Cohomology of stacks of shtukas III

§0 Reminder

Let I be a finite set. Let $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G}^I)$.

$$\begin{array}{ccccc} & & Sht_I & & \mathcal{F}_{I,W} \\ & & \downarrow{\scriptstyle p} & & \\ \overline{\eta}_I & \to & \eta_I & \to & X^I \end{array}$$

generic point

The canonical perverse sheaf $\mathcal{F}_{I,W}$ comes from the geometric Satake equivalence. It is supported on $Sht_{I,W}$, which is a Deligne-Mumford stack locally of finite type, may not be smooth and not be proper.

talk about later

local model of $Sht_{I,W}$: Beilinson-Drinfeld affine grassmanian $Gr_{I,W}$

when W is irreducible, $\mathcal{F}_{I,W} \simeq IC_{Sht_{I,W}}$

We defined complex of cohomology sheaves $\mathcal{H}_{I,W} := Rp_! \mathcal{F}_{I,W}$

degre j cohomology sheaf $\mathcal{H}^j_{I,W} := R^j p_! \mathcal{F}_{I,W}$ over X^I

degre j cohomology group $H^j_{I,W} = \mathcal{H}^j_{I,W}\big|_{\overline{\eta}_I} = H^j_c(Sht_{I,W}|_{\overline{\eta}_I}, \mathcal{F}_{I,W})$

inductive limit of constructible $\overline{\mathbb{Q}}_\ell$-sheaves

$$\mathcal{H}^j_{I,W} = \varinjlim_{\mu} \mathcal{H}^{j,\le\mu}_{I,W} \qquad \mathcal{H}^{j,\le\mu}_{I,W} = R^j\mathfrak{p}_!\left(\mathcal{F}_{I,W}\big|_{\mathrm{Sht}^{\le\mu}_{I,W}}\right)$$

constructible $\overline{\mathbb{Q}}_\ell$-sheaf over X^I

Last time:

$\mathcal{H}^j_{I,W}|_{\overline{\eta_I}}$ is equiped with a canonical action of the partial Frobenius morphisms
with an action of $\pi_1(\eta_I, \overline{\eta_I})$

a canonical action of $FWeil(\eta_I, \overline{\eta_I})$

By Drinfeld's lemma,

Proposition 0: The action of $FWeil(\eta_I, \overline{\eta_I})$ on $\mathcal{H}^j_{I,W}|_{\overline{\eta_I}}$ factors through

$Weil(\eta, \overline{\eta})^I$ ~~$Weil(\overline{F}/F)^I$~~.

$Weil(\eta, \overline{\eta}) = Weil(\overline{F}/F)$

F: function field of X

§1. Smoothness of the cohomology sheaf $\mathcal{H}^j_{I,W}$.

Theorem: For any I and W, the ind-constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{H}^j_{I,W}$ over X^I is ind-~~smooth~~ lisse.

Ind-lisse means:
- can be written as an inductive limit of lisse sheaves

equivalent

- $\forall$ geometric point $\bar{x}, \bar{y}$ of X^I, $\forall$ specialization map $sp: \bar{y} \to \bar{x}$, the induced morphism $sp^*: \mathcal{H}^j_{I,W}|_{\bar{x}} \to \mathcal{H}^j_{I,W}|_{\bar{y}}$ is an isomorphism.

Corollary: The action of $\mathrm{Weil}(\eta,\bar{\eta})^I$ on $\mathcal{H}^j_{I,W}|_{\bar{\eta}_I}$ factors through $\mathrm{Weil}(X,\bar{\eta})^I$.

§2 preparation of the proof of the theorem.

To prove the theorem, we will need the following propositions:

Let $(\bar{\eta})^I := \bar{\eta} \underset{\mathrm{Spec}\,\bar{\mathbb{F}}_q}{\times} \bar{\eta} \times \cdots \underset{\mathrm{Spec}\,\bar{\mathbb{F}}_q}{\times} \bar{\eta}$. Note that this is an integral scheme.

$$\bar{\eta}_I \longrightarrow (\bar{\eta})^I \longrightarrow X^I$$

Proposition 1: $\mathcal{H}^j_{I,W}|_{(\bar{\eta})^I}$ is ind-lisse over $(\bar{\eta})^I$.

(i.e. $\forall$ geometric point $\bar{x}$ of $(\bar{\eta})^I$,

$\forall$ specilization map $sp_{\bar{x}}: \bar{\eta}_I \longrightarrow \bar{x}$,

the induced morphism $sp_{\bar{x}}^*: \mathcal{H}^j_{I,W}|_{\bar{x}} \longrightarrow \mathcal{H}^j_{I,W}|_{\bar{\eta}_I}$ is an isomorphism.)

The proof of the proposition is essentially in V. Lafforgue's paper, where he proved ~~that for~~ the case $\bar{x} = \Delta(\bar{\eta})$, $\Delta: X \xhookrightarrow{diag} X^I$

The proof uses the Eichler-Shimura relations.

Consequence of Proposition 0 and Proposition 1:

recall:

$$\begin{tikzcd}
 & \underset{\text{generic}}{\delta} \arrow[r] \arrow[d] & (\bar\eta)^I \arrow[r] \arrow[d] & \bar\eta \arrow[d] \\
\bar\eta_I \arrow[ur] \arrow[dr] & & & \eta \arrow[d] \\
 & \eta_I \arrow[r] & X^I \arrow[r, "pr_i"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
 & \pi_1(\delta, \bar\eta_I) \arrow[dr] & \mathrm{Ker} \arrow[d] \arrow[r, phantom, "="] & \mathrm{ker} \arrow[d] & & \\
0 \arrow[r] & & \pi_1^{geo}(\eta_I, \bar\eta_I) \arrow[r] \arrow[d] & FWeil(\eta_I, \bar\eta_I) \arrow[r] \arrow[d] & \mathbb{Z}^I \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & & \pi_1^{geo}(\eta, \bar\eta)^I \arrow[r] & Weil(\eta, \bar\eta)^I \arrow[r] & \mathbb{Z}^I \arrow[r] & 0
\end{tikzcd}$$

$\pi_1(\delta, \bar\eta_I) \subset \mathrm{Ker}$.

Prop 0 says that the action of Ker on $\mathcal{H}^j_{I,W}|_{\bar\eta_I}$ is trivial.

$\Rightarrow$ the action of $\pi_1(\delta, \bar\eta_I)$ on $\mathcal{H}^j_{I,W}|_{\bar\eta_I}$ is trivial.

Prop 1 says that $\mathcal{H}^j_{I,W}|_{(\bar\eta)^I}$ is ind-lisse.

Prop 1': $\mathcal{H}^j_{I,W}|_{(\bar\eta)^I}$ is constant.

In general, let $I = I_1 \sqcup I_2$, $\bar s$ a geo. point over a closed point s of X.

Let $(\bar\eta)^{I_1} \times (\bar s)^{I_2} := \underbrace{\bar\eta \underset{\bar{\mathbb{F}}_q}{\times} \cdots \underset{\bar{\mathbb{F}}_q}{\times} \bar\eta}_{I_1\text{ - times}} \underset{\bar{\mathbb{F}}_q}{\times} \underbrace{\bar s \underset{\bar{\mathbb{F}}_q}{\times} \cdots \underset{\bar{\mathbb{F}}_q}{\times} \bar s}_{I_2\text{-times}}$

Similarly as above, we can prove that

proposition 2: $\mathcal{H}^j_{I,W}\big|_{(\bar\eta)^{I_1}\times(\bar\xi)^{I_2}}$ is a constant sheaf over $(\bar\eta)^{I_1}\times(\bar\xi)^{I_2}$.
$\forall\, I = I_1 \sqcup I_2$,

Remark: If a sheaf $\mathcal{F}$ over X^I is of the form $\mathcal{F} = \underset{i\in I}{\boxtimes} \mathcal{F}_i$,

then $\mathcal{F}\big|_{(\bar\eta)^{I_1}\times(\bar\xi)^{I_2}} = \Big(\underset{i\in I_1}{\boxtimes} \mathcal{F}_i\big|_{\bar\eta}\Big) \boxtimes \Big(\underset{i\in I_2}{\boxtimes} \mathcal{F}_i\big|_{\bar\xi}\Big)$ is a constant sheaf.

§3. Idea of proof of the theorem for the case I = singleton

Let $I=\{1\}$ be a singleton. Let $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\widehat{G})$. We have the degree j cohomology sheaf $\mathcal{H}^j_{\{1\},W}$ over X.

In the following, to simplify the notation, we will write ~~写~~ $\mathcal{H}_{\{1\},W}$ instead of $\mathcal{H}^j_{\{1\},W}$.

We want to prove that $\mathcal{H}_{\{1\},W}$ is ~~t~~ ind-lisse, that is to say, $\forall$ geo. point $\bar v$ of X (over a closed point v) and

$\forall$ specialization map $sp: \bar\eta \to \bar v$

$$sp^*: \mathcal{H}_{\{1\},W}\big|_{\bar v} \longrightarrow \mathcal{H}_{\{1\},W}\big|_{\bar\eta}$$

we want to prove that sp^* is an isomorphism.

Reminder : Let W^* be the dual representation of W.

creation operator : Let $\mathbb{1}$ be the trivial representation.

Let $\delta: \mathbb{1} \to W^* \otimes W$

$\mathcal{C}_\delta^{\#,\{2,3\}}$ is the composition of morphisms of sheaves over $X \times X$:

$$\mathcal{H}_{\{1\},W} \boxtimes \underbrace{\overline{\mathbb{Q}}_{\ell X}}_{\substack{\text{constant} \\ \text{sheaf over } X}} \simeq \mathcal{H}_{\{1,0\},W\boxtimes \mathbb{1}} \xrightarrow[\text{functoriality}]{Id_W \boxtimes \delta} \mathcal{H}_{\{1,0\},W\boxtimes(W^*\otimes W)} \overset{\text{fusion}}{\simeq} \mathcal{H}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{X\times \Delta^{\{2,3\}}(X)}$$

$$X\times X \xrightarrow{(Id,\ \Delta^{\{2,3\}})} X\times X\times X$$

annihilation operator :

Let $ev: W\otimes W^* \to \mathbb{1}$

$\mathcal{C}_{ev}^{\flat,\{1,2\}}$ is the composition of morphisms of sheaves over $X\times X$

$$\mathcal{H}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2\}}(X)\times X} \overset{\text{fusion}}{\simeq} \mathcal{H}_{\{0,3\},(W\otimes W^*)\boxtimes W} \xrightarrow[\text{functoriality}]{ev\boxtimes Id_W} \mathcal{H}_{\{0,3\},\mathbb{1}\boxtimes W} \simeq \overline{\mathbb{Q}}_{\ell X}\boxtimes \mathcal{H}_{\{3\},W}$$

$$X\times X \xrightarrow{(\Delta^{\{1,2\}},\ Id)} X\times X\times X$$

"Zorro" Lemma :

Note that the composition

$$W \otimes \bar{\mathbb{Q}}_\ell \xrightarrow{Id \otimes \delta} W \otimes W^* \otimes W \xrightarrow{ev \otimes Id} \bar{\mathbb{Q}}_\ell \otimes W$$

is identity.

By the fonctoriality, we have

"Zorro" Lemma : the composition of morphisms of sheaves over X :

$$\mathcal{H}_{\{1\}, W} \otimes \bar{\mathbb{Q}}_\ell \xrightarrow{\mathcal{C}_\delta^{\#, \{2,3\}}} \mathcal{H}_{\{1,2,3\}, W \boxtimes W^* \boxtimes W}\Big|_{\Delta^{\{1,2,3\}}(X)} \xrightarrow{\mathcal{C}_{ev}^{b, \{1,2\}}} \bar{\mathbb{Q}}_\ell \otimes \mathcal{H}_{\{3\}, W}$$

is identity.

Now we construct a morphism $\mathcal{H}_{\{1\}, W}\big|_{\bar{\eta}} \xrightarrow{\alpha} \mathcal{H}_{\{1\}, W}\big|_{\bar{v}}$:

$$\mathcal{H}_{\{1\}, W}\big|_{\bar{\eta}} \otimes \bar{\mathbb{Q}}_{\ell\, \bar{v}} \xrightarrow{\mathcal{C}_\delta^{\#, \{2,3\}}} \mathcal{H}_{\{1,2,3\}, W \boxtimes W^* \boxtimes W}\Big|_{\bar{\eta} \times \Delta^{\{2,3\}}(\bar{v})}$$

$$\Big\downarrow sp^*_{\{2\}}$$

By Prop 2 applied to $\mathcal{H}_{\{1,2,3\}, W \boxtimes W^* \boxtimes W}$, we have a canonical morphism

$$\mathcal{H}_{\{1,2,3\}, W \boxtimes W^* \boxtimes W}\Big|_{\Delta^{\{1,2\}}(\bar{\eta}) \times \bar{v}} \xrightarrow{\mathcal{C}_{ev}^{b, \{1,2\}}} \bar{\mathbb{Q}}_{\ell\, \bar{\eta}} \otimes \mathcal{H}_{\{3\}, W}\big|_{\bar{v}}$$

Remark: if $\mathcal{H}_{\{1,2,3\},W\boxtimes W^*\boxtimes W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3$, where $\mathcal{F}_i$ are $\overline{\mathbb{Q}}_\ell$-sheaves over X, then $sp^*_{\{2\}}$ is

$$\mathcal{F}_1|_{\bar\eta} \otimes \mathcal{F}_2|_{\bar v} \otimes \mathcal{F}_3|_{\bar v} \xrightarrow{Id \otimes sp^* \otimes Id} \mathcal{F}_1|_{\bar\eta} \otimes \mathcal{F}_2|_{\bar\eta} \otimes \mathcal{F}_3|_{\bar v}$$

In reality, we only know that by Prop 2, the restriction of $\mathcal{H}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}$ is constant over $\bar\eta\times\bar\eta\times\bar\eta$, $\bar\eta\times\bar v\times\bar\eta$, $\bar\eta\times\bar\eta\times\bar v$, $\bar v\times\bar\eta\times\bar\eta$. But this is enough.

Prove that $\alpha \circ sp^* = Id$:

the following diagram is commutative: $\Leftarrow$ Prop 2

$$\begin{tikzcd}
\mathcal{H}_{\{1\},W}|_{\bar v} \otimes \overline{\mathbb{Q}}_{\ell\,\bar v} \arrow[r, "sp^*"] \arrow[dd, "C_S^{\#,\{2,3\}}"] \arrow[ddd, bend right=60, "\text{by ``Zorro'' lemma is identity}"'] & \mathcal{H}_{\{1\},W}|_{\bar\eta} \otimes \overline{\mathbb{Q}}_{\ell\,\bar v} \arrow[d, "C_S^{\#,\{2,3\}}"] \arrow[ddd, bend left=60, "\alpha"] \\
 & \mathcal{H}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}|_{\bar\eta\times\Delta^{\{2,3\}}(\bar v)} \arrow[d, "sp^*_{\{2\}}"] \\
\mathcal{H}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}|_{\Delta^{\{1,2,3\}}(\bar v)} \arrow[ur, "sp^*_{\{1\}}"] \arrow[r, "sp^*_{\{1,2\}}"] \arrow[d, "C_{ev}^{b,\{1,2\}}"] & \mathcal{H}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}|_{\Delta^{\{1,2\}}(\bar\eta)\times\bar v} \arrow[d, "C_{ev}^{b,\{1,2\}}"] \\
\overline{\mathbb{Q}}_{\ell\,\bar v} \otimes \mathcal{H}_{\{3\},W}|_{\bar v} \arrow[r, "\sim"'] & \overline{\mathbb{Q}}_{\ell\,\bar\eta} \otimes \mathcal{H}_{\{3\},W}|_{\bar v}
\end{tikzcd}$$

prove that $sp^* \circ \alpha = id$:

the following diagram is commutative ; $\Leftarrow$ Prop 2

$$\begin{tikzcd}
\mathcal{H}_{\{1\},W}|_{\bar\eta} \otimes \bar{\mathbb{Q}}_{\ell,\bar v} \arrow[r, "\sim"] \arrow[d, "C_\delta^{\sharp,\{2,3\}}"'] \arrow[dddd, bend right, "\alpha"'] & \mathcal{H}_{\{1\},W}|_{\bar\eta} \otimes \bar{\mathbb{Q}}_{\ell,\bar\eta} \arrow[dd, "C_\delta^{\sharp,\{2,3\}}"] \arrow[dddd, bend left, "\text{by ``Zorro'' lemma is identity}"] \\
\mathcal{H}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}|_{\bar\eta\times\Delta^{\{2,3\}}(\bar v)} \arrow[dd, "sp^*_{\{2\}}"'] \arrow[dr, "sp^*_{\{2,3\}}"] & \\
 & \mathcal{H}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}|_{\Delta^{\{1,2,3\}}(\bar\eta)} \arrow[dd, "C_{ev}^{\flat,\{1,2\}}"] \\
\mathcal{H}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}|_{\Delta^{\{1,2\}}(\bar\eta)\times\bar v} \arrow[ur, "sp^*_{\{3\}}"'] \arrow[d, "C_{ev}^{\flat,\{1,2\}}"'] & \\
\bar{\mathbb{Q}}_{\ell,\bar\eta} \otimes \mathcal{H}_{\{3\},W}|_{\bar v} \arrow[r, "sp^*"] & \bar{\mathbb{Q}}_{\ell,\bar\eta} \otimes \mathcal{H}_{\{3\},W}|_{\bar\eta}
\end{tikzcd}$$