Empirical partially Bayes multiple testing and compound χ^2 decisions

BIRS-IASM: Harnessing the power of latent structure models and modern Big Data learning

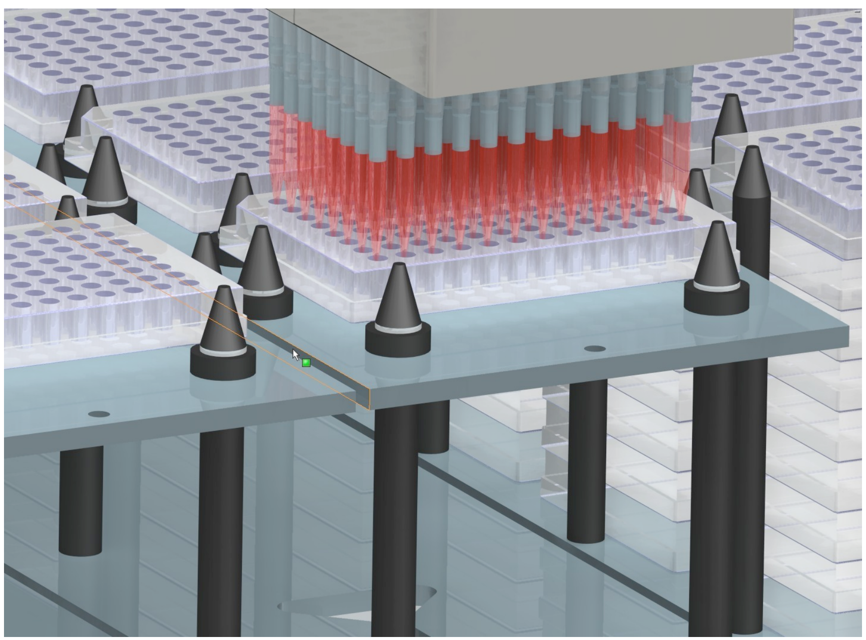
December 12, 2023

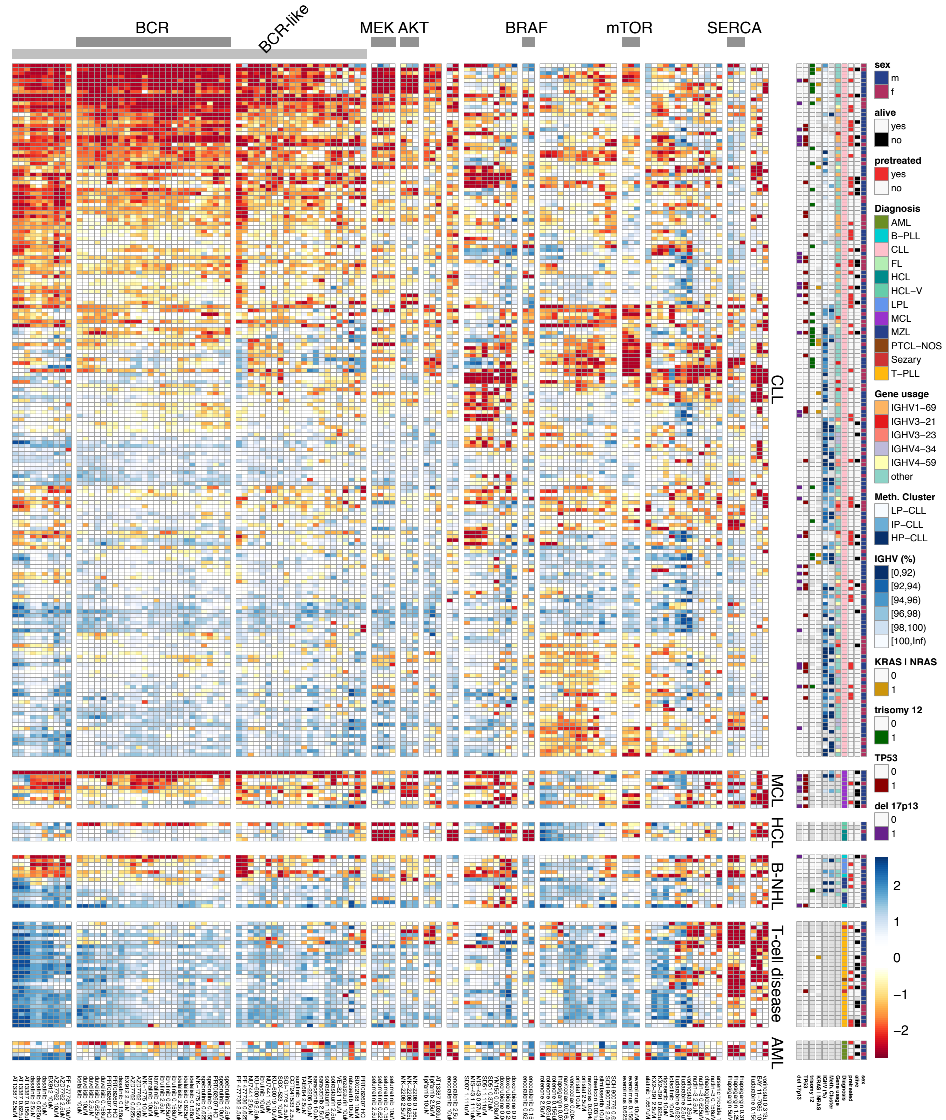
Nikos Ignatiadis
Statistics Department & Data Science Institute, University of Chicago

Joint work with Bodhisattva Sen

Multiple Testing

Many data analysis approaches in high-throughput biology employ item-by-item testing.





Benjamini-Hochberg and false discovery rates

TITLE	CITED BY	YEAR
Controlling the false discovery rate: a practical and powerful approach to multiple testing Y Benjamini, Y Hochberg Journal of the Royal statistical society: series B (Methodological) 57 (1 …	95376	1995

Benjamini-Hochberg and false discovery rates

TITLE	CITED BY	YEAR
Controlling the false discovery rate: a practical and powerful approach to multiple testing Y Benjamini, Y Hochberg Journal of the Royal statistical society: series B (Methodological) 57 (1 …	95376	1995

Limma (DESeq2, edgeR,….)

Moderated estimation of fold change and dispersion for RNA-seq data with DESeq2 MI Love, W Huber, S Anders Genome Biology 15 (12)	48965	2014
edgeR: a Bioconductor package for differential expression analysis of digital gene expression data MD Robinson, DJ McCarthy, GK Smyth Bioinformatics 26 (1), 139-140	30239	2010
limma powers differential expression analyses for RNA-sequencing and microarray studies ME Ritchie, B Phipson, D Wu, Y Hu, CW Law, W Shi, GK Smyth Nucleic acids research 43 (7), e47	22241	2015
Linear models and empirical Bayes methods for assessing differential expression in microarray experiments GK Smyth Statistical applications in genetics and molecular biology 3 (1), Article 3	13222	2004
Limma: linear models for microarray data GK Smyth Bioinformatics and computational biology solutions using R and Bioconductor …	6765	2005

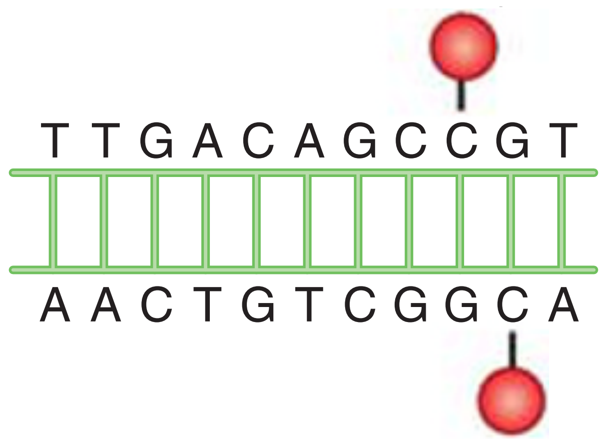
Our contributions

We develop a **nonparametric** framework that **generalizes** and **justifies** limma.

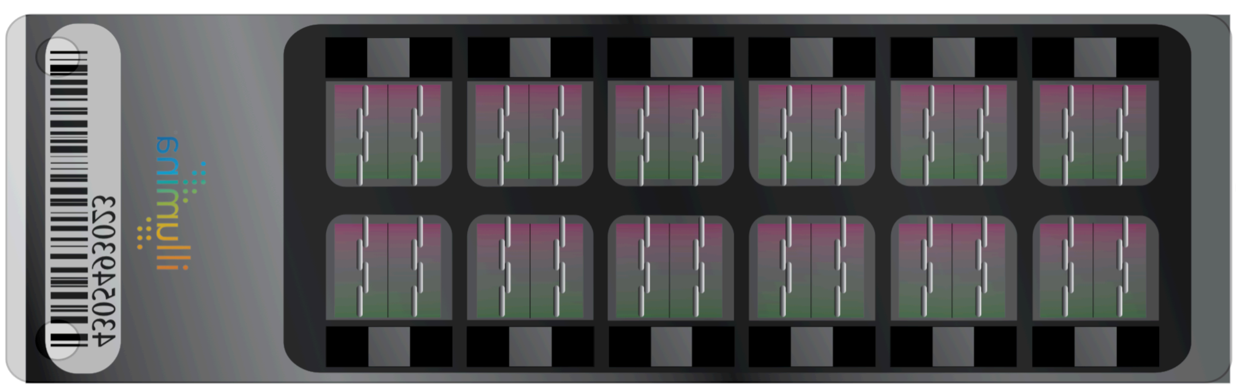
Powerful idea:
“Pretend” **nuisance parameters are i.i.d.** and proceed with empirical partially Bayes.

Differential Methylation

Zhang, Maksimovic, Naselli, Qian, Chopin, Blewitt, Oshlack, and Harrison (2013)



Methylation in **naive** & **activated** T-cells in blood samples from 3 humans



Illumina Infinium® HumanMethylation450 BeadChip

	M28.naive	M28.act_naive	M29.naive	M29.act_naive	M30.naive	M30.act_naive
cg13869341	2.449660	2.187309	2.311837	2.132773	3.040093	3.360123
cg24669183	2.188770	2.296329	1.663033	2.206213	1.943454	2.111070
cg15560884	1.777726	1.612011	1.789361	1.777356	1.721622	1.859812
cg01014490	-3.576590	-5.401990	-4.587255	-4.344729	-5.334906	-3.644528
cg17505339	3.111965	4.158556	3.279807	3.665119	3.034558	3.212872
cg11954957	1.616276	1.796869	2.459432	1.635398	1.915221	2.514348

Finding "interesting" probes

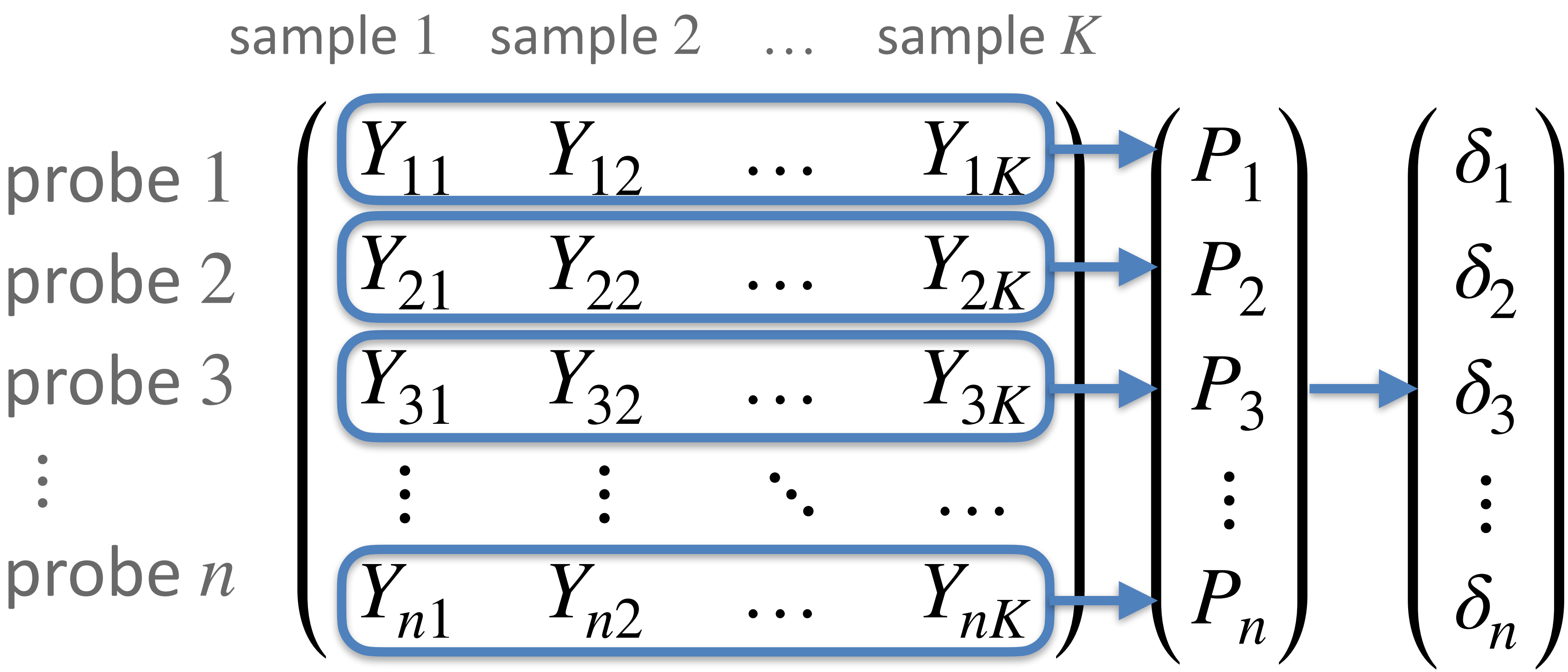
	sample 1	sample 2	…	sample K
probe 1	Y_{11}	Y_{12}	$\cdots$	Y_{1K}
probe 2	Y_{21}	Y_{22}	$\cdots$	Y_{2K}
probe 3	Y_{31}	Y_{32}	$\cdots$	Y_{3K}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$
probe n	Y_{n1}	Y_{n2}	$\cdots$	Y_{nK}

Finding “interesting” probes

sample 1 sample 2 … sample K

$$
\begin{array}{l}
\text{probe } 1 \\
\text{probe } 2 \\
\text{probe } 3 \\
\vdots \\
\text{probe } n
\end{array}
\begin{pmatrix}
Y_{11} & Y_{12} & \cdots & Y_{1K} \\
Y_{21} & Y_{22} & \cdots & Y_{2K} \\
Y_{31} & Y_{32} & \cdots & Y_{3K} \\
\vdots & \vdots & \ddots & \cdots \\
Y_{n1} & Y_{n2} & \cdots & Y_{nK}
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\vdots \\
\delta_n
\end{pmatrix}
$$

Finding “interesting” probes with p-values



The p-value

probe i $\quad \left(Y_{i1} \quad Y_{i2} \quad Y_{i3} \quad Y_{i4} \quad Y_{i5} \quad Y_{i6} \right)$

The p-value

human A

probe i $\quad (Y_{i1} \quad Y_{i2} \quad Y_{i3} \quad Y_{i4} \quad Y_{i5} \quad Y_{i6})$

naive t-cell (Y_{i1}) activated t-cell (Y_{i2})

$$\Delta Y_{iA} := Y_{i2} - Y_{i1}$$

The p-value

human A human B human C

probe i $\left(Y_{i1} \quad Y_{i2} \quad Y_{i3} \quad Y_{i4} \quad Y_{i5} \quad Y_{i6}\right)$

naive t-cell activated t-cell

$\Delta Y_{iA} := Y_{i2} - Y_{i1}$ ΔY_{iB} ΔY_{iC}

The p-value

human A | human B | human C

probe i $\quad (Y_{i1} \quad Y_{i2} \quad Y_{i3} \quad Y_{i4} \quad Y_{i5} \quad Y_{i6})$

naive t-cell $\quad$ activated t-cell

$$\Delta Y_{iA} := Y_{i2} - Y_{i1} \qquad \Delta Y_{iB} \qquad \Delta Y_{iC}$$

$$Z_i := \text{Avg}(\Delta Y_{iA}, \Delta Y_{iB}, \Delta Y_{iC})$$

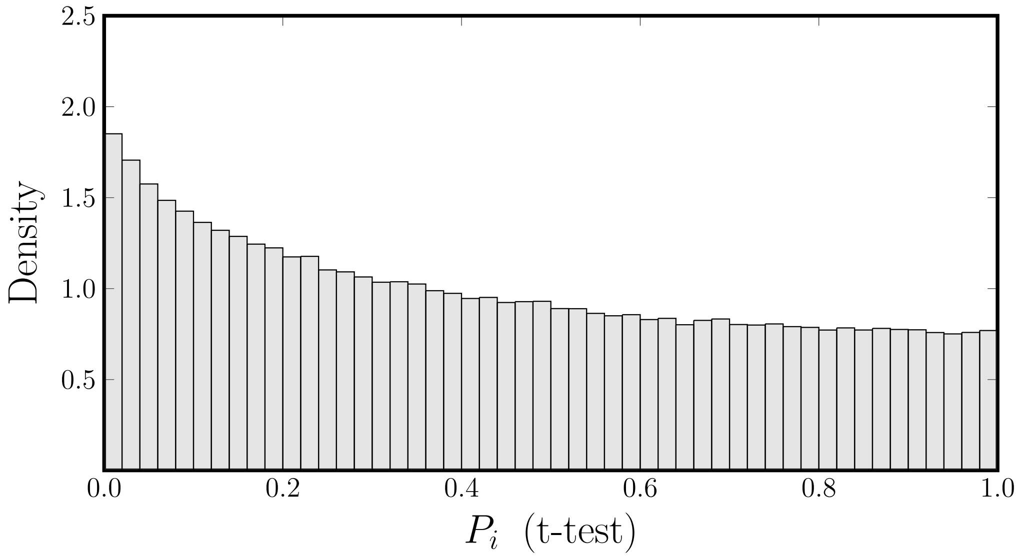
$$S_i^2 := \text{SampleVar}(\Delta Y_{iA}, \Delta Y_{iB}, \Delta Y_{iC})/3$$

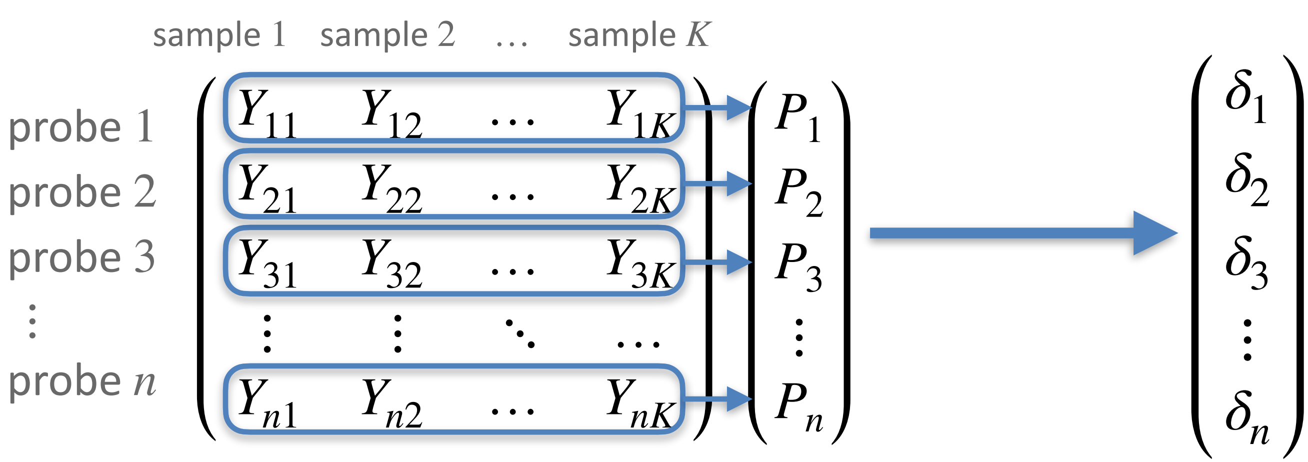
$$T_i := \frac{Z_i}{S_i}$$

If $0 \equiv \mu_i := \mathbb{E}[Z_i]$, then under normality assumptions: $T_i \sim t(2)$.

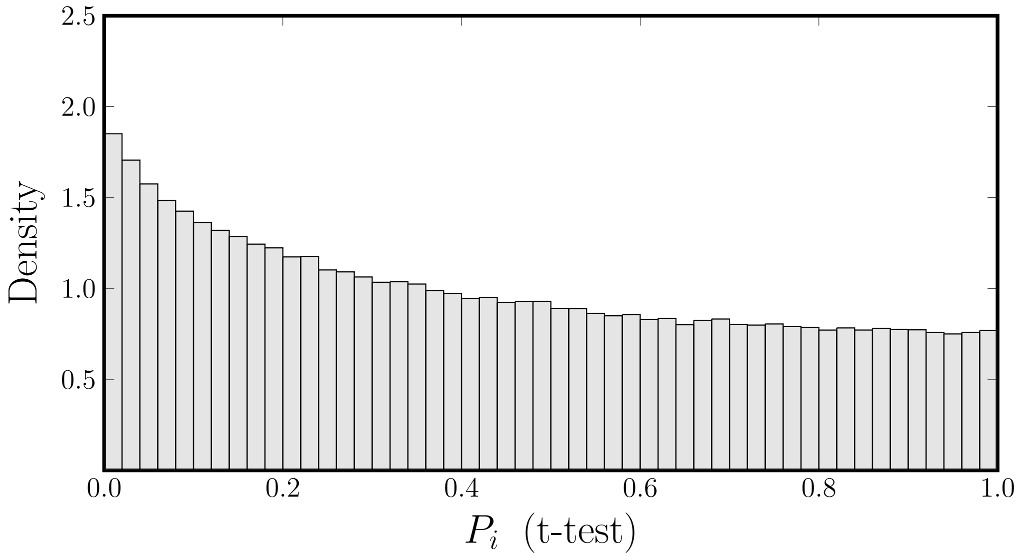
$$P_i := P_{\text{t-test}}(Z_i, S_i^2)$$

t-test p-value histogram

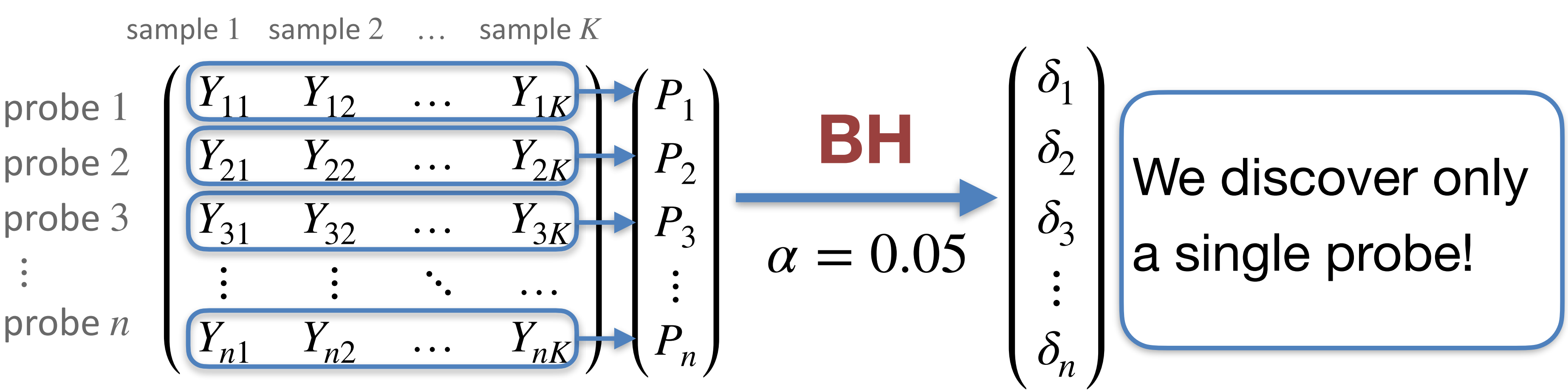




t-test p-value histogram



Benjamini-Hochberg (1995) procedure to control the false discovery rate.



What went wrong? On nuisance parameters.

$$T_i = \frac{Z_i}{S_i} \qquad Z_i \sim \mathcal{N}(\mu_i, \sigma_i^2) \qquad S_i \sim \sqrt{\frac{\sigma_i^2}{2}\chi_2^2}$$

Parameter of interest: μ_i Nuisance parameter: σ_i^2

What went wrong? On nuisance parameters.

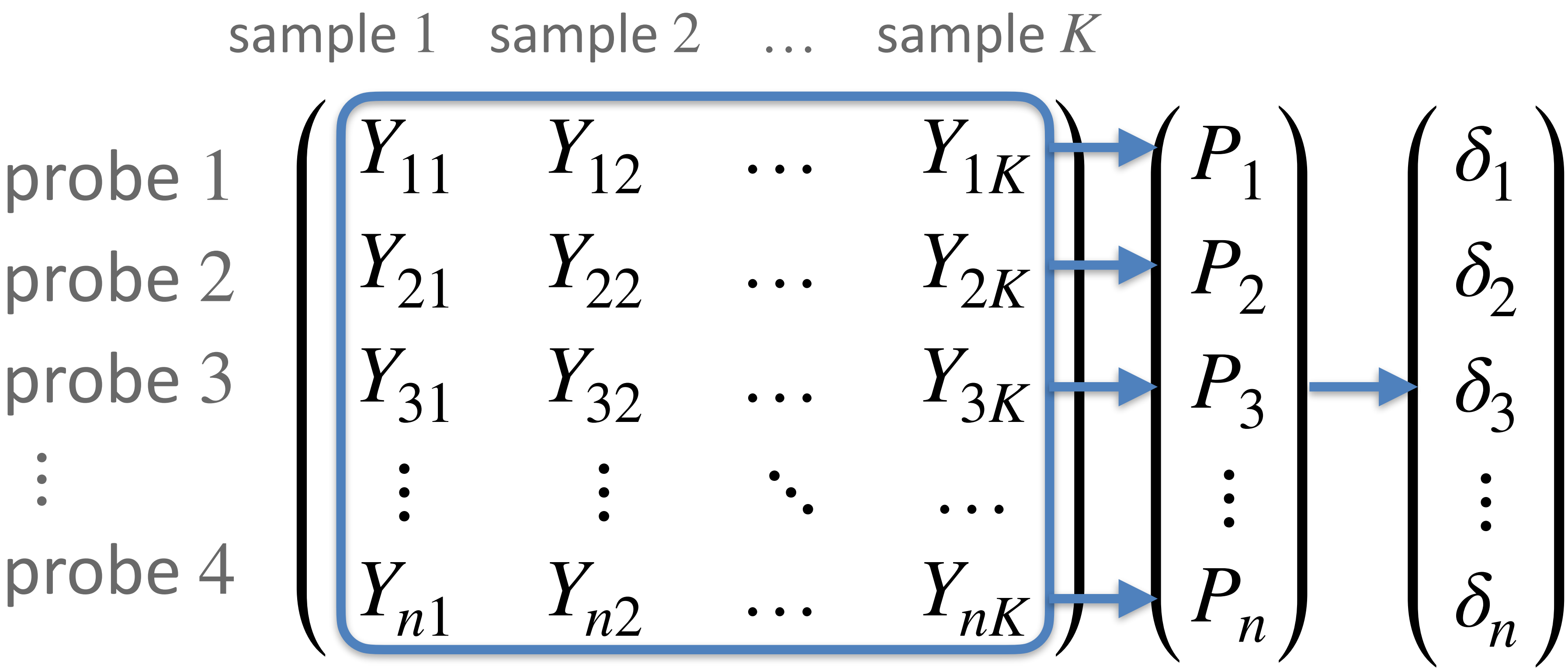
$$T_i = \frac{Z_i \sim \mathcal{N}(\mu_i, \sigma_i^2)}{S_i \sim \sqrt{\frac{\sigma_i^2}{2}\chi_2^2}}$$

Parameter of interest: μ_i Nuisance parameter: σ_i^2

t-test deals with the nuisance parameter, at a cost.

e.g., the single discovery we made, has $S_i^2 = 10^{-8}$, which is the smallest among all 440102 probes.

This work:



Empirical partially Bayes multiple testing

Preview: 549 discoveries by BH

Formal setting

$$(Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu} \chi_\nu^2 \text{ for } i = 1, \ldots, n\,.$$

$\nu \in \mathbb{N}_{\geq 2}$: known degrees of freedom

$\mu_i \in \mathbb{R}$, $\sigma_i^2 \in \mathbb{R}_{>0}$: unknown

Goal: Test the null hypotheses $H_i : \mu_i = 0$, $i = 1, \ldots, n$, while controlling the false discovery rate.

Formal setting

$$(Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu} \chi_\nu^2 \text{ for } i = 1, \ldots, n\,.$$

$\nu \in \mathbb{N}_{\geq 2}$: known degrees of freedom

$\mu_i \in \mathbb{R}$, $\sigma_i^2 \in \mathbb{R}_{>0}$: unknown

Goal: Test the null hypotheses $H_i : \mu_i = 0$, $i = 1, \ldots, n$, while controlling the false discovery rate.

Challenge: n is very large, ν **is very small.**

Borrowing information

$$(Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu} \chi_\nu^2 \text{ for } i = 1, \ldots, n\,.$$

Parameters of interest: μ_i Nuisance parameters: σ_i^2

Idea: Borrow information across i for σ_i^2, but not for μ_i.

Borrowing information

$$(Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu}\chi_\nu^2 \text{ for } i = 1,\ldots,n\,.$$

Parameters of interest: μ_i Nuisance parameters: σ_i^2

Idea: Borrow information across i for σ_i^2, but not for μ_i .

Example: Assume that $\sigma_i^2 = \tilde{\sigma}^2$ for all $i = 1,\ldots,n$, then could compute z-test p-value $P(z, s) = 2(1 - \Phi(|z|/\tilde{\sigma}))$.

Borrowing information

$$(Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu} \chi_\nu^2 \text{ for } i = 1, \ldots, n\,.$$

Parameters of interest: μ_i Nuisance parameters: σ_i^2

Idea: Borrow information across i for σ_i^2, but not for μ_i.

Example: Assume that $\sigma_i^2 = \tilde{\sigma}^2$ for all $i = 1, \ldots, n$, then could compute z-test p-value $P(z, s) = 2(1 - \Phi(|z|/\tilde{\sigma}))$.

Partially Bayes (Cox, 1975)**:**

$$\sigma_i^2 \overset{iid}{\sim} G \quad (\star)$$

Borrowing information

$$(Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu} \chi_\nu^2 \text{ for } i = 1, \ldots, n\,.$$

Parameters of interest: μ_i Nuisance parameters: σ_i^2

Idea: Borrow information across i for σ_i^2, but not for μ_i .

Example: Assume that $\sigma_i^2 = \tilde{\sigma}^2$ for all $i = 1, \ldots, n$, then could compute z-test p-value $P(z, s) = 2(1 - \Phi(|z|/\tilde{\sigma}))$.

Partially Bayes (Cox, 1975)**:**

$$\sigma_i^2 \overset{iid}{\sim} G \qquad (\star)$$

predictive p-values:

Box (1980)

Meng (1994)

Bayarri and Berger (1999, 2000)

…

Limma (Smyth, 2004)

Also:

Cox (1975)

Lönnstedt and Speed (2002)

Lönnstedt (2005)

$$(Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu} \chi_\nu^2 \text{ for } i = 1, \ldots, n\,.$$

$$\frac{1}{\sigma_i^2} \overset{iid}{\sim} \frac{\chi_{\nu_0}^2}{\nu_0 s_0^2}, \qquad \nu_0,\, s_0^2 \in \mathbb{R}_{>0}$$

This work: a nonparametric framework

$$(Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu}\chi^2_\nu \text{ for } i = 1,\ldots,n\,.$$

$$\sigma_i^2 \overset{iid}{\sim} G \quad (\,\star\,)$$

Agenda:

1. Assume $(\,\star\,)$ holds and G is known. How should we proceed? partially Bayes
2. Assume $(\,\star\,)$ holds and G is ***unknown***. How should we proceed? empirical partially Bayes
3. What if $(\,\star\,)$ does not hold and the σ_i^2 are deterministic? compound decisions

Biometrika (1975), **62**, 3, *p.* 651
Printed in Great Britain

A note on partially Bayes inference and the linear model

BY D. R. COX
Department of Mathematics, Imperial College, London

"Optimum tests and confidence limits are based on the conditional distribution of Z_i given S_i^2."

The conditionality principle is in full force

$$\sigma_i^2 \overset{iid}{\sim} G \quad (\star), \quad (Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu} \chi^2_\nu \text{ for } i = 1, \ldots, n.$$

Proposition (I. & Sen): Suppose G is known and not degenerate, then: (Z_i, S_i^2) is minimally sufficient for μ_i and S_i^2 is ancillary.

The conditionality principle is in full force

$$\sigma_i^2 \overset{iid}{\sim} G \ \ (\star), \quad (Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu} \chi^2_\nu \text{ for } i = 1, \ldots, n\,.$$

Proposition (I. & Sen): Suppose G is known and not degenerate, then: (Z_i, S_i^2) is minimally sufficient for μ_i and S_i^2 is ancillary.

The above suggests computing conditional partially Bayes p-values:

Cox (1975), Lu and Stephens (2016)

$$P_i := P_G(Z_i, S_i^2)$$

$$P_G(z, s^2) := \mathbb{E}_G[2(1 - \Phi(|z|/\sigma_i)) \quad | \quad S_i^2 = s^2]\,.$$

The conditionality principle is in full force

$$\sigma_i^2 \overset{iid}{\sim} G \ \ (\star), \quad (Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu}\chi_\nu^2 \ \text{for} \ \ i = 1,\ldots,n\,.$$

Proposition (I. & Sen): Suppose G is known and not degenerate, then: (Z_i, S_i^2) is minimally sufficient for μ_i and S_i^2 is ancillary.

The above suggests computing conditional partially Bayes p-values:

Cox (1975), Lu and Stephens (2016)

$$P_i := P_G(Z_i, S_i^2)$$

$$P_G(z, s^2) := \mathbb{E}_G[2(1 - \Phi(|z|/\sigma_i)) \ \ | \ \ S_i^2 = s^2]\,.$$

Proposition: Under the null $\mu_i = 0$, the partially Bayes p-values are conditionally uniform. $P_i \mid S_i^2 \ \sim U[0,1]\,.$

Step 2.

$$(Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu}\chi_\nu^2 \text{ for } i = 1,\ldots,n\,.$$

$$\sigma_i^2 \overset{iid}{\sim} G \quad (\star)$$

Agenda:

1. Assume ($\star$) holds and G is known. How should we proceed? partially Bayes
2. Assume ($\star$) holds and G is ***unknown***. How should we proceed? empirical partially Bayes
3. What if ($\star$) does not hold and the σ_i^2 are deterministic? compound decisions

AN EMPIRICAL BAYES APPROACH TO STATISTICS

HERBERT ROBBINS
COLUMBIA UNIVERSITY

Empirical Bayes [Robbins (1956), Efron (2010)] presents a powerful framework for learning from others.

If we are facing many simultaneous and related statistical problems, then an empirical Bayesian can mimic an oracle Bayesian that knows the true prior.

AN EMPIRICAL BAYES APPROACH TO STATISTICS

HERBERT ROBBINS
COLUMBIA UNIVERSITY

Empirical Bayes [Robbins (1956), Efron (2010)] presents a powerful framework for learning from others.

If we are facing many simultaneous and related statistical problems, then an empirical Bayesian can mimic an oracle Bayesian that knows the true prior.

Example (limma): $\nu_0,\ s_0^2 \in \mathbb{R}_{>0}$ are unknown, but can be estimated based on $(S_1^2, \ldots, S_n^2)$, e.g., by method of moments (or MLE)

$$\frac{1}{\sigma_i^2} \overset{iid}{\sim} \frac{\chi^2_{\nu_0}}{\nu_0 s_0^2}, \quad (Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu} \chi^2_\nu \ \text{ for } \ i = 1, \ldots, n\,.$$

Nonparametric maximum likelihood

$$\sigma_i^2 \overset{iid}{\sim} G \;\; (\star), \quad (Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu}\chi^2_\nu \text{ for } i = 1, \ldots, n\,.$$

NPMLE: Estimate G by $\hat{G} := \hat{G}(S_1^2, \ldots, S_n^2)$ by the Nonparametric Maximum Likelihood Estimator

Robbins (1950)
Kiefer, and Wolfowitz (1956)
Jewell (1982)
Zhang (2009)
Koenker, and Mizera (2014)

Nonparametric maximum likelihood

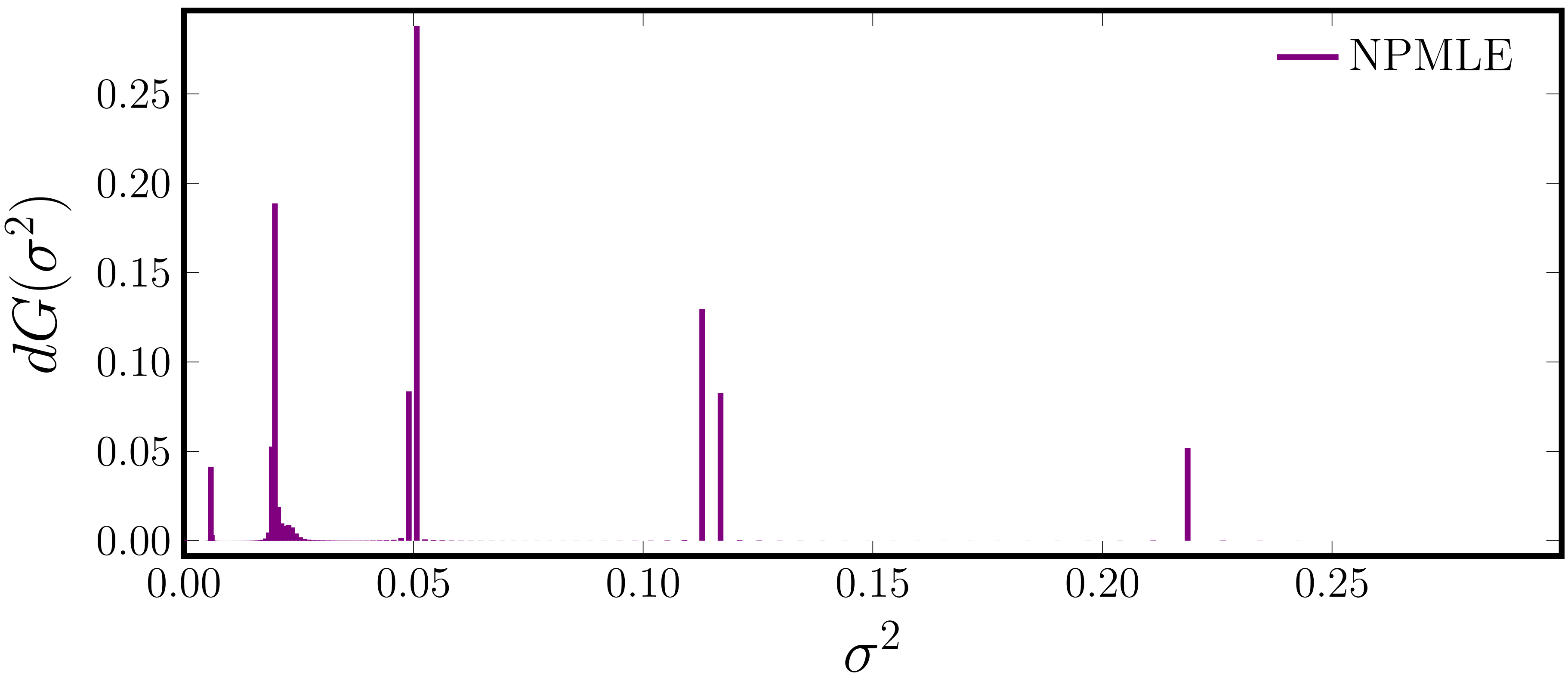
$$\sigma_i^2 \overset{iid}{\sim} G \;\; (\star), \quad (Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu}\chi_\nu^2 \text{ for } i = 1,\ldots,n\,.$$

NPMLE: Estimate G by $\hat{G} := \hat{G}(S_1^2, \ldots, S_n^2)$ by the Nonparametric Maximum Likelihood Estimator

Robbins (1950)
Kiefer, and Wolfowitz (1956)
Jewell (1982)
Zhang (2009)
Koenker, and Mizera (2014)

$$\hat{G} \in \underset{\tilde{G}}{\text{argmax}} \left\{ \prod_{i=1}^{n} f_{\tilde{G}}(S_i^2) \right\}, \; f_{\tilde{G}}(s^2) = \int p(s^2 \mid \sigma^2)\, d\tilde{G}(\sigma^2)\,.$$

Nonparametric maximum likelihood



$$\hat{G} \in \operatorname*{argmax}_{\tilde{G}} \left\{ \prod_{i=1}^{n} f_{\tilde{G}}(S_i^2) \right\}, \; f_{\tilde{G}}(s^2) = \int p(s^2 \mid \sigma^2)\, d\tilde{G}(\sigma^2)\,.$$

Empirical partially Bayes p-values

Oracle partially Bayes p-values: $P_G(z, s^2)$

Empirical partially Bayes p-values: $P_{\hat{G}}(z, s^2)$

Empirical partially Bayes p-values

Oracle partially Bayes p-values: $P_G(z, s^2)$

Empirical partially Bayes p-values: $P_{\hat{G}}(z, s^2)$

Theorem (I. & Sen): Suppose G, $\hat{G}$ have support bounded away from 0 and $+\infty$, $\nu \geq 3$, then:

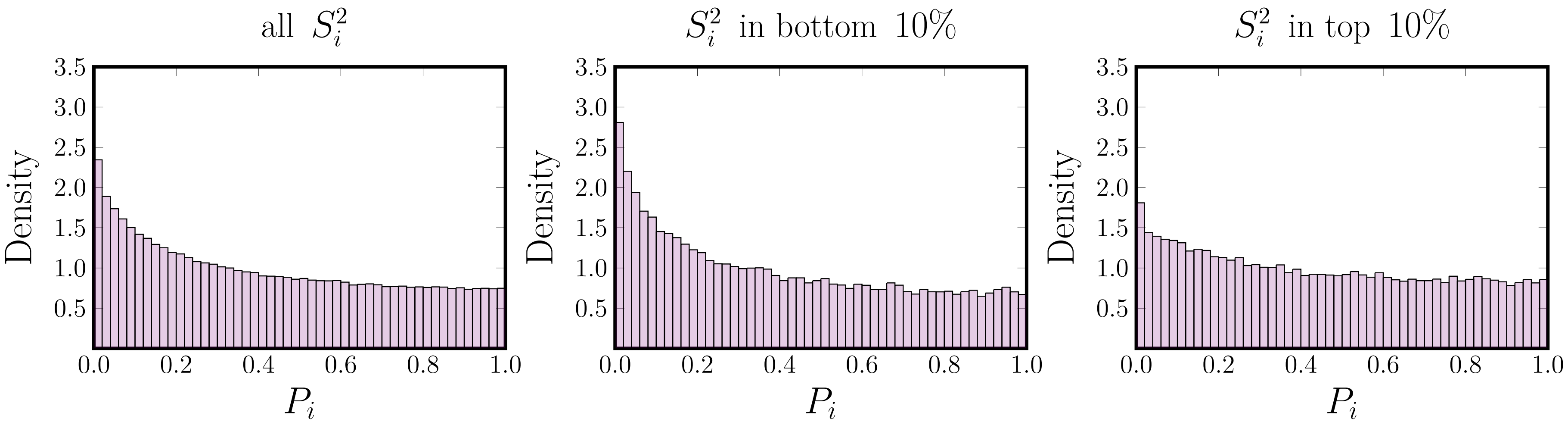
$$\mathbb{E}_G\left[\left|P_{\hat{G}}(Z_i, S_i^2) \wedge 0.99 - P_G(Z_i, S_i^2) \wedge 0.99\right|\right] \lesssim \frac{(\log n)^{5/2}}{\sqrt{n}}.$$

Proposition: For i with $\mu_i = 0$:

$$\limsup_{n \to \infty} \mathbb{E}_G\left[\sup_{t \in [0,1]}\left|\mathbb{P}_G[P_{\hat{G}}(Z_i, S_i^2) \leq t \mid S_1^2, \ldots, S_n^2] - t\right|\right] = 0.$$

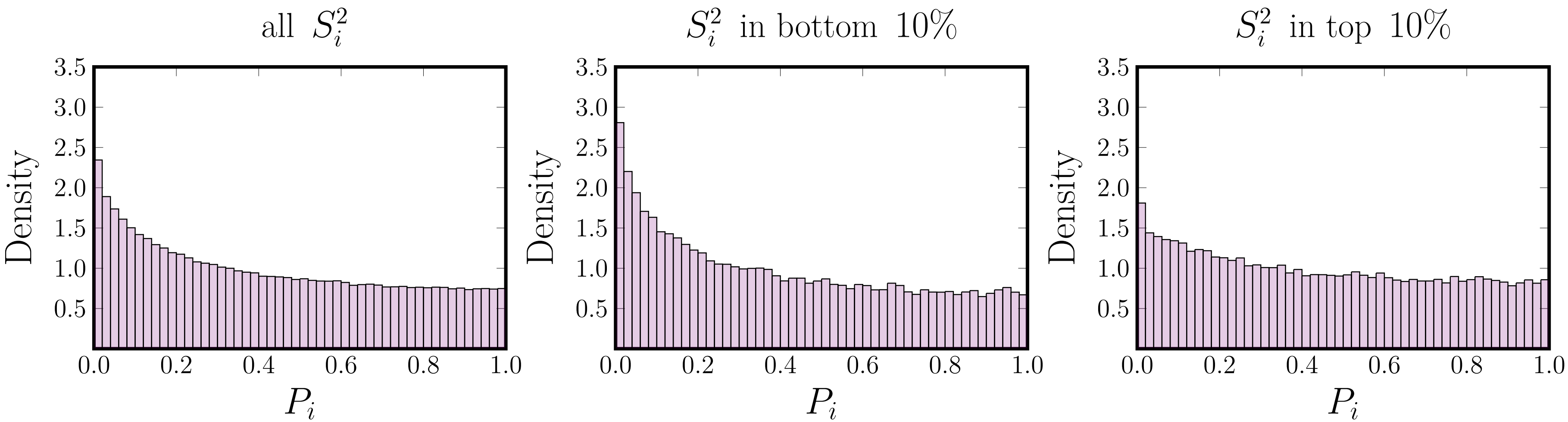
On conditionality

Empirical partially Bayes p-values

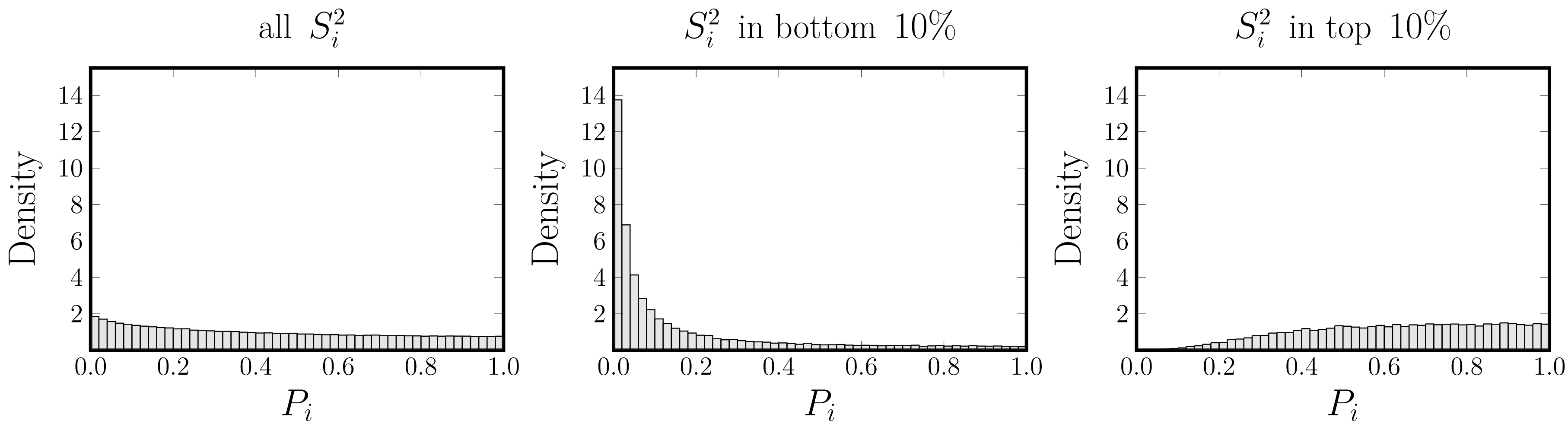


On conditionality

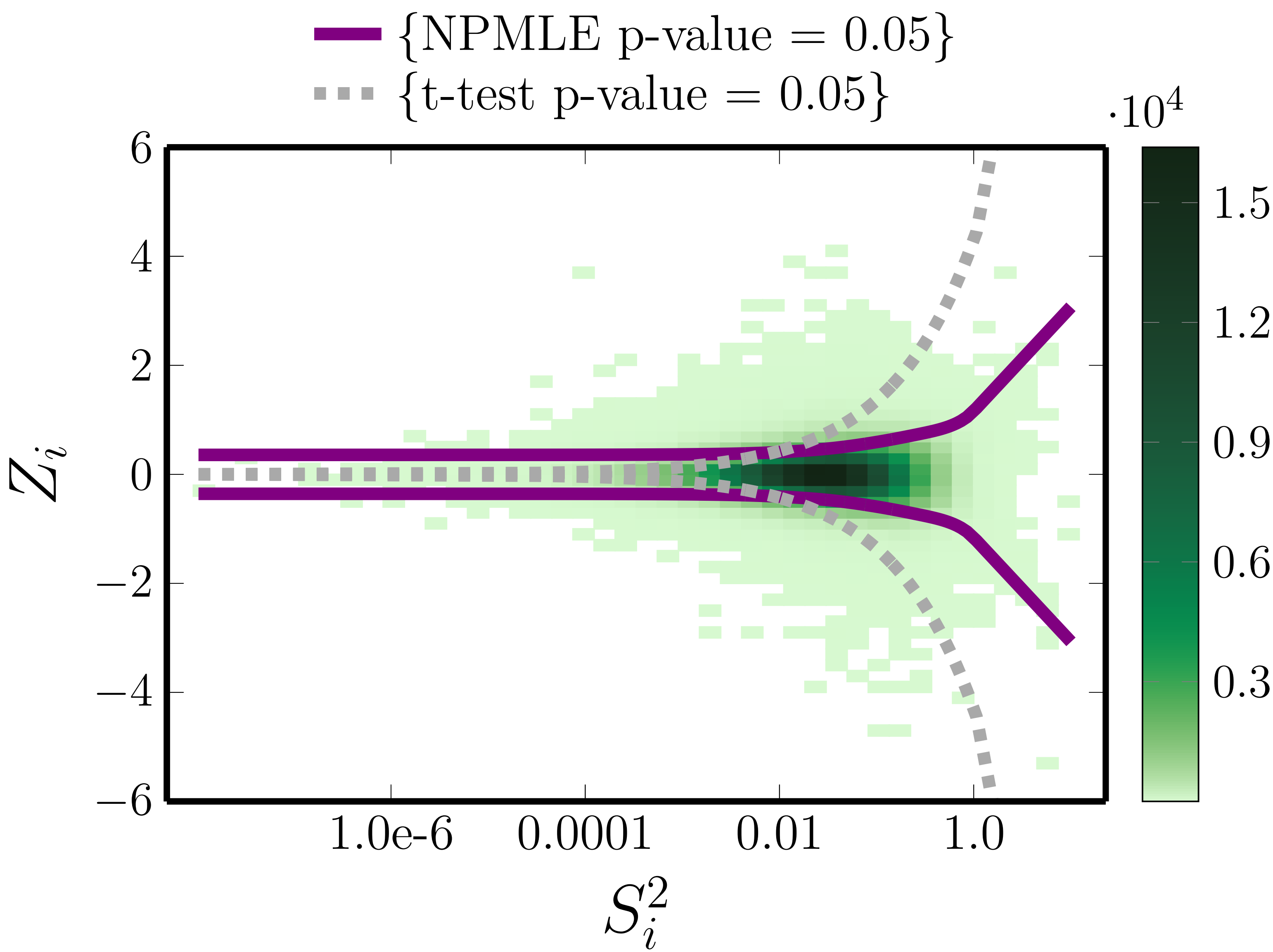
Empirical partially Bayes p-values



Standard t-test p-values



Two-dimensional p-value contours

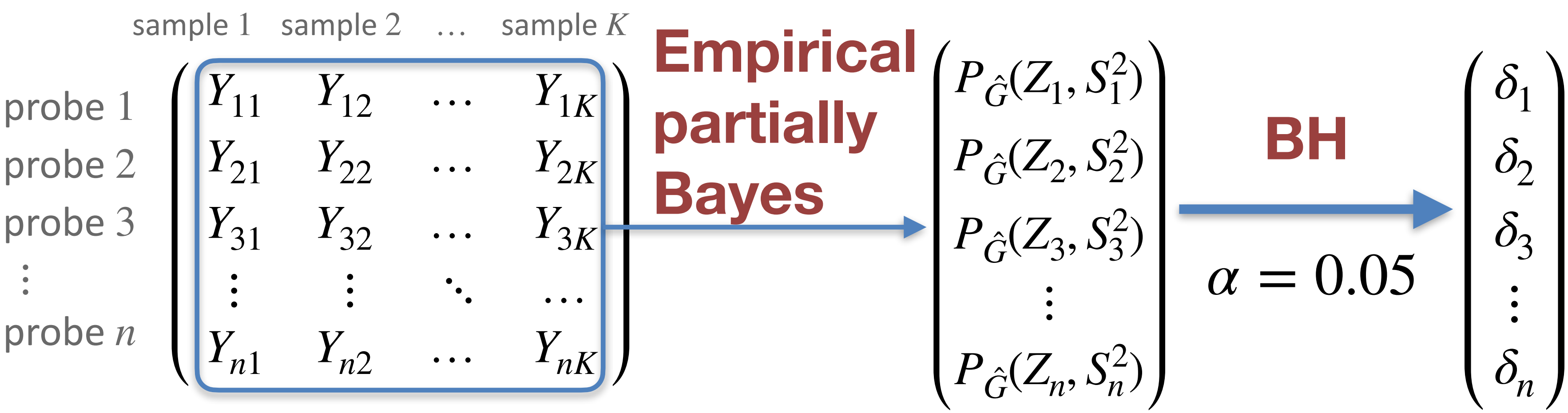


Asymptotic FDR control

$$\begin{array}{r} \\ \text{probe } 1 \\ \text{probe } 2 \\ \text{probe } 3 \\ \vdots \\ \text{probe } n \end{array} \begin{array}{c} \text{sample } 1 \quad \text{sample } 2 \quad \dots \quad \text{sample } K \\ \begin{pmatrix} Y_{11} & Y_{12} & \cdots & Y_{1K} \\ Y_{21} & Y_{22} & \cdots & Y_{2K} \\ Y_{31} & Y_{32} & \cdots & Y_{3K} \\ \vdots & \vdots & \ddots & \cdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{nK} \end{pmatrix} \end{array} \xrightarrow{\text{Empirical partially Bayes}} \begin{pmatrix} P_{\hat{G}}(Z_1, S_1^2) \\ P_{\hat{G}}(Z_2, S_2^2) \\ P_{\hat{G}}(Z_3, S_3^2) \\ \vdots \\ P_{\hat{G}}(Z_n, S_n^2) \end{pmatrix} \xrightarrow[\alpha = 0.05]{\text{BH}} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \vdots \\ \delta_n \end{pmatrix}$$

549 discoveries

Asymptotic FDR control



549 discoveries

Theorem (I. & Sen): Under previous assumption, and a requirement on the effect size of non-zero μ_is, the above procedure controls the FDR asymptotically.

$$\limsup_{n\to\infty} \text{FDR} \leq \alpha\,.$$

Storey, Taylor, and Siegmund (2004)

Step 3:

$$(Z_i, S_i^2) \overset{ind.}{\sim} \mathcal{N}(\mu_i, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu}\chi_\nu^2 \text{ for } i = 1,\ldots,n\,.$$

$$\sigma_i^2 \overset{iid}{\sim} G \quad (\star)$$

Agenda:

1. Assume ($\star$) holds and G is known. How should we proceed? partially Bayes
2. Assume ($\star$) holds and G is ***unknown***. How should we proceed? empirical partially Bayes
3. What if ($\star$) does not hold and the σ_i^2 are deterministic? compound decisions

Compound decisions

So far we assumed that: $\sigma_i^2 \overset{iid}{\sim} G \quad (\star)$

"Let us use a mixed model, even if it might not be appropriate."

van Houwelingen (2014)

Compound decisions

So far we assumed that: $\sigma_i^2 \overset{iid}{\sim} G \quad (\star)$

"Let us use a mixed model, even if it might not be appropriate."

van Houwelingen (2014)

Several of our results hold even if $\sigma_1^2, \ldots, \sigma_n^2$ are in fact deterministic. We build upon techniques from compound decision theory.

Robbins (1951), James, and Stein (1961), Zhang (2003), Armstrong (2022)

Example: Theorem on asymptotic FDR control remains valid,

$$\limsup_{n \to \infty} \mathrm{FDR} \leq \alpha\,.$$

Simulations

$$\sigma_i^2 \overset{iid}{\sim} G, \qquad n = 10{,}000$$

$$\mu_i \mid \sigma_i^2 \overset{ind.}{\sim} 0.9\delta_0 + 0.1\mathcal{N}(0,\, 16\sigma_i^2),$$

$$(Z_i, S_i^2) \mid \mu_i, \sigma_i^2 \overset{ind.}{\sim} \mathcal{N}(\mu_i,\, \sigma_i^2) \otimes \frac{\sigma_i^2}{\nu}\chi_\nu^2\,.$$

Vary G and ν.

We consider the following p-values:

1. Standard t-test p-values.
2. Oracle z-test p-values (under knowledge of σ_i^2) .
3. Empirical partially Bayes p-values.

Apply Benjamini-Hochberg at level $\alpha = 0.1$.

Simulation results

$$\mathbb{P}_G[\sigma_i^2 = 1] = 1$$

$\nu = 2$

	FDR	*Power*
t-test	***0.09***	***0.00***
Oracle z	***0.09***	***0.50***
Empirical partially Bayes	***0.09***	***0.50***

$\nu = 4$

FDR	*Power*
0.09	***0.13***
0.09	***0.50***
0.09	***0.50***

$\nu = 32$

FDR	*Power*
0.09	***0.47***
0.09	***0.50***
0.09	***0.50***

Simulation results

$$\mathbb{P}_G[\sigma_i^2 = 1] = 1$$

$$\nu = 2 \quad \nu = 4 \quad \nu = 32$$

	FDR	Power
t-test	***0.09***	***0.00***
Oracle z	***0.09***	***0.50***
Empirical partially Bayes	***0.09***	***0.50***

FDR	Power
0.09	***0.13***
0.09	***0.50***
0.09	***0.50***

FDR	Power
0.09	***0.47***
0.09	***0.50***
0.09	***0.50***

$$\mathbb{P}_G[\sigma_i^2 = 1] = \mathbb{P}_G[\sigma_i^2 = 10] = 1/2.$$

	FDR	Power
t-test	*0.09*	*0.00*
Oracle z	*0.09*	*0.50*
Empirical partially Bayes	*0.09*	***0.28***

FDR	Power
0.09	*0.13*
0.09	*0.50*
0.09	***0.34***

FDR	Power
0.09	*0.47*
0.09	*0.50*
0.09	***0.50***

Conclusion

Empirical Bayes presents a powerful framework for learning from others.

Opportunities presented by large-scale data that were not available in classical statistics.

Robbins (1956)
Efron (2010)
Stephens (2016)

Empirical partially Bayes multiple testing and compound χ^2 decisions

N.I., and Bodhisattva Sen
arXiv:2303.02887 (2023)