Dynamics of Hénon maps: real, complex and beyond

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1 Overview of the Field

Hénon maps are discrete-time dynamical systems in dimension 2 and are one of the most studied examples of dynamical systems exhibiting chaotic behavior. For example, a quadratic Hénon map can be written as

$$(x, y) \mapsto (ay + x^2 + c, ax)$$

where a and c are given parameters.

This family of maps was introduced by Hénon in the real setting as a simplified model of the Poincaré section of the Lorenz model. For the classical map (that is $(x, y) \mapsto (y - 1.4x^2 + 1, 0.3x)$) an initial point of the plane will either approach a set of points known as the Hénon strange attractor, or diverge to infinity. The Hénon attractor is a fractal, smooth in one direction and a Cantor set in the other direction. This attractor has received great attention, notably around the question of its persistence for nearby parameters (in a measurable sense) in the breakthrough work of Benedicks and Carleson in [14], a full combinatorial proof being given by Berger and Yoccoz in [24].

In higher dimensional discrete holomorphic dynamics, that is the study of iteration of a holomorphic self-map of a complex manifold of dimension at least two, interest in the dynamics of polynomial automorphisms of $\mathbb{C}^2$, and in particular of Hénon maps, arose at the end of the 1980's, thanks to the fundamental work of Friedland and Milnor [34], who proved that every polynomial automorphism of $\mathbb{C}^2$ is affinely conjugate to either an affine map, an elementary map, or a finite compositions of generalized Hénon maps, usually called complex Hénon maps. Hénon maps were also studied thanks to the development of complex dynamics in one variable since Hénon maps appear as a natural generalization of quadratic polynomials (by taking $a \to 0$). In the last forty years, the study of the dynamics of polynomial automorphisms of $\mathbb{C}^2$ has been and continues to be an extremely active area with a productive interaction between questions of dynamics and complex analysis. An intriguing aspect is that even if the maps themselves are simple to write down, the dynamical questions have led to a rather elaborate theory, thanks to the pioneering results of Friedland and Milnor [34], Hubbard and Obertse-Vorth [36], Bedford and Smillie [9], Fornæss and Sibony [35], followed by Lyubich [8], Dujardin [33] and many others. The results obtained have been most successful/complete in the case of dissipative maps (that is $|a| > 1$), while the understanding in the conservative case (that is $|a| = 1$) remains less developed.

In the past decade, deep connections have been found between holomorphic dynamics and arithmetic geometry, particularly through non-archimedean geometry. For example, Dujardin and Favre partially solved the dynamical Manin-Mumford problem for Hénon maps using both height theory and Pesin theory [29]. Such connections have led to significant progress in classical questions in holomorphic dynamics, as well as to new conjectures and discoveries in arithmetic dynamics. In arithmetic dynamics, Hénon maps have proven to be a key object, as for instance in the construction of dynamical height to understand arithmetic phenomena. More precisely, the introduction of height functions for Hénon maps over number fields and function fields by Silverman [53] and Kawaguchi [40] lead to interesting analogs of the Northcott property (see also [37]), the Kawaguchi-Silvermann conjecture on arithmetic degrees, or (complex)-equidistribution results using arithmetic geometry.

The richness of this topic has lead to many striking results and to the introduction of new methods in this areas of dynamics. It has also led to new developments, as in the study of the dynamics of transcendantal diffeomorphisms of the complex plane, and higher dimensional dynamics, just to mention a few examples. The most recent striking result in this area is the recent proof of the existence of wandering domain for complex Hénon maps in 2023 by Berger and Biebler [18], where they mix techniques from real and complex dynamics.

Nevertheless, we organizers felt the four above mentioned perspectives in studying Hénon maps remain compartmentalized with too few interactions between the experts from the different areas of real dynamics, meromorphic complex dynamics, transcendental complex dynamics, and arithmetic dynamics. For instance, while ergodic theory of complex Hénon maps is well understood and relies heavily on pluripotential theory, its efficiency fails for transcendental diffeomorphism due to the lack of compactness. Similarly, whereas several bifurcation phenomena have been exhibited and studied (Newhouse phenomena, blenders, parabolic implosion), we still do not have measures and currents which would give a measurable meaning to this phenomena.

2 Recent Developments and Open Problems

There have been some really exciting new developments in the study of Hénon maps in the past few years, some very recent. This workshop has seen some talks were those developments were exposed by participants. More precisely:

- Bera and Verma recently proved Bedford's conjecture on uniform attracting basins in $\mathbb{C}^2$ ([17]).
- Avila, Lyubich and Zhang have recently constructed a set in the complex Hénon family with a stable homoclinic tangency. This is remarkable since it works for any degree (in particular 2).
- Bianchi and Dinh recently proved the Central Limit Theorem for complex Hénon maps by showing the exponential decay of all order ([19]).
- Boroński and Štimac recently proved the Pruning front conjecture for Hénon maps with positive entropy ([22]).
- Crovisier, Lyubich, Pujals and Yang recently developed a renormalization scheme for two dimensional maps with zero entropy. This allowed them to prove a priori bounds for such systems.

We also had a problem session, chaired by Abate, and here are the questions raised by the participants[1].

2.1 Bedford's open problems

Quasi-hyperbolicity vs uniform hyperbolicity Let f be a holomorphic mapping, and let $\mathcal{S}$ denote its set of (periodic) saddle points.

Condition 1 For each $p \in \mathcal{S}$, there is a metric on E^u_p which is uniformly expanded by Df_p. (This means that there exists $\kappa > 1$ such that for each p, there is a metric $\|\cdot\|_j$ on $E^u_{f^j(p)}$ so that for nonzero $v \in E^u_{f^j(p)}$, we have

$$\|Dfv\|_{j+1} \geq \kappa \|v\|_j$$

[1] We are grateful to the participants for sharing their notes

for each j.)

Condition 2 The unstable manifolds $W^u(p)$ satisfy the proper, locally bounded area condition (see [12]).

In general, each unstable manifold is uniformized by an entire map $\xi_p : \mathbf{C} \to W^u(p) \subset \mathbf{C}^2$ with $\xi_p(0) = p$. If f is a Hénon map, then there is a Green function G^+, and we may use this to define a normalization, where we replace $\xi_p(\zeta)$ by $\hat{\xi}_p(\zeta) = \xi_p(\alpha\zeta)$ so that

$$\max_{|\zeta|\le 1} G^+(\hat{\xi}_p(\zeta)) = 1 \tag{$*$}$$

Condition 3 The normalized maps $\{\hat{\xi}_p : p \in \mathcal{S}\}$ are a normal family of entire mappings.

Note A consequence of Condition 3 is that for all $x \in \overline{\mathcal{S}}$ we may use the sets $W^u(x) := \xi_x(\mathbf{C})$ as unstable manifolds.

For complex Hénon maps, conditions 1, 2, and 3 are equivalent (see [12]), and in case one/all of them hold, we say that f is *quasi-expanding*. Further, f is said to be *quasi-hyperbolic* if both f and f^{-1} are quasi-expanding.

Note The following Theorem from [5] asserts that the difference between quasi-hyperbolicity and (uniform) hyperbolicity is caused by the occurrence of tangencies: *A quasi-hyperbolic map is uniformly hyperbolic if and only if there is no tangency between $W^u(x_1)$ and $W^s(x_2)$ for any x_1, x_2 in the closure of $\mathcal{S}$.*

The answers to Questions 1–5 are all "yes" in the uniformly hyperbolic case, so we ask whether this is true also in the quasi-hyperbolic case.

Question 1 If f is quasi-hyperbolic, then is $\mathrm{int}(K^+)$ the union of a finite number of basins of sink orbits?

Question 2 If f is quasi-hyperbolic, is $J = J^*$?

Question 3 If f is quasi-hyperbolic, then is there no wandering Fatou component?

Question 4 If f is quasi-hyperbolic, do the unstable slices satisfy a John-type condition (as in [11])?

Question 5 If f is quasi-hyperbolic and dissipative, and if J is connected, do the external rays land at J? Is J a finite quotient of the real solenoid?

Surface automorphisms If the map f is an automorphism of a compact, complex surface X, then Conditions 1 and 2 are meaningful, but we do not have a Green function G^+. However, if the dynamical degree of f is $\lambda > 1$, then there is an expanded current T^+ with $f^*T^+ = \lambda T^+$. In this case, we can replace the condition $(*)$ with a condition involving the mass of a slice of the expanded current T^+:

$$\mathrm{Mass}(T^+|_{\xi_p(|\zeta|<1)}) = 1 \tag{$*'$}$$

Now we consider the Condition $3'$, which is Condition 3, but with the normalization $(*')$.

Questions 6 Is Condition $3'$ equivalent to 1 and 2? And do Questions 1, 2, 3 above hold for quasi-hyperbolic surface automorphisms?

Real maps Let us now suppose that f is a real surface automorphism, that is: f and X are both defined over the reals, so we have an invariant restriction $f_{\mathbf{R}}$ to the real points $X_{\mathbf{R}}$. Recall from the Hénon case, that if the entropy of $f_{\mathbf{R}}$ is the same as the entropy of f, then f is quasi-hyperbolic (see [12]). The paper [30] discusses real surface automorphisms with maximal entropy in this sense.

Question 7 If f is a real surface automorphism such that the entropy of $f_{\mathbf{R}}$ is the same as the entropy of f, does it follow that f is quasi-hyperbolic?

2.2 Ishii's open problems: Parameter loci for the Hénon family

Connectedness locus Consider the complex Hénon family:

$$f_{c,b}(x,y) \longmapsto (x^2+c-by, x),$$

where $(c,b) \in \mathbb{C}^2$ is a parameter.[2] Let $J_{c,b}$ be the Julia set of $f_{c,b}$ (we let $J_{c,0}$ be the Julia set of $p(z) = z^2+c$). The *connectedness locus* of $f_{c,b}$ is defined as

$$\mathcal{M} = \big\{(c,b) \in \mathbb{C}^2 : J_{c,b} \text{ is connected}\big\}.$$

In my talk I proposed

Conjecture 1. *$\mathcal{M}$ is disconnected.*

It has been shown that $\mathcal{M} \cap \mathbb{R}^2$ is disconnected [3], which partially supports the conjecture above.

Horseshoe locus We say that $f_{c,b}$ is a *complex hyperbolic horseshoe* if $J_{c,b}$ is a hyperbolic set for $f_{c,b}$ and the restriction $f_{c,b} : J_{c,b} \to J_{c,b}$ is topologically conjugate to the full 2-shift. The *complex hyperbolic horseshoe locus* is defined as

$$\mathcal{H}_{\mathbb{C}} = \big\{(c,b) \in \mathbb{C}^2 : f_{c,b} \text{ is a complex hyperbolic horseshoe}\big\}.$$

One can see that $\mathcal{H}_{\mathbb{C}}$ is not simply connected since the monodromy representation:

$$\rho : \pi_1(\mathcal{H}_{\mathbb{C}}) \longrightarrow \mathrm{Aut}(\{0,1\}^{\mathbb{Z}})$$

of the fundamental group of $\mathcal{H}_{\mathbb{C}}$ (with the base-point at $(c,b) = (-4,0)$) to the group of shift-commuting automorphisms of $\{0,1\}^{\mathbb{Z}}$ is surjective (see, e.g., [13, 1]).

Question 1. *Is the locus $\mathcal{H}_{\mathbb{C}}$ connected?*

For $(c,b) \in \mathbb{R}^2$, we can consider the restriction of $f_{c,b}$ to $\mathbb{R}^2$ and we can analogously define the *real hyperbolic horseshoe locus* $\mathcal{H}_{\mathbb{R}} \subset \mathbb{R}^2$. Main Corollary in [2] shows that $\mathcal{H}_{\mathbb{R}}$ is connected and simply connected (see also [10]).

Isentropes In this section we take $(c,b) \in \mathbb{R}^2$ and denote by $f_{c,b}|_{\mathbb{R}^2}$ the restriction $f_{c,b}|_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$. Let $h_{\mathrm{top}}(f_{c,b}|_{\mathbb{R}^2})$ be the topological entropy of the real Hénon map $f_{c,b}|_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$. For every $0 \le \alpha \le \log 2$, the *isentrope* is defined as

$$\mathcal{E}_\alpha = \big\{(c,b) \in \mathbb{R}^2 : h_{\mathrm{top}}(f_{c,b}|_{\mathbb{R}^2}) = \alpha\big\}.$$

In a topological term, monotonicity of the topological entropy of the real Hénon map $f_{c,b}|_{\mathbb{R}^2}$ can be formulated as

Question 2 (van Strien?)**.** *Is the isentrope $\mathcal{E}_\alpha$ connected for any $0 \le \alpha \le \log 2$?*

Compare [43]. Main Corollary in [2] shows that the maximal one $\mathcal{E}_{\log 2}$ is connected and simply connected (see also [10]).

There are several articles concerning lower bounds for topological entropy of real Hénon maps, e.g., [47, 46]. Among others, the paper [46] rigorously showed that $h_{\mathrm{top}}(f_{c,b}|_{\mathbb{R}^2}) > 0.46469$ for the classical Hénon's parameter, and this bound is believed to be close to optimal. For upper bounds, the paper [8] has shown that $h_{\mathrm{top}}(f_{c,b}|_{\mathbb{R}^2}) < \log 2$ iff the Julia set of $f_{c,b}$ (as a complex dynamical system) is not contained in $\mathbb{R}^2$. However, as far as I know, there is no algorithm which provides rigorous (non-trivial) upper bounds.

Problem 1. *Construct an algorithm to compute a rigorous upper bound for the topological entropy of a real Hénon map $f_{c,b}|_{\mathbb{R}^2}$.*

Probably the only existing formula for (non-trivial) upper bound is given in [56]. However, according to Yomdin himself, the bound in the current form is far from sharp and would not give non-trivial ones.

[2] We include the case $b = 0$ to simplify the presentation.

2.3 Buff's open problems: Hénon maps tangent to the identity

Consider the Hénon map $H_2 : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$H_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x + y^2 \end{pmatrix}.$$

The origin is a fixed point and $H_2^{\circ 2}$ is tangent to the identity at the origin.

Note that H_2 restricts to an orientation reversing diffeomorphism $H_2 : \mathbb{R}^2 \to \mathbb{R}^2$. The dynamics in $\mathbb{R}^2$ is well understood. There is an analytic map $\phi_2 : \mathbb{R} \to \mathbb{R}^2$ such that

$$H_2 \circ \phi_2(t) = \phi_2(t+1) \text{ and } \phi_2(t) \sim \left(\frac{-2}{t}, \frac{-2}{t} \right) \text{ as } t \to +\infty.$$

The curve $\phi_2(\mathbb{R})$ is invariant by H_2 and with $\phi_2(\mathbb{R})$, every orbit converges to the origin in $\mathbb{R}^2$. Outside the origin and $\phi_2(\mathbb{R})$, every orbit diverges to infinity (see Figure 1).

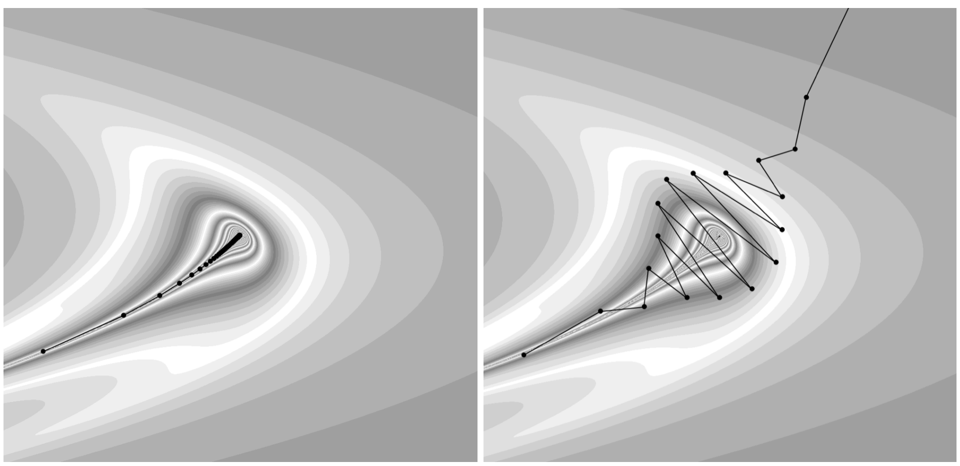

Figure 1: Left: an orbit converging to the origin. Right: an orbit diverging to infinity.

Question 3. *Can we describe the dynamics of H_2 near the origin in $\mathbb{C}^2$?*

Before specifying this question, let us consider the Hénon map $H_3 : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$H_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x + y^3 \end{pmatrix}.$$

This Hénon map also preserves $\mathbb{R}^2$ and the dynamics in $\mathbb{R}^2$ is also completely understood. There is an analytic map $\phi_3 : \mathbb{R} \to \mathbb{R}^2$ such that

$$H_3 \circ \phi_3(t) = -\phi_3(t+1) \text{ and } \phi_3(t) \sim \left(\frac{1}{\sqrt{t}}, \frac{-1}{\sqrt{t}} \right) \text{ as } t \to +\infty.$$

The curves $\phi_3(\mathbb{R})$ and $-\phi_3(\mathbb{R})$ are exchanged by H_3. Within those curves, every orbit converges to the origin. Outside those curves and the origin, every orbit diverges to infinity (see Figure 2 left).

Set $\omega = e^{\mathrm{i}\frac{\pi}{8}}$ so that $\omega^9 = -\omega$ and consider the real planes $\Pi_1 \subset \mathbb{C}^2$ and $\Pi_2 \subset \mathbb{C}^2$ defined by

$$\Pi_1 = \left\{ \begin{pmatrix} \omega x \\ \omega^3 y \end{pmatrix} \ : \ x \in \mathbb{R},\ y \in \mathbb{R} \right\} \text{ and } \Pi_2 = \left\{ \begin{pmatrix} \omega^3 x \\ \omega y \end{pmatrix} \ : \ x \in \mathbb{R},\ y \in \mathbb{R} \right\}.$$

Observe that H_3 exchanges the planes Π_1 and Π_2:

$$H_3 \begin{pmatrix} \omega x \\ \omega^3 y \end{pmatrix} = \begin{pmatrix} \omega^3 y \\ \omega(x - y^3) \end{pmatrix} \text{ and } H_3 \begin{pmatrix} \omega^3 x \\ \omega y \end{pmatrix} = \begin{pmatrix} \omega y \\ \omega^3(x + y^3) \end{pmatrix}.$$

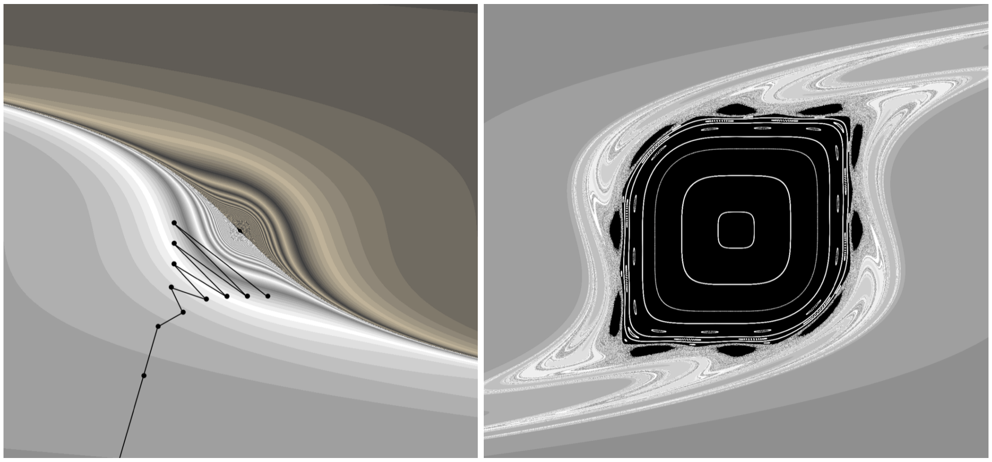

Figure 2: Left: The points are colored according to whether $x+y$ tends to $+\infty$ (dark grey) or to $-\infty$ (light grey). Right: The dynamics of $H_3^{\circ 2}:\Pi_1\to\Pi_1$ exhibits KAM phenomena.

The dynamics of $H_3^{\circ 2}:\Pi_1\to\Pi_1$ is much more complex than that of $H_3:\mathbb{R}^2\to\mathbb{R}^2$ (see Figure 2 right).

The second iterate of H_3 is tangent to the identity at the origin. More precisely

$$H_3^{\circ 2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^3 \\ x^3 \end{pmatrix} + \mathcal{O}\big(\|x,y\|^4\big).$$

It follows that near the origin, the orbits of $H_3^{\circ 2}$ shadow the orbits of the vector field

$$\vec{v}_3 = y^3\partial_x + x^3\partial_y.$$

The vector field is a Hamiltonian vector field. It is tangent to the level curves of the function

$$\Phi_3 = x^4 - y^4.$$

Note that

$$\Phi_3\begin{pmatrix} \omega x \\ \omega^3 y \end{pmatrix} = \mathrm{i}(x^4+y^4)$$

so that the intersection of the level curves of Φ_3 with the real plane Π_1 are topological circles. Those topological circles are invariant by the flow of the vector field $\vec{v}_3$. It follows from the theory of Kolmogorov-Arnold-Moser that in any neighborhood of the origin, there is a set of positive Lebesgue measure of topological circles which are invariant by H_3 and on which H_3 is analytically conjugate to a rotation $\mathbb{R}/\mathbb{Z}\ni t\mapsto t+\theta\in\mathbb{R}/\mathbb{Z}$ with bounded type rotation number $\theta\in(\mathbb{R}\smallsetminus\mathbb{Q})/\mathbb{Z}$. Those invariant circles are separated by small saddle cycles and small elliptic cycles. The analytic conjugacies extend to complex neighborhoods of $\mathbb{R}/\mathbb{Z}$ in $\mathbb{C}/\mathbb{Z}$. This proves that H_3 has lots of Herman rings.

Coming back to our initial problem, observe that the second iterate of H_2 is also tangent to the identity at the origin with

$$H_2^{\circ 2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2 \\ x^2 \end{pmatrix} + \mathcal{O}\big(\|x,y\|^3\big).$$

It follows that near the origin, the orbits of $H_2^{\circ 2}$ shadow the orbits of the vector field

$$\vec{v}_2 = y^2\partial_x + x^2\partial_y.$$

The vector field is also a Hamiltonian vector field. The vector field $\vec{v}_2$ is tangent to the level curves of the function

$$\Phi_2 = x^3 - y^3.$$

We can no longer apply the theory of Kolmogrov-Arnold-Moser since there is no invariant real-plane on which the level curves of Φ_2 are topological circles. However, we may wonder whether the complex dynamics of H_2 exhibits KAM phenomena.

We say that H_2 has small cycles if for any neighborhood U of the origin $\mathbf{0}$ in $\mathbb{C}^2$, there exists a cycle of H_2 which is entirely contained in $U \smallsetminus \{\mathbf{0}\}$.

Question 4. *Does H_2 have small cycles?*

Question 5. *Does H_2 have both small saddle cycles and small elliptic cycles?*

We say that H_2 has a Herman ring with rotation number $\theta \in (\mathbb{R} \smallsetminus \mathbb{Q})/\mathbb{Z}$ if there exists an annulus $V = \big\{z \in \mathbb{C}/\mathbb{Z} \,:\, \mathrm{Im}(z) < h\big\}$ with $h > 0$, a holomorphic map $\phi : V \to \mathbb{C}^2$, and an integer $n \geq 2$ such that

$$\forall z \in V, \quad H_2^{\circ n} \circ \phi(z) = \phi(z + \theta).$$

Question 6. *Does H_2 have a Herman ring?*

If the answer is yes, we may consider the set $\Theta \in (\mathbb{R} - \mathbb{Q})/\mathbb{Z}$ of rotation numbers θ such that H_2 has a Herman ring with rotation θ.

Question 7. *Does Θ have positive Lebesgue measure? More precisely, is 0 a Lebesgue density point of Θ?*

We believe that the answers to the previous questions are all affirmative. Regarding the following question, we do not have an opinion.

Question 8. *Assume H_2 has a Herman ring with bounded type rotation number θ. Is it possible to find parameters $a \in \mathbb{D} \smallsetminus \{0\}$ arbitrarily close to 1 such that the dissipative Hénon map $H : \mathbb{C}^2 \to \mathbb{C}^2$ defined by*

$$H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ay \\ x + y^2 \end{pmatrix}.$$

has a Herman ring with rotation number θ?

2.4 Dujardin's open problem

For polynomials and rational maps in dimension 1, there is a well-known list of exceptional examples whose Julia sets and dynamical properties are unexpectedly regular: Chebychev polynomials, monomial mappings and Lattès examples. For polynomial automorphisms of $\mathbb{C}^2$ with positive entropy, it is expected that no such exceptional example exists. In this direction, Bedford and Kim proved in [6, 7] that for such an automorphism, neither J^+ nor J^- can be a smooth C^1 submanifold, nor a semi-analytic set.

The question is about a quantitative reinforcement of the Bedford-Kim theorem. Let $f_{a,c} : (z, w) \mapsto (z^2 + c - aw, az)$ be the standard complex Hénon map. For (a, c) close to $(0, 0)$, $f_{a,c}$ is a small perturbation of the monomial map $(z^2, 0)$, whose Julia set is smooth, and in this case $J^+_{a,c}$ is a topological 3-manifold.

Question 9. *Give an asymptotic expansion of the Hausdorff dimension of $J^+_{a,c}$ as (a, c) tends to $(0, 0)$. In particular is there a uniform lower bound of $\dim(J^+_{a,c})$ of the form $\dim(J^+_{a,c}) \geq 3 + h(a)$ with $h(a) > 0$ in the neighborhood of $c = 0$?*

Note that for (a, c) close to $(0, 0)$, $f_{a,c}$ is uniformly hyperbolic so it is likely that $(a, c) \mapsto \dim(J^+_{a,c})$ is real analytic.

2.5 Štimac's open problems: Questions about the symbolic dynamics of the real Hénon and Lozi maps

Let us consider the real Hénon and Lozi maps $H_{a,b}, L_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$,

$$H_{a,b}(x, y) = (1 + y - ax^2, bx),\; L_{a,b}(x, y) = (1 + y - a|x|, bx),$$

respectively.

In [44], the authors developed the kneading theory for the Lozi maps $L_{a,b}$ with $(a,b) \in \mathcal{M}$, where $\mathcal{M} = \{(a,b) \in \mathbb{R}^2 : b > 0,\ a\sqrt{2} - b > 2,\ 2a + b < 4\}$ is the Misiurewicz set of parameters (for details see [44] and [45]). A kneading sequence $\bar{k}$ is defined as the itinerary of a turning point T, where turning points are points of the transversal intersections of the x-axis and the unstable manifold W^u of the fixed point X of the attractor. The kneading set $\mathfrak{K} = \{\bar{k}^n : n \in \mathbb{Z}\}$ is the set of all kneading sequences $\bar{k}^n$, $n \in \mathbb{Z}$, and every kneading sequence $\bar{k} = \bar{k}^n$, for some $n \in \mathbb{Z}$, has the following form:

$$\bar{k} = {}^{\infty}{+}w \pm \cdot \overrightarrow{k_0},$$

where ${}^{\infty}{+} = \cdots + {+}{+}$, $w = w_0 \dots w_m$, for some $m \in \mathbb{N}_0$, $\overrightarrow{k_0} = k_0 k_1 k_2 \dots$, $w_0 = -$, $k_0 = +$, $w_i, k_j \in \{-,+\}$ for $i = 1, \dots m$ and $j \in \mathbb{N}$, and the dot shows where the 0th coordinate is. Here for $\pm$ one can substitute any of $+$ and $-$.

In [44], the authors prove that $\mathfrak{K}$ characterizes all itineraries of all points of the attractor of $L_{a,b}$. The proof is given in two steps. First, the authors prove that $\mathfrak{K}$ characterizes all itineraries of all points of the unstable manifold W^u (what is called W^u-admissibility):

Theorem 1. *A sequence ${}^{\infty}{+}\overrightarrow{p_n}$, where $\overrightarrow{p_n} = p_n p_{n+1} \dots$ such that $p_n = -$ for some $n \in \mathbb{Z}$, is W^u-admissible if and only if for every kneading sequence ${}^{\infty}{+}w \pm \cdot \overrightarrow{k_0}$, such that $w = p_n p_{n+1} \dots p_{n+m}$ for some $m \in \mathbb{N}_0$, we have $\sigma^{m+2}(\overrightarrow{p_n}) \preceq \overrightarrow{k_0}$, where $\preceq$ is the parity-lexicographical ordering.*

Now that it is known which sequences are W^u-admissible, and since the symbolic space is equipped with the product topology, the following theorem holds:

Theorem 2. *A sequence $\bar{p} = \dots p_{-2} p_{-1} \cdot p_0 p_1 \dots$ is admissible if and only if for every positive integer n there is a W^u-admissible sequence $\bar{q} = \dots q_{-2} q_{-1} \cdot q_0 q_1 \dots$ such that $p_{-n} \dots p_n = q_{-n} \dots q_n$.*

Question 10. *Describe the set of kneading sequences $\mathfrak{K}$.*

In [38], Ishii developed formulas that can be used to obtain a relationship between parameters a, b, a turning point $T = (x_T, 0)$ of the Lozi map $L_{a,b}$, and its itinerary $\bar{k}$ (that is a kneading sequence of $L_{a,b}$). This relationship is $p(a,b,\bar{k}) = x_T = q(a,b,\bar{k})$, where $p = p(a,b,\bar{k})$ is given in formula [37, (4.2)] and $q = q(a,b,\bar{k})$ is given in formula [37, (4.3)]. Therefore, every kneading sequence $\bar{k}$ gives an equation

$$p(a,b,\bar{k}) = q(a,b,\bar{k}). \tag{1}$$

Numerical experiments show that if one has two kneading sequences, $\bar{k}^0$ of the rightmost turning point T_0 and $\bar{k}^{-1}$ of the leftmost turning point T_{-1}, and if these two turning points lie in the stable manifolds of some periodic points with small periods, then it is possible to calculate a and b from the corresponding two equations, implying that these two kneading sequences govern all the other kneading sequences of the kneadig set $\mathfrak{K}$.

Example Let $\bar{k}^0 = {}^{\infty}{+} \pm \cdot + - - +^{\infty}$ and $\bar{k}^{-1} = {}^{\infty}{+} - \pm \cdot (+-)^{\infty}$. The equation $p(a,b,\bar{k}^0) = q(a,b,\bar{k}^0)$ reads

$$a^4 - 6a^2 - 4a + 4b^2 + a^2 b + (a^3 + 2a - ab)\sqrt{4b + a^2} = 0, \tag{2}$$

and the equation $p(a,b,\bar{k}^{-1}) = q(a,b,\bar{k}^{-1})$ reads

$$\frac{4(-a^2 - 2b^2 + 2b + a\sqrt{a^2 - 4b})}{a - 2b - \sqrt{a^2 - 4b}} - \left(2 + a - \sqrt{a^2 + 4b}\right)\left(3a - \sqrt{a^2 + 4b}\right) = 0. \tag{3}$$

Using the "NSolve" command of Wolfram Mathematica produces a unique solution to this system of equations in the region $a \in [1,2]$, $b \in [0,1]$. This solution is approximately $a = 1.655319602968851744592$, $b = 0.2765071079677260998121$, see Figure 3.

Question 11. *Whether two kneading sequences determine a unique pair of parameters (a,b), and in that way govern all the other kneading sequences of $\mathfrak{K}$ in general?*

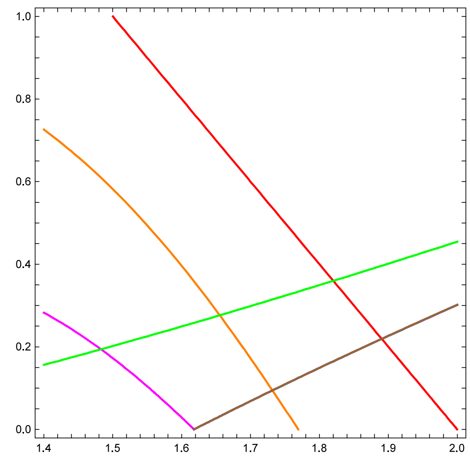

Figure 3: Graph of (2) is in orange and of (3) in green. Graph of (1) for $\bar{k} = {}^{\infty}+ \pm \cdot (+--)^{\infty}$ is in magenta and for $\bar{k} = {}^{\infty}+ - \pm \cdot (+--)^{\infty}$ in brown. Graph of the line $2a + b = 4$, that is a boundary line of the Misiurewicz set, is in red.

Very recently, in [22], the authors developed the kneading theory for the Hénon maps $H_{a,b}$ within the set of parameters $\mathcal{WY}$ (for details see [22] and [55]).

Question 12. *Describe the set of kneading sequences $\mathfrak{K}$ of the Hénon map $H_{a,b}$, with $(a, b) \in \mathcal{WY}$.*

Question 13. *Whether two kneading sequences of the Hénon map $H_{a,b}$, with $(a, b) \in \mathcal{WY}$, determine a unique pair of parameters (a, b), and in that way govern all the other kneading sequences in $\mathfrak{K}$ of $H_{a,b}$?*

2.6 Favre's open problems: towards algebraic geometry and complex differential geometry

Hénon maps and the group of polynomial automorphisms of $\mathbb{A}^2$. Any Hénon map $H_{a,P}(x, y) := (ay + P(x), x)$ where $a \in \mathbb{C}^*$ and $P \in \mathbb{C}[x]$ is a polynomial of degree $d \geq 2$ induces a polynomial automorphism of the affine plane $\mathbb{A}^2_{\mathbb{C}}$. It is a theorem by Jung [39] that the group $\mathrm{Aut}[\mathbb{A}^2_{\mathbb{C}}]$ of polynomial automorphisms $\mathbb{A}^2_{\mathbb{C}}$ is generated by affine transformations and triangular maps of the form $E_P(x, y) := (x, y + P(y))$. In a celebrated article [34], Friedland and Milnor have deduced from Jung's theorem the following dichotomy. Pick any $f \in \mathrm{Aut}[\mathbb{A}^2_{\mathbb{C}}]$. Then either f is conjugated to a composition of Hénon maps $H_{a_1,P_1} \circ \cdots \circ H_{a_k,P_k}$ and $\deg(f^n) \asymp (d_1 \cdots d_k)^n$ for all n; or $\deg(f^n)$ remains bounded and f is conjugated to an affine map or to a triangular map.

Let us state the following general problem in vague terms.

Question 14. *Describe the growth type of the sequence $\{\deg(f^n)\}$ for any given $f \in \mathrm{Aut}[\mathbb{A}^d_{\mathbb{C}}]$, $d \geq 3$.*

Very few results are known. Recall that Russakovski and Shiffman observed that

$$\deg(f^{n+m}) \leq \deg(f^n) \deg(f^m)$$

for all $n, m \geq 0$ so that the following limit $\lambda(f) := \lim_n \deg(f^n)^{1/n}$ exists. We refer to it as the dynamical degree of f.

The case of cubic automorphisms on $\mathbb{A}^3_{\mathbb{C}}$, and the case of automorphisms obtained as a composition of an affine transformation and a triangular one were considered by Blanc and Van Santen [20, 21]. Their computations lead them to formulate the following intriguing problem. A weak Perron number is an algebraic integer $\lambda \geq 1$ such that all its Galois conjugates satisfy $|\mu| \leq \lambda$.

Question 15. *Is the dynamical degree of any polynomial automorphism of $\mathbb{A}^d_{\mathbb{C}}$ equal to a weak Perron number of degree $\leq d - 1$?*

It has been proven in [27], that $\lambda(f)$ is an algebraic number of degree ≤ 6 for any $f \in \mathrm{Aut}[\mathbb{A}^3_{\mathbb{C}}]$.
The case $\lambda(f) = 1$ is also particularly interesting.

Question 16. *Suppose $f \in \mathrm{Aut}[\mathbb{A}^d_{\mathbb{C}}]$ satisfies $\lambda(f) = 1$. Is it true that $\deg(f^n) \asymp n^k$ for some $k \in \mathbb{N}$? Moreover, if $k \geq 1$, does f preserve any rational fibration?*

Urech proved that $\deg(f^n)$ tends to infinity whenever it is unbounded, see [54]. His result was made stronger in [26]: there exists a universal function $\sigma \colon \mathbb{N} \to \mathbb{N}$ such that $\limsup \sigma = \infty$ and $\deg(f^n) \geq \sigma(n)$. Cantat and Xie raised the following weaker form of the previous problem.

Question 17. *Suppose that $\lambda(f) = 1$, and $\deg(f^n)$ is unbounded for some $f \in \mathrm{Aut}[\mathbb{A}^d_{\mathbb{C}}]$. Does there exist $C > 0$ such that $\deg(f^n) \geq Cn$?*

Let $\mathrm{Tame}(3)$ be the subgroup generated by affine and triangular transformations of $\mathbb{C}^3$. A theorem of Shestakov and Umirbaev [51] states that $\mathrm{Tame}(3)$ is a strict subgroup of $\mathrm{Aut}[\mathbb{A}^3_{\mathbb{C}}]$ (as opposed to the 2-dimensional situation). Decisive progress on the structure of $\mathrm{Tame}(3)$ was recently made by Lamy and Przytycky [41], who constructed a $\mathrm{CAT}(0)$-complex $\mathcal{C}$ over which $\mathrm{Tame}(3)$ acts by isometries.

Question 18. *Is it possible to characterize those $f \in \mathrm{Tame}(3)$ for which $\lambda(f) = 1$ in terms of their action on $\mathcal{C}$?*

Hénon maps and compact complex manifolds. Consider any composition of Hénon maps $f = H_{a_1,P_1} \circ \cdots \circ H_{a_k,P_k}$ as in the previous section. Recall that f extends to the projective plane $\mathbb{P}^2_{\mathbb{C}}$ as a birational map contracting the line at infinity to the super-attracting fixed point $p = [1 : 0 : 0]$. The topology of the basin of attraction of this point $\Omega(f) := \{q \in \mathbb{C}^2,\ f^n(q) \to p\}$ has been explored by Hubbard and Oberste-vorth [36]. They also observed that the map f acts properly discontinuously on $\Omega(f)$ so that the space of orbits $S(f) := \Omega(f)/\langle f\rangle$ is naturally a complex surface. One can then construct a compact complex surface $\check{S}(f)$ having an isolated normal singularity at a point $0 \in \check{S}(f)$ such that $\check{S}(f) \setminus \{0\}$ is biholomorphic to $S(f)$. The minimal resolution of $\check{S}(f)$ is a compact complex surface $\bar{S}(f)$ that is non-kähler, contains no smooth rational curve of self-intersection -1, and satisfies $b_1(\bar{S}(f)) = 1$. In Kodaira's classification of surfaces [4], it belongs to the class VII_0 which is arguably the most mysterious class of compact complex surfaces. Dloussky and Oeljeklaus [32] have investigated when these surfaces carry global holomorphic vector fields.

Question 19. *Pick $f, g \in \mathrm{Aut}[\mathbb{A}^2_{\mathbb{C}}]$ such that $\lambda(f), \lambda(g) \geq 2$. Suppose that $\bar{S}(f)$ is biholomorphic to $\bar{S}(g)$. Does this imply that f is conjugated to g?*

It follows from [Fa, Proposition 2.1] that under the preceding assumptions, f and g have the same degree and the same jacobian.

Surfaces $\bar{S}(f)$ carry only finitely many rational curves that are all contracted to the singular point $0 \in \check{S}(f)$. One can also prove that it carries a unique holomorphic foliation which is induced by the Levi flats of the Green function G_+ on $\Omega(f)$.

An interesting feature of the complex surface $\bar{S}(f)$ is that it admits a family of charts (U_i, ϕ_i) where U_i is an open cover of $\bar{S}(f)$, and $\phi_i \colon U_i \to \mathbb{C}^2$ is an open immersion such that ϕ_{ij} is the restriction to an open domain of a birational self-map of $\mathbb{P}^2_{\mathbb{C}}$.

A complex manifold which admits a holomorphic atlas whose transition maps are restriction of birational maps of $\mathbb{P}^d_{\mathbb{C}}$ is said to carry a birational structure.

The following problem is extracted from [31].

Conjecture 2. *Any non-kähler compact complex surface admits a birational structure.*

This question is extremely challenging, and reduces to the case of VII_0 surfaces. One can ask whether any deformation S of a surface $\bar{S}(f)$ associated to a polynomial automorphism f as above admits a birational structure. This is true when the surface satisfies $b_2(S) \leq 3$, see [31].

Analogs of the construction of $\bar{S}(f)$ have been explored by Oeljeklaus and Renaud in [48] for some quadratic polynomial automorphisms of $\mathbb{C}^3$, and further expanded by Ruggiero [49, Chapter 4]. A polynomial automorphism $f \in \mathrm{Aut}[\mathbb{A}^3_{\mathbb{C}}]$ is said to be regular if the indeterminacy locus $I(f)$ of its extension to $\mathbb{P}^3_{\mathbb{C}}$ is disjoint from $I(f^{-1})$. This notion was introduced by Sibony in [52]. Let $\Omega(f)$ be the basin of attraction of $I(f^{-1})$: this is an open f-invariant set over which f acts properly discontinuously. As above, denote by $S(f)$ the quotient space $\Omega(f)/\langle f\rangle$.

Problem 2. *Let $f \in \mathrm{Aut}[\mathbb{A}^d_{\mathbb{C}}]$ be any regular polynomial automorphism.*

1. *Prove that one can find a compact complex manifold $\bar{S}(f)$ and an open immersion $S(f) \subset \bar{S}(f)$ such that the complement $\bar{S}(f) \setminus S(f)$ is a divisor.*
2. *Prove that $\bar{S}(f)$ is unique up to bimeromorphism.*
3. *Describe complex objects on $\bar{S}(f)$ (analytic subvarieties, vector fields, holomorphic foliations, positive closed currents,...). Compute its deformation space.*

It is unclear how to extend this construction to a larger class of polynomial automorphisms of $\mathbb{A}^3_{\mathbb{C}}$. However when $\lambda(f)^2 > \lambda(f^{-1})$ Dang and Favre [27] have built an invariant valuation which suggests the following question.

Question 20. *Let $f \in \mathrm{Aut}[\mathbb{A}^3_{\mathbb{C}}]$ be any polynomial automorphism such that $\lambda(f)^2 > \lambda(f^{-1})$. Prove the existence of a projective compactification X of $\mathbb{A}^3_{\mathbb{C}}$ such that the induced birational map $f\colon X \dashrightarrow X$ admits a super-attracting fixed point p on the divisor at infinity.*

Once such a compactification has been found, one can consider the basin of attraction Ω of the point p and try to construct a compactification of the space of f-orbits in Ω as above.

3 Presentation Highlights

We asked to four experts to give introductory lectures of 1,5 hours on the different aspects of Hénon maps covered by our workshop:
Eric Bedford *Complex aspects of Hénon maps*
Pierre Berger *Real aspects of Hénon maps*
Patrick Ingram *Arithmetic aspects of Hénon maps*
Miriam Benini *Transcendental aspects of Hénon maps*

These introductory lectures gave overviews on each aspect, focusing on fundamental techniques and pointing out open questions. We think that this allowed our participants to have a better understanding of the specific problems of the above mentioned aspects of Hénon maps.

Other talks were more specialized and 45 minutes long. We give here below the list in alphabetical order.
Sayani Bera *Uniform attracting basins and Hénon maps.* We discuss the construction and the convergence properties of Green's function, for (appropriate) non-autonomous families of Hénon maps. Further, we discuss the proof of Bedford's conjecture on uniform attracting basins in $\mathbb{C}^2$, which is an important consequence of the above results.

Finally if time permits we discuss the generalisation of the above technique to any n-dimensions, using weak shift-like maps.
Fabrizio Bianchi *Every complex Hénon map satisfies the Central Limit Theorem* We show that every Hölder observable satisfies the Central Limit Theorem (CLT) with respect to the measure of maximal entropy of every complex Hénon map. The proof is based on a general criterion to ensure the validity of the CLT, valid for abstract measurable dynamical systems. This is a joint work with Tien-Cuong Dinh.
Jeffrey Diller *Equidistribution without stability* I will discuss recent work with Roland Roeder concerning rational surface maps that we call 'toric'. These include all monomial maps and some recent examples with Bell and Jonsson of rational maps that have transcendental first dynamical degrees. Our main theorem is an equidistribution result for preimages of curves that works precisely when the map in question is 'bad' in the sense that it cannot be birationally conjugated to an algebraically stable map.

Romain Dujardin *Structure of hyperbolic Hénon maps with disconnected Julia sets.* For hyperbolic polynomial automorphisms with disconnected Julia sets, we give a description of the connected components of J which is strongly reminiscent of the classical Branner-Hubbard theory for 1D polynomials. This is joint work with Misha Lyubich.

Charles Favre *Pseudo-automorphisms vs automorphisms* (J.w with Alexandra Kuznetsova.) A bimeromorphic map f on a polarized family of abelian varieties over the unit disk is always conjugated to a pseudo-automorphism. We shall discuss when it is possible to find a relatively ample model over which f becomes regular.

Denis Gaidashev *Renormalization for real Henon-like maps* We will give an overview of renormalization for real Henon-like maps, both dissipative and area-preserving, concentrating on rigidity and geometry of invariant sets.

Thomas Gauthier *Stability and rigidity for marked points in families of Hénon maps* In this talk I will discuss a joint work with Gabriel Vigny where we study stability properties of marked points in algebraic families of Hénon maps. A marked point is forward-stable (resp. backward-stable) if its forward orbit (seen as a sequence of holomorphic functions of the parameter) is equicontinuous. In analogy with the case of families of endomorphisms of the projective spaces, we show that a marked point is forward-stalbe if and only if it is backward-stable if and only if it is persistently periodic, provided the family is algebraic and not trivial.

Reimi Irokawa *Non-archimedean and hybrid dynamics of Hénon maps* To study of the meromorphic degeneration of dynamics, the theory of hybrid spaces, established by Boucksom and Jonsson, is known to be a strong tool. In this talk, we apply this theory to the dynamics of Hénon maps; for a famiy of Héenon maps$\{H_t\}_t$ parametrized by a unit punctured disk meromorphically degenerating at the origin, we show that as $t \to 0$, the family of the invariant measures $\{\mu_t\}$ "weakly converges" to the measure on the Berkovich affine plane which is naturally defined by the family $\{H_t\}_t$, in the sense of the theory of hybrid spaces.

Yutaka Ishii *On the connectedness locus for the complex Henon family.* In this talk we discuss the following two topics:

(i) the construction of a hyperbolic complex Henon map with connected Julia set which is non-planar,

(ii) a numerical algorithm to detect the disconnectivity of Julia sets.

As a consequence, we obtain a certain topological property of the connectedness locus for the complex Henon family. This is a joint work in progress with Zin Arai (Tokyo Institute of Technology).

Mattias Jonsson *The complex dynamics of birational maps defined over number fields*

Rohini Ramadas *Degenerations and irreducibility problems.* Per_n is a (nodal) Riemann surface parametrizing degree-2 rational functions with an n-periodic critical point. The n-th Gleason polynomial G_n is a polynomial in one variable with $\mathbb{Z}$-coefficients, whose roots correspond to degree-2 polynomials with an n-periodic critical point (i.e. to the period-n components of the Mandelbrot set). Two long-standing open questions are: (1) Is Per_n connected? (2) Is G_n is irreducible over $\mathbb{Q}$? We show that if G_n is irreducible over $\mathbb{Q}$, then Per_n is connected. In order to do this, we find a smooth point with $\mathbb{Q}$-coordinates on a compactification of Per_n. This smooth $\mathbb{Q}$-point represents a special degeneration of degree-2 rational maps, and as such admits an interpretation in terms of tropical geometry.

Sonja Štimac *The pruning front conjecture, wandering domains and a classification of Hénon maps in the presence of strange attractors.* I will talk about recent results on topological dynamics of Hénon maps obtained in joint work with Jan Boroński. For a parameter set generalizing the Benedicks-Carleson parameters (the Wang-Young parameter set) we obtain the following: The existence of wandering domains (answering a question of Lyubich, Martens and van Strien); The pruning front conjecture (due to Cvitanović, Gunaratne, and Procacci); A kneading theory (realizing a conjecture by Benedicks and Carleson); A classification: two Hénon maps are conjugate on their strange attractors if and only if their sets of kneading sequences coincide, if and only if their folding patterns coincide. The classification result relies on a further development of the authors' recent inverse limit description of Hénon attractors in terms of densely branching trees. (Joint work with Jan P. Boroński)

Johan Taflin *Bifurcations of complex horseshoes.* A horseshoe is a very classic example in hyperbolic dynamics. In this talk, I will consider holomorphic dynamical systems where bifurcations of complex horseshoes appear. In this context, there is a link between - a counting problem of tangencies and - the variations of the Lyapunov exponent of the maximum entropy measure. Moreover, we will see that this counting problem is easy to solve explicitly.

Raluca Tanase *Critical loci in the Hénon family*

Michael Yampolsky *KAM-renormalization and Herman rings for 2D maps* I will describe extending the renormalization horseshoe we have recently constructed with N. Goncharuk for analytic diffeomorphisms of the circle to their small two-dimensional perturbations. As one consequence, Herman rings with rotation numbers of bounded type survive on a codimension one set of parameters under small two-dimensional ”Henon-like” perturbations.
Jonguk Yang *Infinitely Renormalizable Unimodal Hénon Maps* We generalize the renormalization theory of unimodal intervals maps to dimension two, so that it can be applied to the study of real Hénon maps. The key step is to identify the higher-dimensional analog of critical orbits using Pesin theory. As the main result, we will give an explicit and complete description of the non-uniform partial hyperbolicity of Hénon maps that are properly dissipative, infinitely renormalizable and unimodal.
Zhiyuan Zhang *Newhouse phenomenon in the complex Hénon family* In a work in progress with Avila and Lyubich, we show that there are maps in the complex Hénon family with a stable homoclinic tangency. This is due to a new mechanism on the stable intersections between two dynamical Cantor sets generated by two classes of conformal IFSs on the complex plane.

4 Progress of the meeting

4.1 Hybrid, pros and cons

We had planned a purely in-person workshop, since being in the same place and far from usual life duties is the best way to ensure that all participants can discuss together and focus on the topic of the workshop. Finally, this meeting was hybrid with 35 in-person participants and approximatively 35 online-participants. Moreover, four talks were given by online-participants (Benini, Bera, Berger, Zhang). This was a practical solution as it allowed us to have talks given by participants who had to cancel their in-person attendance at the last minute, or by participants who could not have made it otherwise.

All lectures and discussions were streamed live and all talks were recorded in order to be available to those in a less comfortable time zone or who could not attend a talk for other reasons.

We had 10 to 25 participants simultaneously connected to zoom, depending on the talks. We believe this is a success. Nevertheless, while during few talks some online participants asked questions to the speaker, most of the time there was essentially no interaction between in-person participants and online-participants. In our proposal, we had suggested ways to allow all participants to be able to attend in the best way possible, but these were not realizable with the facilities we had at Juniper hotel. BIRS provided technical assistance on site at Juniper hotel, but we would have needed: a better lecture room, better cameras and a better audio system to allow a really seamless communication between virtual and in-person participants. Several virtual-participants told us that they could not hear well questions from the audience, and sometimes even the speaker. Moreover, the cameras were not self-moving, which means that sometimes the speaker had moved to another board while the camera was still filming in another direction. Organizers cannot take care also of this aspects, as they are interested in being able to listen to the talks, like the other participants.

4.2 Young participants, EDI activities and career development

We put all our efforts in having a diverse group of participants and we planned to have at least one third of participants from under-represented groups in STEM: women, gender-fluid persons, persons with disabilities, and some minority groups like Blacks or African Americans, Hispanics or Latin Americans, and American Indians or Alaska Natives. We had 12 female in-person participants out of 35 and 8 female virtual-participants out of 35.

We were particularly vigilant to have as many young participants as possible and encouraged them to speak. Despite all our efforts, we only had 5 junior in-person participants and 5 junior virtual-participants.

We would like to stress on the fact that all those participants who needed a visa to enter Canada could not come in-person. This had a huge impact on the diversity of our group (we had no in-person participant from China and India for example) and in particular affected almost all the junior participants who could not come in-person.

We tried to increase the the visibility of minorities and junior participants, and in particular we had 6 female speakers and two junior speakers. Moreover, prior to the workshop, we asked to all participants (and in particular those not giving a talk) whether they could give a “5-minutes talk” (in the spirit of what is done at MSRI now SLMSI) where they would give a short and accessible presentation on their research interests. We held the “5-minutes talk” session on the first day of the workshop and it was a success. In particular, four junior participants (3 PhD students and a post-doc) presented their research.

We also held a Career Development Panel destined to junior participants and open to everyone. This career development session has been especially fruitful and interesting, and junior participants told us they appreciated it. Several questions and some issues came up and we are happy to write that we could address and solve most of them.

4.3 Juniper Hotel

The meeting was not held at BIRS usual facility in Banff but at Juniper hotel. Though the accommodation was more than ok, the meeting room was not well equipped for a math conference: we lacked large boards and the disposition of the room was far from optimal, as we would have preferred a classical class room. We have already mentioned above the technical problems for holding a hybrid meeting. Moreover, we lacked of space for discussions in small groups. The location is also kind of isolated and, even if it is possible to walk to Banff, everyone mentioned that the “highway-along” walk to go to town is far from being pleasant.

Another suggestion from some of the participant is to try to have a “conference dinner”. These social events are indeed very useful to increase cohesion and connections.

Nevertheless, the feedback we got from our participants is excellent. Most of them enjoyed their week and we are eager to see in the near future some new progress on Hénon maps coming out from the discussions started at Banff.

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