Generalized Cluster Structures and Periodic Difference Operators

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References:
[GSV] https://arxiv.org/abs/1605.05705 (Proc. London Math. Soc. '2018)
https://arxiv.org/abs/1912.00453. (Int. Math. Res. Not. '2020)
https://arxiv.org/abs/2004.05118 (J. LMS 2022)

[FGT]: Generalized cluster structures and cyclic symmetry in three settings (in preparation)

Cast of characters:

- Periodic difference operators / infinite periodic finite band matrices (e.g. Jacobi matrices)
  - spectral theory, integrable systems (periodic Toda lattice, discrete KP hierarchy)...
- Poisson varieties with group actions (Poisson-Lie groups, Poisson homogeneous spaces)
  - integrable systems, representation theory, quantum groups.
- (Generalized) cluster algebras
  - combinatorics, Lie theory, rep'ns of algebras, (higher) Teichmüller theory, ...

Cluster Structures on Poisson Varieties.

Let $(V, \{\cdot,\cdot\})$ be a Poisson variety that admits a toric action.

We want to construct a family of coordinate systems (clusters) $\mathbf{F} = \{x = (x_1, \ldots, x_n)\}$ such that

1. For any $x \in \mathbf{F}$ and any i, x_i is a regular function on V homogeneous w.r.t. the toric action.
2. Every $x \in \mathbf{F}$ is compatible with $\{\cdot,\cdot\}$: $\{x_i, x_j\} = \omega_{ij} x_i x_j$, $\omega_{ij} \in \mathbb{Z}$ (*log-canonical Poisson bracket*)
3. Any $x, x' \in \mathbf{F}$ are connected via Laurent polynomial transformation
4. Any regular function on V is a Laurent polynomial in terms of every $x \in \mathbf{F}$

Ingredients

Poisson Side

- $\left(V^{n+m} \supset (\mathbb{C}^*)^m, \alpha, \{\,\}\right)$

$\mathbf{x} = (x_1, \dots, x_n; x_{n+1}, \dots, x_{n+m})$

$\{x_i, x_j\} = \omega_{ij}\, x_i x_j \quad i,j = 1, \dots, m+n$ $\leftarrow B_Q \Omega = [\lambda \mathbb{1}_n ; 0] \rightarrow$

$\Omega = (\omega_{ij})_{i,j=1}^{n+m}$

x_i – homogeneous under $(\mathbb{C}^\times)^m$-action

$x_i \in \mathcal{O}(V)$

- $\tilde{\mathbf{x}}_k = (\mathbf{x} \setminus \{x_k\}) \cup \tilde{x}_k \quad k \in \{1, \dots, n\}$

$\{\tilde{x}_i, \tilde{x}_j\} = \tilde{\omega}_{ij}\, \tilde{x}_i \tilde{x}_j$

$\tilde{x}_i$ – homogeneous under $(\mathbb{C}^\times)^m$-action $\leftarrow B_{\tilde{Q}} \tilde{\Omega} = [\lambda \mathbb{1}_n ; 0] \rightarrow$

$\tilde{x}_i \in \mathcal{O}(V)$

$\underbrace{\qquad}_{\text{regularity}}$

- Completeness:

Any elements in $\mathcal{O}(V)$ is a Laurent Poly. in terms of any $\mathbf{x}$, polynomial in $x_{n+1}, \dots, x_{n+m}$.

$\longleftrightarrow$ $\mathcal{O}(V) \cong$ Upper Cluster Algebra $\overline{\mathcal{A}}(\mathbf{x}; Q)$.

$\longleftrightarrow$ $\mathcal{O}(V) = \overline{\mathcal{A}}(\mathbf{x}; Q)$

Cluster Algebra Side

- Initial seed:

$(\mathbf{x}, \underbrace{Q}_{\text{quiver}})$

$\rightarrow (\,\cancel{\circlearrowleft}\;\; \cancel{\rightleftarrows}\,)$

$Q \rightsquigarrow B_Q = n \times (n+m)$ adjacency matrix

- Exchange Relation

$x_k \cdot \tilde{x}_k = \prod_{(j \to k) \in Q} x_j + \prod_{(k \to j) \in Q} x_j$

+

Quiver Mutation

$Q \rightsquigarrow \tilde{Q}$

(explained below)

$\rightsquigarrow (\tilde{\mathbf{x}}, \tilde{Q})$ – adjacent seed.

Example 1 Cluster algebra of finite type A_{n-1}

(Yang - Zelevinsky realization)

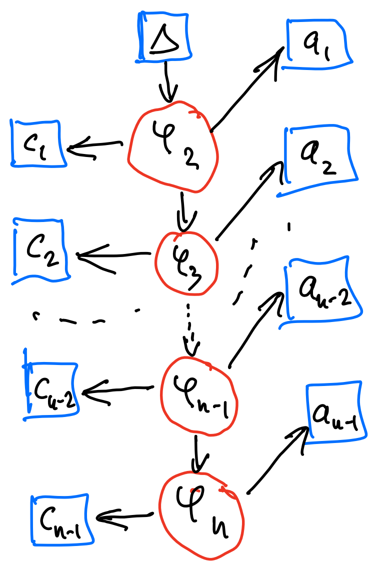
$$G = GL_n \supset G^{c,c^{-1}} = \underbrace{B_+ c B_+ \cap B_- c^{-1} B_-}_{(c = s_1 \cdots s_{n-1})} = \left\{ X = \begin{bmatrix} b_1 & a_1 & & 0 \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ 0 & & c_{n-1} & b_n \end{bmatrix}, \quad \begin{matrix} a_i \neq 0 \\ b_i \neq 0 \\ \Delta \neq 0 \end{matrix} \right\}$$

$\{\ ,\ \}$ - standard P.-L.

Coxeter Double Bruhat Cell

Initial Seed:

$$\boxed{\varphi_i = \det X^{[i,\dots,n]}_{[i,\dots,n]}}$$



Cluster transformations: Desnanot - Jacobi Identities

Plücker, Whitehead, Ptolemy, Hirota ...

THM (Yang-Zelevinsky)

$$\{\text{cluster variables}\} \longleftrightarrow \{\det X^{[i,j]}_{[i,j]},\ 1 \le i \le j \le n\}$$

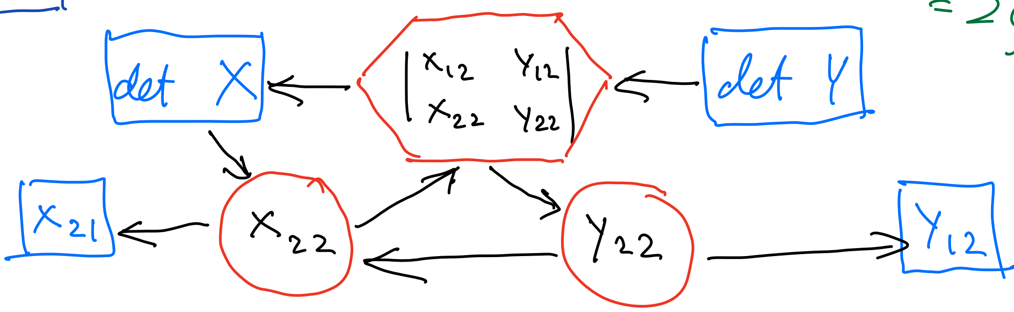
Example 2 — Drinfeld double of GL_n.

$$\underset{G}{\underset{\shortparallel}{D(GL_n)}} = GL_n \times GL_n = \underset{\substack{\text{standard} \\ \text{P.-L. } \{\,,\}}}{\mathrm{diag}(G)} \times \underset{\substack{\text{dual} \\ \text{P.-L. group}}}{G^*}$$

- $D(GL_n)$ is itself a Poisson-Lie group with $\{\,,\}$ quadratic in terms of matrix entries of $(X,Y) \in D(GL_n)$

$\boxed{n=2}$ Initial seed:

$$\det(\lambda X + \mu Y) = \sum \underbrace{c_i(X,Y)}\lambda^{i}\mu^{n-i}$$



But: „Usual" cluster transformation at ⬡ does not produce a regular function.

Remedy: Replace by a „longer relation"

$$\begin{vmatrix} x_{12} & y_{12} \\ x_{22} & y_{22} \end{vmatrix} \cdot \begin{vmatrix} y_{21} & y_{22} \\ x_{21} & x_{22} \end{vmatrix} = \det(-y_{22}X + x_{22}Y)$$

$$= \underbrace{\det X}\cdot y_{22}^2 + c(X,Y)\, y_{22}\cdot x_{22} + \underbrace{\det Y}\cdot x_{22}^2$$

Casimirs

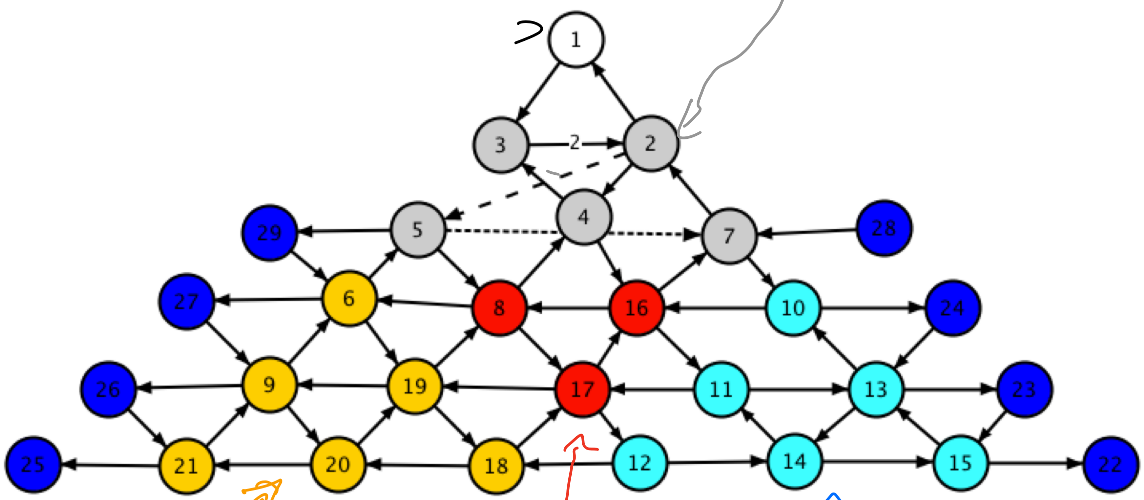
This is a generalized cluster transformation.

$\boxed{n = \text{any}}$

Relation of length $(n+1)$:

$$\varphi_1 \overleftarrow{\varphi}_1 = \det\left((-1)^{n-1}\varphi_2 \cdot X + \varphi_3 Y\right) = \sum_{k=0}^{n} \pm c_k(X,Y)\, \varphi_2^k \varphi_3^{n-k}$$

Funny looks



dense minors of X mixed determinants —/— of Y

The picture above „contains":

(i) regular complete generalized cluster structure on $D(GL_n)$

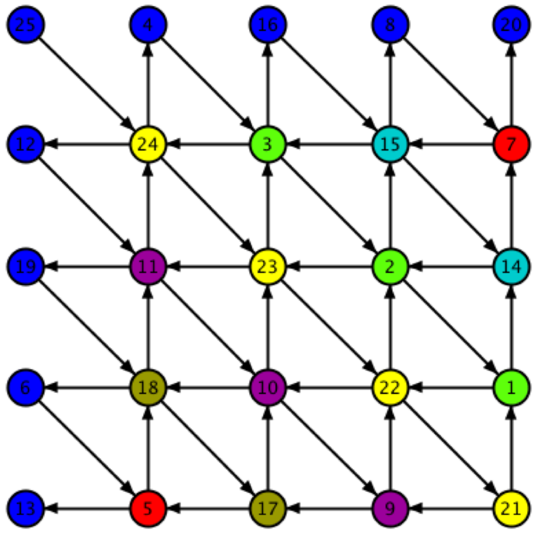
(ii) regular complete ~~generalized~~ cluster structure on GL_n.

(iii) regular complete generalized cluster structure on GL_n^*.

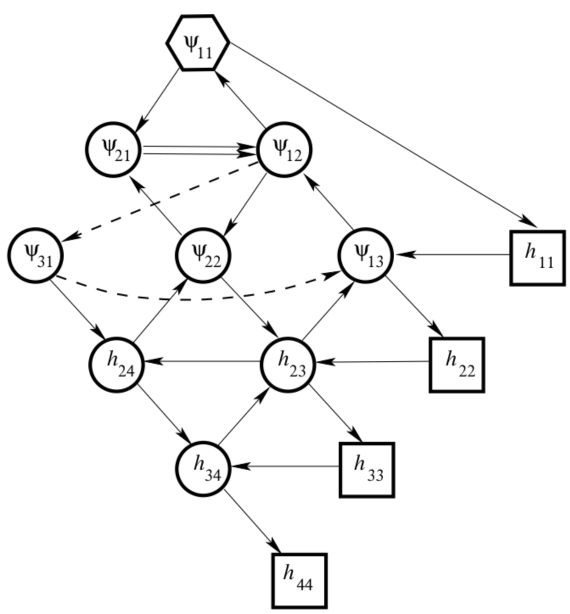
— // — conj. class in $GL_n \cap B_+B_-$.

— // — $B_+ w_0$

(ii) :



(iii) :



Variables: functions of $U = X^{-1}Y$

Key Identity

Let A be an $n \times n$ matrix. For $u, v \in \mathbb{C}^n$, define matrices

$$\Gamma(u) = \left[u \; Au \; A^2 u \ldots A^{n-1}u\right],$$

$$\Gamma_1(u,v) = \left[v \; u \; Au \ldots A^{n-2}u\right], \quad \Gamma_2(u,v) = \left[Av \; u \; Au \ldots A^{n-2}u\right].$$

Let w be the last row of the classical adjoint of $\Gamma_1(u,v)$, i.e.

$$w\Gamma_1(u,v) = (\det \Gamma_1(u,v))\, e_n^T .$$

Define $\Gamma^*(u,v)$ to be the matrix with rows $w, wA, \ldots, wA^{n-1}$. Then

$$\det\left(\det \Gamma_1(u,v)A - \det \Gamma_2(u,v)\mathbf{1}\right) = (-1)^{\frac{n(n-1)}{2}} \det \Gamma(u) \det \Gamma^*(u,v).$$

polynomials in A

Initial Seed for a Generalized Cluster Structure

(G.-S.-V. modification of the definition by Chekhov - Shapiro)

- Each non-frozen vertex $i \in Q$ now has a multiplicity d_i and a string of coefficients $p_i = (p_{i0} = 1, p_{i1}, \ldots, p_{id_i - 1}, p_{id_i} = 1)$ attached.

- Generalized Exchange Relation:

$$x_k \tilde{x}_k = \sum_{r=0}^{d_k} p_{kr}\, u_{k;>}^r\, v_{k;>}^{[r]}\, u_{k;<}^{d_k - r}\, v_{k;<}^{[d_k - r]}$$

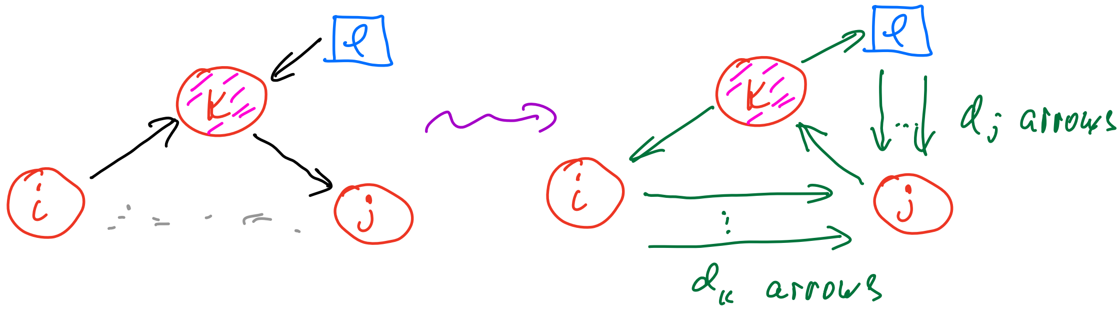
$$u_{k;>} = \prod_{k \to i} x_i, \quad u_{k;<} = \prod_{i \to k} x_i$$

$$v^{[r]}_{k,>} = \prod_{\boxed{i}} x_i^{\left\lfloor \frac{r\cdot \#(k\to i)}{d_k} \right\rfloor}, \quad v^{[r]}_{k,<} = \prod_{\boxed{i}} x_i^{\left\lfloor \frac{r\cdot \#(i\to k)}{d_k} \right\rfloor}$$

- Coefficient Mutation

$$\tilde{p}_{kr} = p_{k,d_k-r}$$

- Modified Quiver Mutation



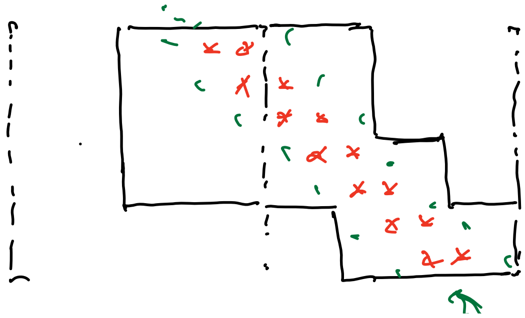
- Under these definitions all nice properties of cluster algebras remain intact (Chekhov-Shapiro, Rupel-Nakanishi, Nakanishi, G-S-V)

Source of Examples: Periodic Staircase Matrices

$$\mathcal{Y} = \begin{bmatrix} \ddots & \ddots & & \\ & \underset{\sim}{A} & \underset{\sim}{B} & \\ & & A & B \\ & & \ddots & \ddots \end{bmatrix}$$

, A, B — $n\times n$ of special form, s.t. $\mathcal{Y}$ has k inner diagonals

E.g.:

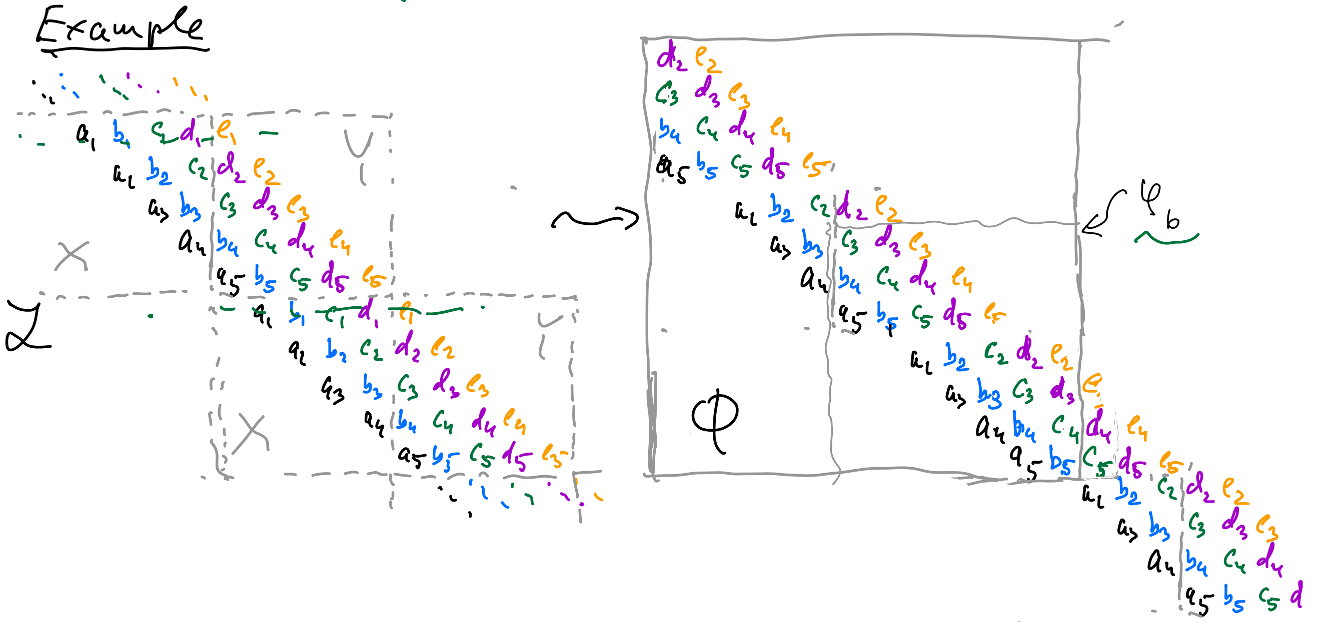
$[A \,\vdots\, B] =$ 

$(n=7)$

(2 inner diagonals)

- Remove 1st row from each block row
  $\rightsquigarrow$ Reducible matrix with maximal irred. component $\Phi(\mathcal{L})$ of size $N \sim k \cdot n$

Example



- Define $\varphi_i = \det \Phi(\mathcal{L})^{[i,N]}_{[i,N]}$

- Then $\det(\lambda B + \mu A) = \lambda^{n-k} \sum_{i=0}^{k} c_i(A,B)\mu^i \lambda^{k-i}$

and „master identity" $\Longrightarrow$

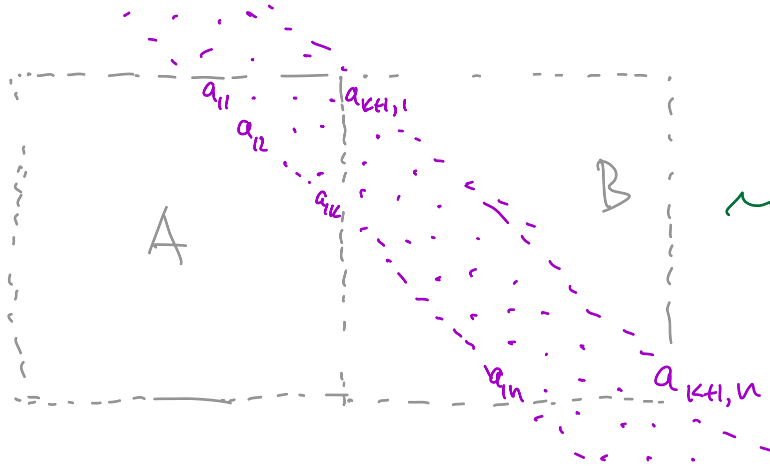
$$\varphi_1 \cdot \varphi_1^* = \sum_{i=0}^{k} c_i(A,B) \left((-1)^{n-1} \det B^{[2,n]}_{[2,n]} \cdot \varphi_{n+1}\right)^i \varphi_2^{k-i}$$

polynomial in A, B

- Can one use this as a generalized cluster transformation? YES!!!

Example 3 — Periodic Band Matrices

- $\mathcal{L} =$



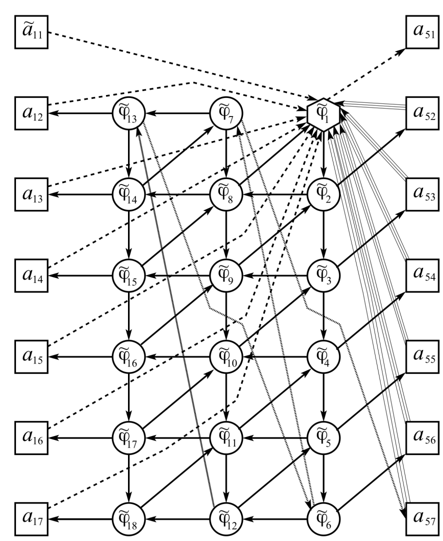
$\rightsquigarrow \phi(\mathcal{L})$

- $N = (n-1)(k-1)$
- $\{\ ,\ \}$: $\mathcal{L}_{k,n} = \{M(\lambda) = A + \lambda B\}$ is a Poisson submanifold in $GL_n[\lambda]$ w.r.t. trigonometric R-matrix bracket

$$\{M(\lambda) \overset{\otimes}{,} M(\mu)\} = [R(\lambda,\mu), M(\lambda) \otimes M(\mu)]$$

$$R(\lambda,\mu) = \frac{\lambda+\mu}{\lambda-\mu} \sum e_{kk} \otimes e_{kk} + \frac{2}{\lambda-\mu} \sum_{l<m} (\mu e_{lm} \otimes e_{ml} + \lambda e_{ml} \otimes e_{lm})$$

THM A family $\{\varphi_i,\ i = 1, \dots, (k-1)(n-1)\}$ together with the quiver



defines a regular complete generalized cluster structure compatible with the $\{\ ,\ \}$ above.

$\boxed{k=2}$ **COMPARE!** Cluster algebra of finite type A_{n-1}
(Yang-Zelevinsky realization)

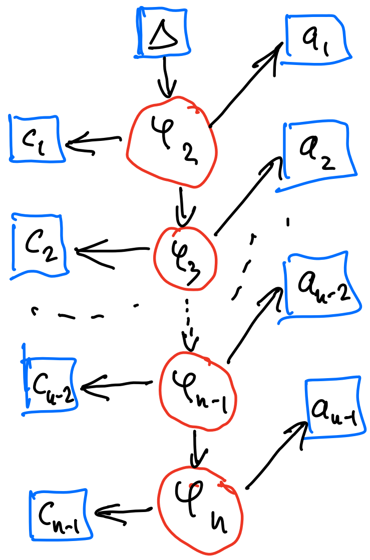
$G = GL_n \supset \underbrace{G^{c,c^{-1}}}_{(c = s_1 \cdots s_{n-1})} = \left\{ X = \begin{bmatrix} b_1 & a_1 & & 0 \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ 0 & & c_{n-1} & b_n \end{bmatrix}, \quad \begin{matrix} a_i \neq 0 \\ b_i \neq 0 \\ \Delta \neq 0 \end{matrix} \right\}$

$\{\ ,\ \}$ - standard P.-L.

Coxeter Double Bruhat Cell

Initial Seed:

$$\varphi_i = \det X^{[i,\dots,n]}_{[i,\dots,n]}$$



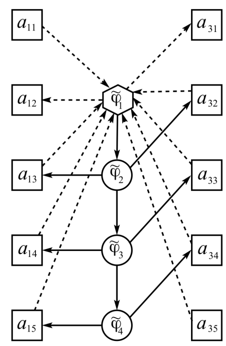
Cluster transformations:
Desnanot-Jacobi Identities

Plücker, Whitehead, Ptolemy, Hirota ...

THM (Yang-Zelevinsky)
$\{\text{cluster variables}\} \longleftrightarrow \{\det X^{[i,j]}_{[i,j]},\ 1 \le i \le j \le n\}$

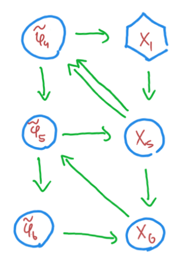
Periodic Case: Analogue of the Yang-Zelevinsky Thm.
(3-diagonal periodic matrices)

THM For $k=2$, the generalized cluster structure above is of finite type C_n and
$\{\text{cluster variables}\} \longleftrightarrow \{\text{distinct dense principal minors of } \mathcal{L} \text{ of size} < n\}$.



Remark • A monoidal categorification of the generalized cluster structure of the type was used by A. Gleitz in the study of rep'ns of $U_q(\widehat{sl_2})$ at roots of unity (following the work of Hernandez and Leclerc)!

- The Grothendieck ring of certain tensor subcategory of fin.dim rep'ns of $U_q(\widehat{sl_2}) \cong$ gen. cl. algebra of type $C_{\ell-1}$ (ℓ = order of q^2).
- Gleitz conjectured a similar result for $U_q(\widehat{SL_3})$.
- Gleitz's conjectured generalized quiver is mutation equivalent to our quiver for $\mathcal{L}_{3,n}$ (with frozens set to 1)



Three Settings for Cyclic Symmetry

<u>Recall</u> (in the case of „usual" cluster structures)
$\exists$ quasi cluster isomorphisms btw

$$\mathbb{C}\left(Gr(k,n)\right), \quad \mathbb{C}\left(\underbrace{B^{(n-k)}_{(k)}}\right), \quad \mathbb{C}\left(\underbrace{K_0[\mathcal{C}_{n-k-1}]}\right)$$

Band matrices of width $(n-k)$ with $(k+1)$ diagonals

Grothendieck ring of the Hernandez-Leclerc category of rep's of $U_q(\widehat{sl}_{n-k-1})$

I.p.: $\quad Gr(k,n) \longrightarrow B^{(n-k)}_{(k)}$

$$X \longmapsto \Big(\underbrace{\Delta_{[i,i+k]\setminus j}(X)}\Big)_{i,j} \longrightarrow \underbrace{V_{\omega_{j-i}}}_{\text{fund. repn.}}(q^{j+i-3})$$

<u>Cyclic symmetry</u>: $\bullet\ B^{(n-k)}_{(k)} \rightsquigarrow \mathcal{L}_{k,\ell}$

$(k+1)$-diag., ℓ-periodic band matrices

$\bullet\ Gr(k,n) \rightsquigarrow Gr(k,n)^{\rho^\ell} \supset D_n(k,\ell)$

fixed point locus under ℓth power of the cyclic shift

irred. component containing pos. points (studied by C. Fraser)

$\bullet\quad q \longrightarrow \varepsilon = (2\ell)$th root of unity

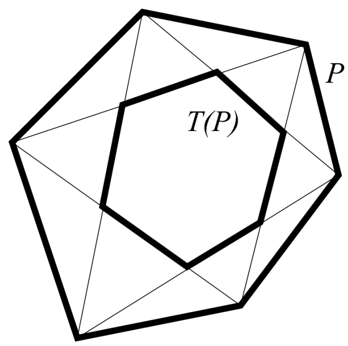
- Condition $k \leq \ell$ needs to be removed, but the initial cluster for $\mathcal{L}_{k,\ell}$ is defined in the same way, with master identity, responsible for a generalized mutation, extended
  (matrix pencil $\rightsquigarrow$ matrix polynomial)

- Isospectrality: coeffts of the generalized cluster relation in $D_n(k,\ell)$ do not depend on an element $X \in$ (conjectured by C. Fraser)

- Generalized cluster structure in an appropriate Grothendieck ring for $U_\varepsilon^{res}(\widehat{sl}_k)$

Connections / Applications Further Directions

- Spaces of twisted projective polygons — phase spaces of pentagram maps and their generalizations.



- Cluster structure in spaces of matrix-valued polynomials compatible with the trigonometric R-matrix Poisson bracket.

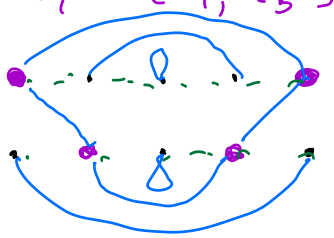
All of the above require replacing bidiagonal block periodic matrices with arbitrary band width block periodic matrices and extending

- construction of the initial cluster ✓
- key identity ✓
- generalized cluster transform. ✓

• Exotic cluster structures in $GL(n)$

<u>Example</u> Belavin-Drinfeld P.-L. $\{\ ,\ \}$

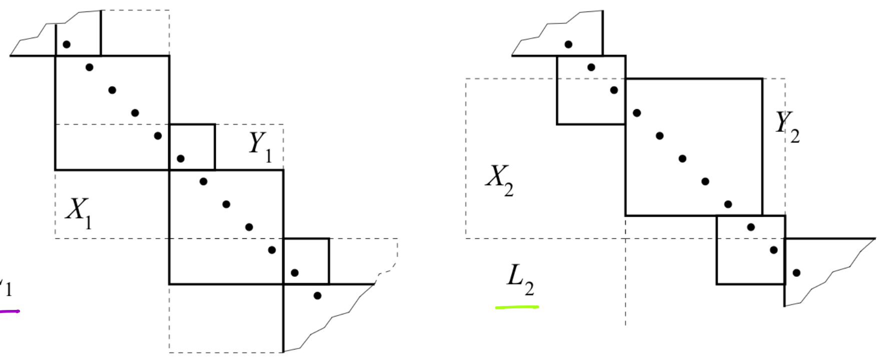
$\Gamma_1 = \{\alpha_1, \alpha_5\} \longrightarrow \Gamma_2 = \{\alpha_2, \alpha_4\}$

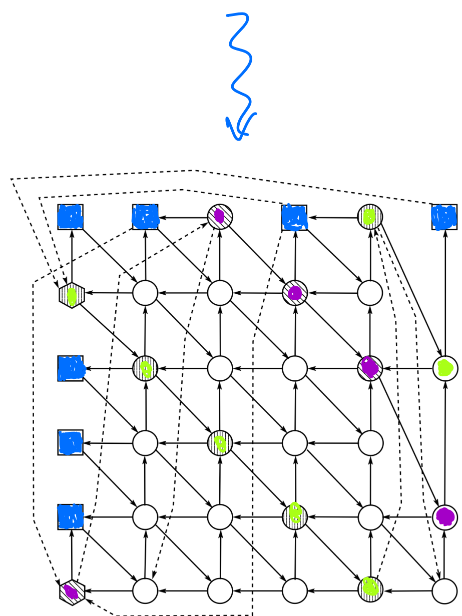


$\rightsquigarrow$ (cf. Yakimov's combinatorial data)

$\rightsquigarrow$ 2 periodic matrices

$\Longleftrightarrow$ 2 generalized relations





Thank

You!