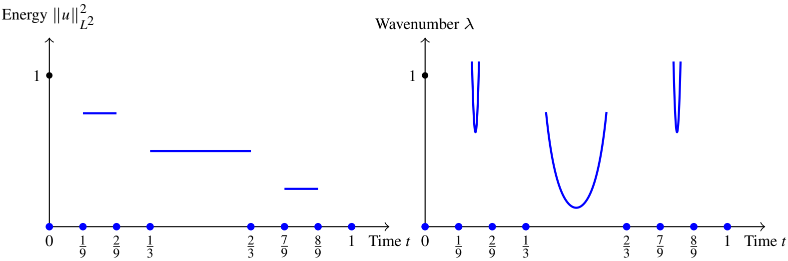


Figure: Countably many limiting solutions of the Euler equation (blue).

Cantor staircase



(a) The energy.

(b) The wavenumber.

Figure: Cantor staircase example.

Upper bound on the energy dissipation for time-periodic forces f^ν

$f^\nu \to f$ in $L_t^\infty L^2$ as $\nu \to 0$:

$$\frac{\epsilon \ell}{U^3} \leq c_1 + c_2 Re^{-1}.$$

C. Foias (97), C. Doering and C. Foias (02)

$$\epsilon := \operatorname*{Lim}_{T\to\infty} \frac{1}{T} \int_0^T \nu \|\nabla u^\nu\|_{L^2}^2 \, dt,$$

$$U^2 = \operatorname*{Lim}_{T\to\infty} \frac{1}{T} \int_0^T \|u^\nu\|_{L^2}^2 \, dt, \quad Re = \frac{U\ell}{\nu}$$

There exist $f^\nu \to f$ in $L_t^\infty L^2$ and initial data u_{in}, such that

$$\frac{\epsilon \ell}{U^3} \geq c_3, \qquad \forall \nu > 0,$$

for some absolute constant $c_3 > 0$.

Anomalous Dissipation revisited

2008 C, Constantin, Friedlander, and Shvydkoy.

$$\lim_{q\to\infty} \int_0^1 \lambda_q^{\frac{1}{3}} \|\Delta_q u\|_{L^3}\, dt = 0,$$

$\implies$ no Anomalous Dissipation:

$$\|u(t)\|_{L^2}^2 = \|u(0)\|_{L^2}^2 + 2\int_0^1 (f,u)\, dt, \qquad t \in [0,1].$$

For any $\{a_q\}$ such that

$$\sum_q a_q < \infty,$$

there exists u not satisfying the energy equality such that

$$\int_0^1 \lambda_q^{\frac{1}{3}} \|\Delta_q u\|_{L^\infty}\, dt = a_q^{-1}.$$

Dissipation anomally

Theorem

There is a constant $c > 0$ *and a family of smooth solutions to the 3D NSE* $\{u^\nu(t)\}_\nu$ *on time interval* $[T_1, T_2] = [1/2, 1]$ *with viscosity* $\nu > 0$*, the force* $f^\nu \to f$ *in* $C([T_1, T_2]; C^\alpha)$ *for all* $0 < \alpha < 1$*, and initial data* $u^\nu(T_1) = u_{\mathrm{in}} \in L^2$*, and satisfying the following.*
For any level of anomalous dissipation $\mathcal{E} \in [0, E_0)$ *there exists a sequence* $\nu_m \to 0$ *such that* u^{ν_m} *converges weakly in* L^2 *to some weak solution of the Euler equation* $u_{\mathcal{E}} \in L^\infty(T_1, T_2; L^2)$

$$u^{\nu_m} \to u_{\mathcal{E}} \qquad \text{in} \qquad C_w([T_1, T_2]; L^2),$$

converges strongly on the complement of the Cantor set:

$$u^{\nu_m}(t) \to u_{\mathcal{E}}(t) \qquad \text{in} \qquad L^2, \qquad \forall t \in [T_1, T_2] \setminus C,$$

and exhibits the dissipation anomaly (when $\mathcal{E} > 0$*):*

$$D(T_2) = 2 \limsup_{m \to \infty} \nu_m \int_{T_1}^{T_2} \|\nabla u^{\nu_m}\|_{L^2}^2 \, dt = \mathcal{E}, \tag{21}$$

Moreover, the limiting energy $E(t)$ *and anomalous dissipation* $D(t)$ *of* $u_{\mathcal{E}}(t)$ *are continuous on* $[T_1, T_2]$*, and*

$$\|u_{\mathcal{E}}(t)\|_{L^2}^2 = \begin{cases} E(t) = \lim_{m \to \infty} \|u^{\nu_m}\|_{L^2}^2, & \forall t \in [T_1, T_2] \setminus C, \\ 0, & \forall t \in [T_1, T_2] \cap C. \end{cases}$$