Around symmetries of K3 surfaces

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1 Overview of the Field

K3 Surfaces are simply connected complex algebraic surfaces whose canonical bundle is trivial. They play a central role in the theory of algebraic surfaces—similar to the role played by elliptic curves in dimension one—as they exhibit both interesting geometric and arithmetic properties. The archetypal example of a K3 surface is given by the zero set of a quartic polynomial in projective space. In that case, the choice of coefficients, 19 up to a projectivity, illustrates one of the 19–dimensional segments of their algebraic moduli. Another important example of K3 surfaces is given by the Kummer construction: one obtains a K3 surface by taking the quotient of a torus by the group generated by inversions consisting of elements preserving the volume form and then taking a suitable resolution.

While in theory K3 surfaces are completely classified through the Torelli type theorem of Piatetski-Shapiro, Shafarevich, Burns and Rapoport, the linearization of their geometry does not give a clear interpretation of their many interesting properties. And although there has been tremendous progress in understanding the geometry of K3 surfaces and of their moduli, there remain several open problems which need to be addressed. In particular, let us mention two of these. The first is to understand the geometry of K3 surfaces that contain a specific curve on the surface (known as the Mukai program). The second is to classify the possible automorphisms (in the finite and infinite case) of these surfaces, and use these to understand their geometry.

When talking about K3 surfaces, one can not avoid studying what happens in higher dimensions. Higher dimensional generalisation of K3 surfaces are irreducible holomorphic symplectic manifolds and Calabi-Yau manifolds. Also for these varieties an investigation of symmetries and moduli spaces is an important and active research area, with many connections to the geometry of K3 surfaces. Although, the workshop focused mainly on K3 surfaces, but we had several talks that made the bridge to higher dimension and opened the way to several fruitful discussions.

2 Recent Developments and Presentation Highlights

In this section we recall the title and abstracts of the talks given during the workshop. We had 16 talks given by more advanced mathematicians and 4 talks given by younger mathematicians. This choice was made so that PhD students and postdocs would have an opportunity to present their research and so to better be able to interact with senior participants. Furthermore, special attention was given to diversity of speakers, in order to have a more inclusive workshop.

Most talks at the workshop were devoted to important recent discoveries in the area of automorphisms of K3 surfaces, and of irreducible holomorphic symplectic varieties. The talks listed below are grouped by subject to give a better overview of the recent state of the art. Some talks could have been included in several sections.

2.1 Automorphisms

Claudia Correa Deisler, University of Tarapacá, Chile *A general result on Cox rings of K3 surfaces*

The problem of finding a presentation for the Cox ring of a Mori dream space (Cox ring is finitely generated) is interesting and difficult. For K3 surfaces, it is known that the Cox ring is finitely generated if and only if its effective cone is polyhedral, or equivalent if its automorphism group is finite. K3 surfaces with this last property have been classified in a series of classical works and for those with Picard number ≥ 3 there are a finite number of families. In this talk we will see a general theorem about Cox rings of K3 surfaces (not necessarily Mori dream) and some examples where we have applied this result together with other techniques based on exact sequences of the Koszul type, allowing us to obtain the degrees of a set of generators of the Cox ring of K3 Mori dream surfaces. This work is in collaboration with M. Artebani, A. Laface and X. Roulleau.

Dino Festi, University of Milan, Italy *K3 surfaces with two involutions and low Picard number*

Having an automorphism is a non-trivial property for a complex K3 surface. Indeed, if X is a generic complex K3 surface of degree $d \geq 4$, then the only automorphism of X is the identity. If X is a generic of degree $d = 2$, then X admits only one involution beside the identity map. Hence a natural question arises: given a fixed positive even integer d, how special is it for a K3 surface of degree d to admit an involution?

Alice Garbagnati, University of Milan, Italy *Hodge structures of bidouble covers of rational surfaces.*

A bidouble cover is a Galois cover whose Galois group is $(\mathbb{Z}/2\mathbb{Z})^2$. If $X \to Y$ is a bidouble cover there are 3 intermediate covers $Y_{i=1,2,3}$, which are $\mathbb{Z}/2\mathbb{Z}$-quotient of X. We will consider bidouble covers X of a rational surface Y and we will assume that the surfaces Y_i are either surfaces with $h^2 = 0$ or K3 surfaces. Under this condition the transcendental Hodge structure of X splits into the direct sum of sub-Hodge structures of K3-type, each of them geometrically related with a K3 surface. This gives a strong control on the Hodge structure of X and allows one to discuss some of the classic problems related with the Hodge structures of the second cohomology group of surfaces: we first discuss conditions under which X satisfies the Mumford Tate conjecture and the Tate conjecture or enjoys the infinitesimal Torelli property. Then, we provide a classification of the smooth bidouble covers of a minimal rational surface satisfying the previous conditions and we discuss how to modify and generalize the previous construction by considering either singular bidouble covers or the so-called iterated bidouble covers. The talk is based on a joint work with Matteo Penegini.

Ana Victoria Quedo, IMPA, Brasil and University of Poitiers, France, and Daniela Paiva, IMPA, Brasil *Automorphisms of quartic surfaces and Cremona transformations*

Given a smooth quartic K3 surface $S \subset \mathbb{P}^3$, Gizatullin was interested in which automorphisms of S are induced by Cremona transformations of P^3. Later on, Oguiso answered it for some interesting examples and he posed the following natural question:

Is every automorphism of finite order of any smooth quartic surface $S \subset \mathbb{P}^3$ induced by a Cremona transformation?

In this talk, we will give a negative answer to this question by constructing a family of smooth quartic K3 surfaces S_n with Picard number two such that $Aut(S_n) = D_\infty$ together with an involution of S_n that is not derived by any element of $Bir(P^3)$. More precisely, we will prove that no element of $Aut(S_n)$ is induced by an element of $Bir(\mathbb{P}^3)$.

Xavier Roulleau, Univeristy of Angers, France *Kummer structures and construction of automorphisms on some generalised Kummer surfaces*

Joint with with A. Sarti. Let B be an abelian surface. Suppose that there exists an order 3 automorphism j_A acting on A symplectically. The quotient surface A/J_A has nine cuspidal singularities. The minimal resolution $Km(A, j_A)$ of A/J_A is called a generalized Kummer surface; it contains a configuration $\mathcal{C}$ of type $9A_2$, which means nine disjoint pairs of (-2)-curves C, C' such that $CC' = 1$, which curves are above the singularities. A Kummer structure on a generalized Kummer surface X is an isomorphism class of pairs (B, j_B) where B is an abelian surface and j_B an order 3 automorphism such that the associated generalized Kummer surface $Km(B, j_B)$ is isomorphic to X. One wants to understand these generalized Kummer structures. Thanks to results of Barth, the data of a Kummer structure is equivalent to the orbit under the automorphism group of A of nine disjoint A_2-configurations. Using the Pell-Fermat equation and some geometric results, we construct new $9A_2$-configurations $\mathcal{C}'$ on a given generalised Kummer surface. Using the Torelli theorem, one obtains the following alternative: a) Either there exists no automorphism sending $\mathcal{C}$ to $\mathcal{C}'$, then we get a non-trivial Kummer structure, b) Or there exists an automorphism sending $\mathcal{C}$ to $\mathcal{C}'$, and then we can describe the action of such automorphism on the Néron-Severi lattice and obtain in that way non-trivial elements of the automorphism group of the generalized Kummer surface.

Matthias Schuett, University of Hannover, Germany *Finite symplectic automorphism groups of supersingular K3 surfaces*

Finite symplectic automorphism groups on complex K3 surfaces have famously been classified by Mukai, based on work of Nikulin. I will report on joint work with Hisanori Ohashi which aims to extend this to positive characteristic. While the tame case retains a close connection to the Mathieu group M_{23} (extending work of Dolgachev-Keum), we will develop a unified approach using symmetries of the Leech lattice which also covers all wild cases.

2.2 Symplectic varieties

Lucas Li Bassi, University of Poitiers, France and University of Milan, Italy *Cubic threefolds and hyperkahler manifolds with a non-symplectic automorphism*

In a paper of 2019 Boissière–Camere–Sarti prove that there exists an isomorphism between the moduli space of smooth cubic threefolds and the moduli space of hyperkähler fourfolds of $K3^{[2]}$-type with a non-symplectic automorphism of order three, whose invariant lattice is generated by a class of square 6. Then, the authors study the degeneration of the automorphism along a generic nodal hyperplane, proving the existence of a birational map from the locus of nodal cubic threefolds to some moduli space of hyperkähler fourfolds of $K3^{[2]}$-type with a non-symplectic automorphism of order three belonging to a different family. In the exceptional locus of this morphism there are several interesting cubic threefolds. Therefore, I will present a generalization of their result to some non-generic nodal cases.

Chiara Camere, University of Milan, Italy *Prym fibrations as irreducible symplectic varieties*

In this talk, I will first recall the construction of Lagrangian fibrations by Prym varieties starting from a K3 surface with a non-symplectic involution. Then I will discuss a criterion to ensure that the normalizazion of such a fibration is an irreducible symplectic variety. This is joint work in progress with E. Brakkee, A. Grossi, L. Pertusi, G. Saccà and A. Viktorova.

Claudio Onorati, University of Milan, Italy *Monodromy of singular symplectic moduli spaces of sheaves*

I will report on a joint work in progress with A. Perego and A. Rapagnetta on the monodromy group of singular moduli spaces of sheaves on K3 surfaces. This extends previous results by Markman (smooth case) and myself (case in which there exists a symplectic desingularisation).

Pablo Quezada, Universidad de La Frontera, Chile *IHS Manifolds of $K3^{[2]}$-type with an action of $Z_3^4 : A_6$*

In this talk we will study IHS manifolds of $K3^{[2]}$-type with a symplectic action of $\mathbb{Z}_3^4 : A_6$, the symplectic group with the biggest order, and such that they also admit a non-symplectic automorphism. We will show that there are three IHS manifolds that satisfies this, with two of them admitting two possible actions, and particularly we will show that there is a unique IHS manifold of $K3^{[2]}$-type with finite automorphism group of order 174960, the biggest possible order for the automorphism group of a IHS manifold of $K3^{[2]}$-type. This is a joint work with Paola Comparin and Romain Demelle.

Gregory Sankaran, University of Bath, UK *Towards projective models of generalised Kummers*

Several projective constructions of families of holomorphic symplectic varieties are known but they are all of K3 type. I will describe work in progress towards constructing an explicit family of holomorphic symplectic 4-folds of generalised Kummer type.

Davide Cesare Veniani, University of Stuttgart *Symplectic rigidity of O'Grady's manifolds*

Mukai classified all symplectic groups of automorphisms of K3 surfaces as possible subgroups of one of the Mathieu groups. Since then, the proof of Mukai's theorem has been simplified using lattice theoretical techniques, and extended to higher dimensional hyperkähler manifolds. In two joint works with L. Giovenzana, A. Grossi and C. Onorati, we studied possible cohomological actions of symplectic automorphisms of finite order on the two sporadic deformation types found by O'Grady in dimension 6 and 10. In particular, we showed that all symplectic automorphisms in dimension 10 are trivial. In my talk, I will explain the connection between our proof and the sphere packing problem.

2.3 Moduli and parameters spaces

Samuel Boissière, University of Poitiers, France *The Fano variety of lines of a cuspidal cyclic cubic fourfold*

In the framework of the compactification of the moduli spaces of prime order non-symplectic automorphisms of irreducible holomorphic symplectic manifolds, a key question is to understand the geometry of limit automorphisms. Starting from a nodal degeneration of cubic threefolds, the general member of the family of Fano varieties of lines of the triple covering branched over the cubic is an IHS manifold equipped with the automorphism induced by the covering. It degenerates to a variety whose singular locus is a K3 surface. I will present recent results obtained in collaboration with Chiara Camere and Alessandra Sarti that explain how the geometry of this K3 surface permits to define a limit automorphism in a suitable moduli space parametrizing pairs of IHS manifolds with automorphism.

Noam D. Elkies, Harvard University, USA *The $W(E_6)$-invariant quintic fourfold and other moduli spaces*

Building on the work of Shioda and Shioda-Usui on "excellent families" of rational elliptic surfaces, we study several related families of rational elliptic surfaces with two additive fibers. In each case the moduli space is a complement of hyperplanes in a hypersurface S, which is also the moduli space of suitably polarized K3 surfaces X obtained by quadratic base change ramified at the additive fibers. We illustrate this with the example of additive fibers of types II and IV, where S is the quintic fourfold invariant under the Weil group of E_6, and X is a quartic surface with a 40_{12} configuration of lines. Other examples includes cases where S is the Segre cubic or Igusa quartic threefold, the self-dual sextic fourfold with $W(D_6)$ automorphism, and new moduli spaces of dimension 5 and 6.

Klaus Hulek, Univeristy of Hannover, Germany *Ball quotients and moduli spaces*

Many moduli spaces can be described as ball quotients. Examples include the Deligne-Mostow varieties, moduli of cubic surfaces and certain moduli spaces of lattice-polarized K3 surfaces. Here I will discuss the geometry of some of these examples, including their topology and different (partial) resolutions. I will also comment on the relationship with the Minimal Model Program.

Andreas Leopold Knutsen, University of Bergen, Norway *Severi varieties of Enriques surfaces*

Given a (smooth) projective (complex) surface S and a complete linear (or algebraic) system of curves on S, one defines the Severi varieties to be the (possibly empty) subvarieties parametrizing nodal curves in the linear system, for any prescribed number of nodes. These were originally studied by Severi in the case of the projective plane. Afterwards, Severi varieties on other surfaces have been studied, mostly rational surfaces, K3 surfaces and abelian surfaces, often in connection with enumerative formulas computing their degrees. Interesting questions are nonemptiness, dimension, smoothness and irreducibility of Severi varieties. Whereas it is know that a general primitively polarized K3 surface contains nodal curves of every possible geometric genus g, and that the dimension of the corresponding Severi varieties are precisely g, very little has been known on special K3 surfaces, such as for instance the ones admitting an Enriques involution, that is, a fixed point free involution, so that their quotients are Enriques surfaces. In this talk I will present recent results about Severi varieties on Enriques surfaces, obtained with Ciliberto, Dedieu and Galati.

2.4 Physics

Tyler Kelly, University of Birmingham, UK *Applying exoflops to Calabi-Yau varieties*

Landau-Ginzburg (LG) models consist of the data of a quotient stack X and a regular complex-valued function W on X. Here, geometry is encapsulated in the singularity theory of W. One can find that LG models are deformations of many Calabi-Yau varieties in some sense. For example, if the Calabi-Yau is a hypersurface in a smooth projective variety Z cut out by a polynomial f, then one can take X to be the canonical bundle of Z with function W=uf, where u is the bundle coordinate—when the hypersurface is smooth, the critical locus of uf will indeed just be the hypersurface. Exoflops were introduced by Aspinwall as a way to effectively find new birational models of the quotient stack to get new geometries. They effectively create new GIT problems of partial compactifications of X, expanding the tractable birational geometries related to Z. We will explain this technique, provide some foundational results about this, and then provide some new applications proven recently for Calabi-Yau varieties with nontrivial scaling symmetry groups. This talk contains joint work with D. Favero (UMinn), C. Doran (Bard), A. Malter (Birmingham).

Andreas Malmendier, Utah State University, USA *On 2–elementary K3 surfaces and string dualities*

Inose surfaces (associated with Kummer surfaces of Jacobians of smooth genus-two curves) provide a purely geometric interpretation for a certain duality in string theory. Building on this foundation, I will explain how algebraic K3 surfaces obtained from abelian varieties provide a fascinating arena for string compactification and string dualities as they are not-trivial spaces, but are sufficiently simple to analyze most of their properties in detail. I will then describe recent results where we used families of lattice polarized K3 surfaces of Picard rank 10, 14, and 16 to provide a geometric interpretation, called geometric two-isogeny, for the so-called F-theory/heterotic string duality in eight dimensions with up to 4 Wilson lines. This is joint work with A. Clingher.

2.5 Arithmetic

Cecilia Salgado, University of Groningen, The Netherlands *Non–thin rank jumps for elliptic K3 surfaces*

We discuss recent progress on the variation of the Mordell-Weil rank in families of elliptic curves over number fields. In the case of elliptic K3 surfaces, we show, under certain conditions, that the set of fibres for which the Mordell-Weil rank is strictly larger than the generic rank is not thin, as a subset of the base of the fibration. This is based on joint work with Hector Pasten.

Ursula Whitcher, American mathematical society, USA *Hypergeometric decomposition of symmetric K3 pencils*

We study the hypergeometric functions associated to five one-parameter deformations of K3 quartic hypersurfaces in projective space, each admitting a symplectic group action. We match the Picard–Fuchs differential equations to factors of the zeta function, and we write the result in terms of global L-functions. We obtain a complete, explicit description of the motives for these pencils in terms of hypergeometric motives. This talk describes joint work with Charles Doran, Tyler Kelly, Adriana Salerno, Steven Sperber, and John Voight.

3 Outcome of the Meeting

During this workshop, participants were provided with a conducive environment for fostering meaningful interactions and knowledge exchange. The scheduled sessions served as the main platform for sharing ideas and expertise, but equally valuable were the informal exchanges that took place during the long coffee breaks and during meal times. These less structured settings allowed participants to engage in deeper conversations and build professional connections. In particular, the shifted meal schedules allowed participants to interact with other mathematicians with whom they might never have met otherwise.

Attendees, recognizing shared interests and complementary skill sets, expressed eagerness to collaborate on specific projects or initiatives. The workshop played a pivotal role in the formation of at least two new collaborations, though there may have been others, which were not reported.

All in all, the conference is best summarized by the testimonial of Klaus Hulek:
"The topic of the workshop was very interesting. I enjoyed the talks and learned from them. This will be helpful for my future research. I appreciated that the organizers made a major effort to also have more junior speakers. The level of female participation was very high (and much above comparable conferences)."