Categorifying zeta and L-functions

Andrew J. Kobin

ajkobin@emory.edu

Moduli, Motives & Bundles

September 22, 2022

Joint work with Jon Aycock

Introduction

Based on

A Primer on Zeta Functions and Decomposition Spaces

Andrew Kobin

Many examples of zeta functions in number theory and combinatorics are special cases of a construction in homotopy theory known as a decomposition space. This article aims to introduce number theorists to the relevant concepts in homotopy theory and lays some foundations for future applications of decomposition spaces in the theory of zeta functions.

Comments: 23 pages; minor changes and additional references added
Subjects: **Number Theory (math.NT)**; Algebraic Geometry (math.AG); Category Theory (math.CT)
MSC classes: 11M06, 11M38, 14G10, 18N50, 16T10, 06A11, 55P99
Cite as: arXiv:2011.13903 **[math.NT]**
(or arXiv:2011.13903v2 **[math.NT]** for this version)

and

Categorifying quadratic zeta functions

Jon Aycock, Andrew Kobin

The Dedekind zeta function of a quadratic number field factors as a product of the Riemann zeta function and the L-function of a quadratic Dirichlet character. We categorify this formula using objective linear algebra in the abstract incidence algebra of the division poset.

Comments: 27 pages
Subjects: **Number Theory (math.NT)**
MSC classes: 11M06, 11M41, 18N50, 06A11, 16T10
Cite as: arXiv:2205.06298 **[math.NT]**
(or arXiv:2205.06298v1 **[math.NT]** for this version)

as well as work in progress with J. Aycock.

Introduction

Motivation: How are different zeta and L-functions related? Do they fit into a common framework?

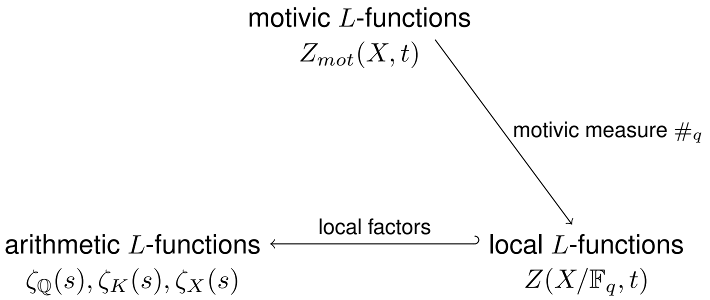
motivic L-functions
$Z_{mot}(X, t)$

arithmetic L-functions
$\zeta_{\mathbb{Q}}(s), \zeta_K(s), \zeta_X(s)$

local L-functions
$Z(X/\mathbb{F}_q, t)$

Introduction

Motivation: How are different zeta and L-functions related? Do they fit into a common framework?



Arithmetic Functions

A good starting place is always with the Riemann zeta function:

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Arithmetic Functions

This is an example of a Dirichlet series:

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

We will focus on the formal properties of Dirichlet series.

The coefficients $f(n)$ assemble into an **arithmetic function** $f : \mathbb{N} \to \mathbb{C}$. (Think: F is a generating function for f.)

Then $\zeta_{\mathbb{Q}}(s)$ is the Dirichlet series for $\zeta : n \mapsto 1$.

Arithmetic Functions

The space of arithmetic functions $A = \{f : \mathbb{N} \to \mathbb{C}\}$ form an algebra under convolution:

$$(f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).$$

This identifies the algebra of formal Dirichlet series with A:

$$\begin{aligned} A &\longleftrightarrow DS(\mathbb{Q}) \\ f &\longmapsto F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \\ f * g &\longmapsto F(s)G(s) \\ \zeta &\longmapsto \zeta_{\mathbb{Q}}(s) \end{aligned}$$

Arithmetic Functions over Number Fields

For a number field $K/\mathbb{Q}$, its zeta function can be written

$$\zeta_K(s) = \sum_{\mathfrak{a} \in I_K^+} \frac{1}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{\#\{\mathfrak{a} \mid N(\mathfrak{a}) = n\}}{n^s}$$

where $I_K^+ = \{\text{ideals in } \mathcal{O}_K\}$ and $N = N_{K/\mathbb{Q}}$.

Arithmetic Functions over Number Fields

As with $\zeta_{\mathbb{Q}}(s)$, we can formalize certain properties of $\zeta_K(s)$ in the algebra of arithmetic functions $A_K = \{f : I_K^+ \to \mathbb{C}\}$ with

$$(f * g)(\mathfrak{a}) = \sum_{\mathfrak{b} | \mathfrak{a}} f(\mathfrak{b}) g(\mathfrak{a}\mathfrak{b}^{-1}).$$

This admits an algebra map to $DS(\mathbb{Q})$:

$$\begin{aligned} N_* : A_K &\longrightarrow A \cong DS(\mathbb{Q}) \\ f &\longmapsto \left(N_* f : n \mapsto \sum_{N(\mathfrak{a})=n} f(\mathfrak{a}) \right) \\ \zeta &\longmapsto N_* \zeta \leftrightarrow \zeta_K(s) \end{aligned}$$

Arithmetic Functions over Number Fields

As with $\zeta_{\mathbb{Q}}(s)$, we can formalize certain properties of $\zeta_K(s)$ in the algebra of arithmetic functions $A_K = \{f : I_K^+ \to \mathbb{C}\}$ with

$$(f * g)(\mathfrak{a}) = \sum_{\mathfrak{b}|\mathfrak{a}} f(\mathfrak{b})g(\mathfrak{a}\mathfrak{b}^{-1}).$$

This admits an algebra map to $DS(\mathbb{Q})$:

$$\begin{aligned} N_* : A_K &\longrightarrow A \cong DS(\mathbb{Q}) \\ f &\longmapsto \left(N_* f : n \mapsto \sum_{N(\mathfrak{a})=n} f(\mathfrak{a}) \right) \\ \zeta &\longmapsto N_*\zeta \leftrightarrow \zeta_K(s) \end{aligned}$$

Interpretation: N allows us to build Dirichlet series for arithmetic functions over K.

Varieties over Finite Fields

Let X be an algebraic variety over $\mathbb{F}_q$. Its point-counting zeta function is the power series

$$Z(X,t) = \exp\left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right]$$

Historically, this is called a zeta function because it has:

- a product formula $Z(X,t) = \prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}}$
- a functional equation
- an expression as a *rational function*
- a Riemann hypothesis which is a theorem!

Varieties over Finite Fields

Once again, we can formalize certain properties of $Z(X,t)$ in an algebra of arithmetic functions.

Let $Z_0^{\text{eff}}(X)$ be the set of effective 0-cycles on X, i.e. formal $\mathbb{N}_0$-linear combinations of closed points of X, written $\alpha = \sum m_x x$.

We say $\beta \leq \alpha$ if $\beta = \sum n_x x$ with $n_x \leq m_x$ for all $x \in |X|$.

Let $A_X = \{f : Z_0^{\text{eff}}(X) \to \mathbb{C}\}$ be the algebra of arithmetic functions with

$$(f * g)(\alpha) = \sum_{\beta \leq \alpha} f(\beta) g(\alpha - \beta).$$

We call the distinguished element $\zeta : \alpha \mapsto 1$ the *zeta function* of X.

Varieties over Finite Fields

Let $A_X = \{f : Z_0^{\text{eff}}(X) \to \mathbb{C}\}$ be the algebra of arithmetic functions with

$$(f * g)(\alpha) = \sum_{\beta \leq \alpha} f(\beta) g(\alpha - \beta).$$

This time, there's no map to $DS(\mathbb{Q})$...

Varieties over Finite Fields

Let $A_X = \{f : Z_0^{\text{eff}}(X) \to \mathbb{C}\}$ be the algebra of arithmetic functions with

$$(f * g)(\alpha) = \sum_{\beta \leq \alpha} f(\beta) g(\alpha - \beta).$$

This time, there's no map to $DS(\mathbb{Q})$... but there's a map to the algebra of formal power series:

$$A_X \longrightarrow A_{\operatorname{Spec} \mathbb{F}_q} \cong \mathbb{C}[[t]]$$
$$f \leftrightarrow \sum_{n=0}^{\infty} f(n) t^n$$
$$f \longmapsto \text{“}\deg_*(f)\text{”}$$
$$\zeta \longmapsto \text{“}\deg_*(\zeta)\text{”} \leftrightarrow Z(X, t)$$

What’s really going on?

What's really going on?

A, A_K and A_X are examples of **reduced incidence algebras**, which come from a much more general simplicial framework.

Incidence Algebra of a Poset

Classically, a locally finite poset $(\mathcal{P}, \leq)$ admits an **incidence algebra**, the k-algebra

$$I(\mathcal{P}) = \{k\text{-linear maps } f : \mathrm{Int}(\mathcal{P}) \to k\}$$

with multiplication given by convolution

$$(f * g)([x, y]) = \sum_{z \in [x,y]} f([x, z])g([z, y]).$$

Incidence Algebra of a Poset

Classically, a locally finite poset $(\mathcal{P}, \leq)$ admits an **incidence algebra**, the k-algebra

$$I(\mathcal{P}) = \{k\text{-linear maps } f : \mathrm{Int}(\mathcal{P}) \to k\}$$

with multiplication given by convolution

$$(f * g)([x, y]) = \sum_{z \in [x,y]} f([x, z])g([z, y]).$$

Think: elements in $I(\mathcal{P})$ are like arithmetic functions on the intervals in $\mathcal{P}$.

Reduced Incidence Algebra of a Poset

Definition

The **reduced incidence algebra** of $\mathcal{P}$ is the subalgebra $\widetilde{I}(\mathcal{P}) \subseteq I(\mathcal{P})$ of functions that are constant on isomorphism classes of intervals.

Think: elements in $\widetilde{I}(\mathcal{P})$ are like arithmetic functions on the isomorphism classes of intervals in $\mathcal{P}$.

Reduced Incidence Algebra of a Poset

Definition

The **reduced incidence algebra** of $\mathcal{P}$ is the subalgebra $\widetilde{I}(\mathcal{P}) \subseteq I(\mathcal{P})$ of functions that are constant on isomorphism classes of intervals.

Think: elements in $\widetilde{I}(\mathcal{P})$ are like arithmetic functions on the isomorphism classes of intervals in $\mathcal{P}$.

Example

For the division poset $(\mathbb{N}, |)$, every interval is isomorphic to $[1, n]$ for some n. For $f \in \widetilde{I}(\mathbb{N}, |)$, write $f(n) := f([1, n])$. Then

$$\widetilde{I}(\mathbb{N}, |) \cong DS(\mathbb{Q})$$

$$f \longmapsto \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

Fact: the zeta function $\zeta : [x, y] \mapsto 1$ always lives in $\widetilde{I}(\mathcal{P})$.

Numerical Incidence Algebras

Idea (due to Gálvez-Carrillo, Kock and Tonks): zeta functions don't just come from posets, but from higher homotopy structure.

In this talk: zeta functions come from decomposition sets.

In general: zeta functions come from decomposition spaces.

Numerical Incidence Algebras

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \rightleftarrows S_1 \begin{smallmatrix}\leftarrow\\\rightarrow\\\leftarrow\\\rightarrow\\\leftarrow\end{smallmatrix} S_2 \cdots .$$

Example

A poset $\mathcal{P}$ determines a simplicial set $N\mathcal{P}$ with:

- 0-simplices $=$ elements $x \in \mathcal{P}$
- 1-simplices $=$ intervals $[x, y]$
- 2-simplices $=$ decompositions $[x, y] = [x, z] \cup [z, y]$
- etc.

Numerical Incidence Algebras

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \leftrightarrows S_1 \leftrightarrows S_2 \cdots .$$

Example

More generally, any category $\mathcal{C}$ determines a simplicial set $N\mathcal{C}$ with:

- 0-simplices $=$ objects x in $\mathcal{C}$
- 1-simplices $=$ morphisms $x \xrightarrow{f} y$ in $\mathcal{C}$
- 2-simplices $=$ decompositions $x \xrightarrow{h} y = x \xrightarrow{f} z \xrightarrow{g} y$
- etc.

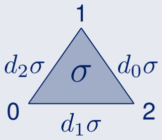
Numerical Incidence Algebras

A certain type of simplicial set called a **decomposition set** defined by Gálvez-Carrillo, Kock and Tonks admits a notion of incidence algebra.

Definition

The **numerical incidence coalgebra** of a decomposition set S is the free k-vector space $C(S) = \bigoplus_{x \in S_1} kx$ with comultiplication

$$\begin{aligned} C(S) &\longrightarrow C(S) \otimes C(S) \\ x &\longmapsto \sum_{\substack{\sigma \in S_2 \\ d_1\sigma = x}} d_2\sigma \otimes d_0\sigma. \end{aligned}$$



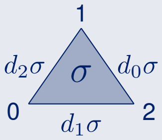
Numerical Incidence Algebras

A certain type of simplicial set called a **decomposition set** defined by Gálvez-Carrillo, Kock and Tonks admits a notion of incidence algebra.

Definition

The **numerical incidence algebra** of a decomposition set S is the dual vector space $I(S) = \mathrm{Hom}(C(S), k)$ with multiplication

$$\begin{aligned} I(S) \otimes I(S) &\longrightarrow I(S) \\ f \otimes g &\longmapsto (f * g)(x) = \sum_{\substack{\sigma \in S_2 \\ d_1\sigma = x}} f(d_2\sigma) g(d_0\sigma). \end{aligned}$$



Numerical Incidence Algebras

In $I(S) = \mathrm{Hom}(C(S), k)$, there is a distinguished element called the **zeta function** $\zeta : x \mapsto 1$.

Numerical Incidence Algebras

In $I(S) = \mathrm{Hom}(C(S), k)$, there is a distinguished element called the **zeta function** $\zeta : x \mapsto 1$.

Key takeaways:

(1) A zeta function is $\zeta \in I(S)$ for some decomposition set S.

(2) Familiar zeta functions like $\zeta_K(s)$ and $Z(X,t)$ are constructed from some $\zeta \in \widetilde{I}(S)$ by pushing forward to another reduced incidence algebra which can be interpreted in terms of generating functions:

$$\text{e.g.} \quad \widetilde{I}(\mathbb{N}, |) \cong DS(\mathbb{Q}), \qquad \text{e.g.} \quad \widetilde{I}(\mathbb{N}_0, \leq) \cong k[[t]].$$

(3) Some properties of zeta functions can be proven in the incidence algebra directly:

$$\text{e.g.} \quad \zeta_{\mathbb{Q}}(s) = \prod_p \frac{1}{1-p^{-s}} \longleftrightarrow \widetilde{I}(\mathbb{N}, |) \cong \bigotimes_p \widetilde{I}(\{p^k\}, |).$$

Numerical Incidence Algebras

Okay, so far: $\zeta_{\mathbb{Q}}(s), \zeta_K(s), Z(X,t)$, etc. lift to the same framework.

Next: how can we get them talking to each other?

Objective Linear Algebra

The construction of $I(S)$ can be generalized further using the formalism of **objective linear algebra** (“linear algebra with sets”):

Numerical	Objective
basis B	set B
vector v	set map $v : X \to B$
matrix M	span $B \xleftarrow{s} M \xrightarrow{t} C$
vector space V	slice category $\mathrm{Set}_{/B}$
linear map with matrix M	linear functor $t_! s^* : \mathrm{Set}_{/B} \to \mathrm{Set}_{/C}$
tensor product $V \otimes W$	$\mathrm{Set}_{/B} \otimes \mathrm{Set}_{/C} \cong \mathrm{Set}_{/B \times C}$

Objective Linear Algebra

The construction of $I(S)$ can be generalized further using the formalism of **objective linear algebra** (“linear algebra with sets”):

Numerical	Objective
basis B	set B
vector v	set map $v : X \to B$
matrix M	span $B \xleftarrow{s} M \xrightarrow{t} C$
vector space V	slice category $\mathrm{Set}_{/B}$
linear map with matrix M	linear functor $t_! s^* : \mathrm{Set}_{/B} \to \mathrm{Set}_{/C}$
tensor product $V \otimes W$	$\mathrm{Set}_{/B} \otimes \mathrm{Set}_{/C} \cong \mathrm{Set}_{/B\times C}$

$\longleftarrow$

To recover vector spaces, take $V = k^B$ and take cardinalities.

Abstract Incidence Algebras

How do we construct $I(S)$ as an "objective vector space"?

Numerical	Objective
basis B	set B
vector space V	slice category $\mathrm{Set}_{/B}$

Abstract Incidence Algebras

How do we construct $I(S)$ as an "objective vector space"?

Numerical	Objective
basis B	set B
vector space V	slice category $\mathrm{Set}_{/B}$
basis S_1	set S_1

Abstract Incidence Algebras

How do we construct $I(S)$ as an “objective vector space”?

Numerical	Objective
basis B	set B
vector space V	slice category $\mathrm{Set}_{/B}$
basis S_1	set S_1
$C(S) =$ free vector space on S_1	slice category $C(S) := \mathrm{Set}_{/S_1}$

Abstract Incidence Algebras

How do we construct $I(S)$ as an "objective vector space"?

Numerical	Objective
basis B	set B
vector space V	slice category $\mathrm{Set}_{/B}$
basis S_1	set S_1
$C(S)$ = free vector space on S_1	slice category $C(S) := \mathrm{Set}_{/S_1}$
dual space $I(S) = \mathrm{Hom}(C(S), k)$	dual space $I(S) := \mathrm{Lin}(\mathrm{Set}_{/S_1}, \mathrm{Set})$

Abstract Incidence Algebras

How do we construct $I(S)$ as an “objective vector space”?

Numerical	Objective
basis B	set B
vector space V	slice category $\mathrm{Set}_{/B}$
basis S_1	set S_1
$C(S) =$ free vector space on S_1	slice category $C(S) := \mathrm{Set}_{/S_1}$
dual space $I(S) = \mathrm{Hom}(C(S), k)$	dual space $I(S) := \mathrm{Lin}(\mathrm{Set}_{/S_1}, \mathrm{Set})$

So an element $f \in I(S)$ is a linear functor $f = t_! s^* : \mathrm{Set}_{/S_1} \to \mathrm{Set}$ represented by a span

$$f = \left(S_1 \xleftarrow{\;s\;} M \xrightarrow{\;t\;} * \right)$$

Abstract Incidence Algebras

So an element $f \in I(S)$ is a linear functor $f = t_! s^* : \mathrm{Set}_{/S_1} \to \mathrm{Set}$ represented by a span

$$f = \left(\begin{array}{ccccc} & & M & & \\ & \swarrow^{s} & & \searrow^{t} & \\ S_1 & & & & * \end{array} \right)$$

Example

The **zeta functor** is the element $\zeta \in I(S)$ represented by

$$\zeta = \left(\begin{array}{ccccc} & & S_1 & & \\ & \swarrow^{\mathrm{id}} & & \searrow & \\ S_1 & & & & * \end{array} \right)$$

Abstract Incidence Algebras

Example

For two elements $f, g \in I(S)$ represented by

$$f = \begin{pmatrix} & M & \\ {\scriptstyle s}\swarrow & & \searrow \\ S_1 & & * \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} & N & \\ {\scriptstyle t}\swarrow & & \searrow \\ S_1 & & * \end{pmatrix}$$

the convolution $f * g \in I(S)$ is represented by

$$(f * g) = \begin{pmatrix} & & & P & & & \\ & & \swarrow & & \searrow & & \\ & S_2 & & & & M \times N & \\ {\scriptstyle d_1}\swarrow & & \searrow{\scriptstyle (d_2, d_0)} & & {\scriptstyle s \times t}\swarrow & & \searrow \\ S_1 & & & S_1 \times S_1 & & & * \end{pmatrix}$$

Abstract Incidence Algebras

Advantages of the objective approach:

- Intrinsic: zeta is built into the object S directly
- General: most* zeta functions can be produced this way
- Functorial: to compare zeta functions, find the right map $S \to T$
- Structural: proofs are categorical, avoiding choosing elements (e.g. computing local factors of zeta functions explicitly is difficult)
- It's pretty fun to prove things!

Quadratic Zeta Functions

For a quadratic number field $K/\mathbb{Q}$, the zeta function $\zeta_K(s)$ satisfies

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s)L(\chi, s)$$

where $L(\chi, s)$ is the L-function attached to the Dirichlet character $\chi = \left(\frac{D}{\cdot}\right)$, where $D =$ disc. of K.

Quadratic Zeta Functions

For a quadratic number field $K/\mathbb{Q}$, the zeta function $\zeta_K(s)$ satisfies

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s)L(\chi, s)$$

where $L(\chi, s)$ is the L-function attached to the Dirichlet character $\chi = \left(\frac{D}{\cdot}\right)$, where $D =$ disc. of K.

Theorem (Aycock–K.)

This formula lifts to an equivalence of linear functors in $\widetilde{I}(\mathbb{N}, |)$:

$$N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+$$

where $N : (I_K^+, |) \to (\mathbb{N}, |)$ is the norm and $\chi^+, \chi^- \in I(\mathbb{N}, |)$.

In the numerical incidence algebra, this becomes

$$N_*\zeta_K = \zeta_{\mathbb{Q}} * (\chi^+ - \chi^-) = \zeta_{\mathbb{Q}} * \chi.$$

Sketch of Proof

$$\boxed{N_* \zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+}$$

Let $S = (\mathbb{N}, |)$ and $T = (I_K^+, |)$, so that $N : T \to S$ induces

$$N_* : \widetilde{I}(T) \longrightarrow \widetilde{I}(S), \quad f \longmapsto \left(N_* f : n \mapsto \sum_{N(\mathfrak{a})=n} f(\mathfrak{a}) \right).$$

Sketch of Proof

$$\boxed{N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+}$$

Each term in the formula is represented by a span:

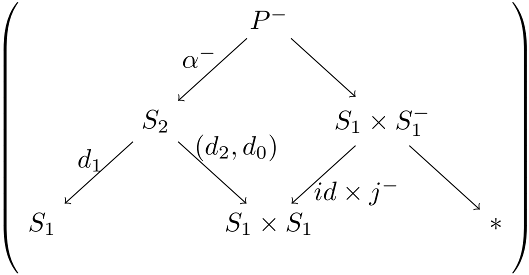
$$N_*\zeta_K = \left(S_1 \xleftarrow{N} T_1 \rightarrow * \right)$$

Sketch of Proof

$$\boxed{N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+}$$

Each term in the formula is represented by a span:

$\zeta_{\mathbb{Q}} * \chi^- =$



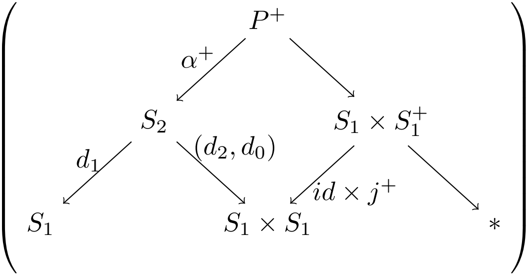
for a certain “vector” $j^- : S_1^- \to S_1$ representing χ^-.

Sketch of Proof

$$\boxed{N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+}$$

Each term in the formula is represented by a span:

$\zeta_{\mathbb{Q}} * \chi^+ =$



for a certain “vector” $j^+ : S_1^+ \to S_1$ representing χ^+.

Sketch of Proof

$$\boxed{N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+}$$

So the formula is an equivalence of the following spans:

$$\left(\begin{array}{ccc} & T_1 \coprod P^- & \\ {\scriptstyle N \sqcup d_1 \circ \alpha^-}\swarrow & & \searrow \\ S_1 & & * \end{array}\right) \cong \left(\begin{array}{ccc} & P^+ & \\ {\scriptstyle d_1 \circ \alpha^+}\swarrow & & \searrow \\ S_1 & & * \end{array}\right)$$

These are shown to be equivalent prime-by-prime and then assembled into the global formula. $\square$

Higher Order Zeta Functions

More generally, for any Galois extension $K/\mathbb{Q}$, $\zeta_K(s)$ factors into a product of L-functions

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s) \prod_{\chi \neq 1} L(\chi, s)$$

where χ are the nontrivial irreducible characters of $\mathrm{Gal}(K/\mathbb{Q})$.

Problem: values of $\chi(n)$ land in μ_n in general, so they can't be categorified with sets.

Higher Order Zeta Functions

More generally, for any Galois extension $K/\mathbb{Q}$, $\zeta_K(s)$ factors into a product of L-functions

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s) \prod_{\chi \neq 1} L(\chi, s)$$

where χ are the nontrivial irreducible characters of $\mathrm{Gal}(K/\mathbb{Q})$.

Solution (in progress with J. Aycock): upgrade to simplicial G-representations ($G = G_{\mathbb{Q}}$).

Higher Order Zeta Functions

Actually, let's go for broke: for a(n admissible) G_K-representation V, we define an "L-functor"

$$L(V) = \left(\begin{array}{ccc} & \bigoplus_{n=0}^{\infty} V_n & \\ \swarrow & & \searrow \\ \bigoplus_{n=0}^{\infty} R_K & & R_K \end{array} \right)$$

($R_K =$ a certain representation ring incorporating Frobenius actions)

Theorem (Additivity)

For two (admissible) G-representations V, W, there is an equivalence

$$L(V \oplus W) \cong L(V) * L(W)$$

in the incidence algebra $I(\mathbb{Q})$ of L-functors of G-representations.

Higher Order Zeta Functions

Actually, let's go for broke: for a(n admissible) G-representation V, we define an "L-functor"

$$L(V) = \left(\begin{array}{ccc} & \bigoplus_{n=0}^{\infty} V_n & \\ \swarrow & & \searrow \\ \bigoplus_{n=0}^{\infty} R_K & & R_K \end{array} \right)$$

($R_K =$ a certain representation ring incorporating Frobenius actions)

Conjecture (Artin Induction)

For a(n admissible) G_K-representation V, there is an equivalence

$$L\left(\mathrm{Ind}_{G_K}^{G} V\right) \approx N_* L(V)$$

where $N_* : I(K) \to I(\mathbb{Q})$ is the pushforward along the norm map and $\approx$ is "trace equivalence".

Elliptic Curves

For an elliptic curve $E/\mathbb{F}_q$, the zeta function $Z(E,t)$ can be written

$$Z(E,t) = \frac{1 - a_q t + qt^2}{(1-t)(1-qt)} = Z(\mathbb{P}^1, t) L(E,t).$$

Theorem (Aycock–K., '22+ϵ)

In the reduced incidence algebra $\widetilde{I}(Z_0^{\mathrm{eff}}(E))$, there is an equivalence of linear functors

$$\pi_* \zeta_E + \zeta_{\mathbb{P}^1} * L(E)^- \cong \zeta_{\mathbb{P}^1} * L(E)^+$$

where $\pi : E \to \mathbb{P}^1$ is a fixed double cover and $L(E)^+, L(E)^- \in I(Z_0^{\mathrm{eff}}(\mathbb{P}^1))$.

Pushing forward to $\widetilde{I}(Z_0^{\mathrm{eff}}(\operatorname{Spec} \mathbb{F}_q)) \cong k[[t]]$, it already reads

$$\pi_{E,*} \zeta_E = \pi_{\mathbb{P}^1,*} \zeta_{\mathbb{P}^1} * L(E).$$

Motivic Zeta Functions

For any k-variety X, $Z_{mot}(X,t) = \sum_{n=0}^{\infty} [\mathrm{Sym}^n X] t^n$ decategorifies to other zeta functions by applying motivic measures (point counting, Euler characteristic, etc.)

Das–Howe ('21) lift $Z_{mot}(X,t)$ to a numerical incidence algebra

$$\widetilde{I}_{mot}(\Gamma^{\bullet,+}(X)) = \prod_{n=0}^{\infty} K_0(\mathtt{Var}_{/\Gamma^n X})$$

where $\Gamma^n X$ are the divided powers of X.

Motivic Zeta Functions

For any k-variety X, $Z_{mot}(X,t) = \sum_{n=0}^{\infty} [\mathrm{Sym}^n X] t^n$ decategorifies to other zeta functions by applying motivic measures (point counting, Euler characteristic, etc.)

Das–Howe ('21) lift $Z_{mot}(X,t)$ to a numerical incidence algebra

$$\widetilde{I}_{mot}(\Gamma^{\bullet,+}(X)) = \prod_{n=0}^{\infty} K_0(\mathtt{Var}_{/\Gamma^n X})$$

where $\Gamma^n X$ are the divided powers of X.

Idea (in progress): lift $Z_{mot}(X,t)$ to an objective incidence algebra $I(\Gamma^{\bullet,+}(X))$ in the category of simplicial k-varieties. Passing to K_0 recovers Das and Howe's construction.

More Dreams

Here are some other things I want to do:

- Study the zeta function of an algebraic stack $\mathcal{X} \to X$ in terms of ζ_X, e.g. over $\mathbb{F}_q$, Behrend defines $Z(\mathcal{X}, t)$ for such a stack.
- Lift motivic L-functions to the objective level and prove formulas, e.g Artin induction.
- Realize archimedean factors of completed zeta functions as elements of abstract incidence algebras, e.g. the factor at ∞ $\zeta_\infty(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$ lives in a certain Hecke algebra.

Key insight: decomposition sets $\rightsquigarrow$ decomposition spaces

Thank you!