Minuscule Multiples & Reverse Plane Partitions

- Semistandard Young tableaux and irreducible components of Springer fibers model highest weight crystals in a compatible way.
- We present a generalization of these correspondences to *ADE minuscule Demazure crystals*.
- With Elek, Kamnitzer and Morton-Ferguson our generalization uses reverse plane partitions in place of tableaux and *quiver Grassmannians* of preprojective algebra modules in place of flags.
- Do reverse plane partitions play with good bases or clusters?

Heaps, crystals and preprojective algebra modules

- Goal: to extend (*partially* and in a type independent way) the crystal isomorphism $\mathrm{Irr}\ F(A) \to Y(\lambda)$
  - $A : \mathbb{C}^N \to \mathbb{C}^N$ order n nilpotent of Jordan type λ
  - $F(A) = \{V_0 \subset V_1 \subset \cdots \subset V_m = \mathbb{C}^N : AV_i \subset V_{i-1}\}$
  - $Y(\lambda)$ is the set of SSYT of shape λ in $\{1, 2, \ldots, m\}$
  - *Partial*: λ minuscule or *minuscule witness* for some $w \in W$
- Th.2: a crystal isomorphism $\mathrm{Irr}\ G(w, n) \to R(w, n)$
- Th.1: a crystal isomorphism $R(w, n) \to B(n\lambda)$

Minuscule

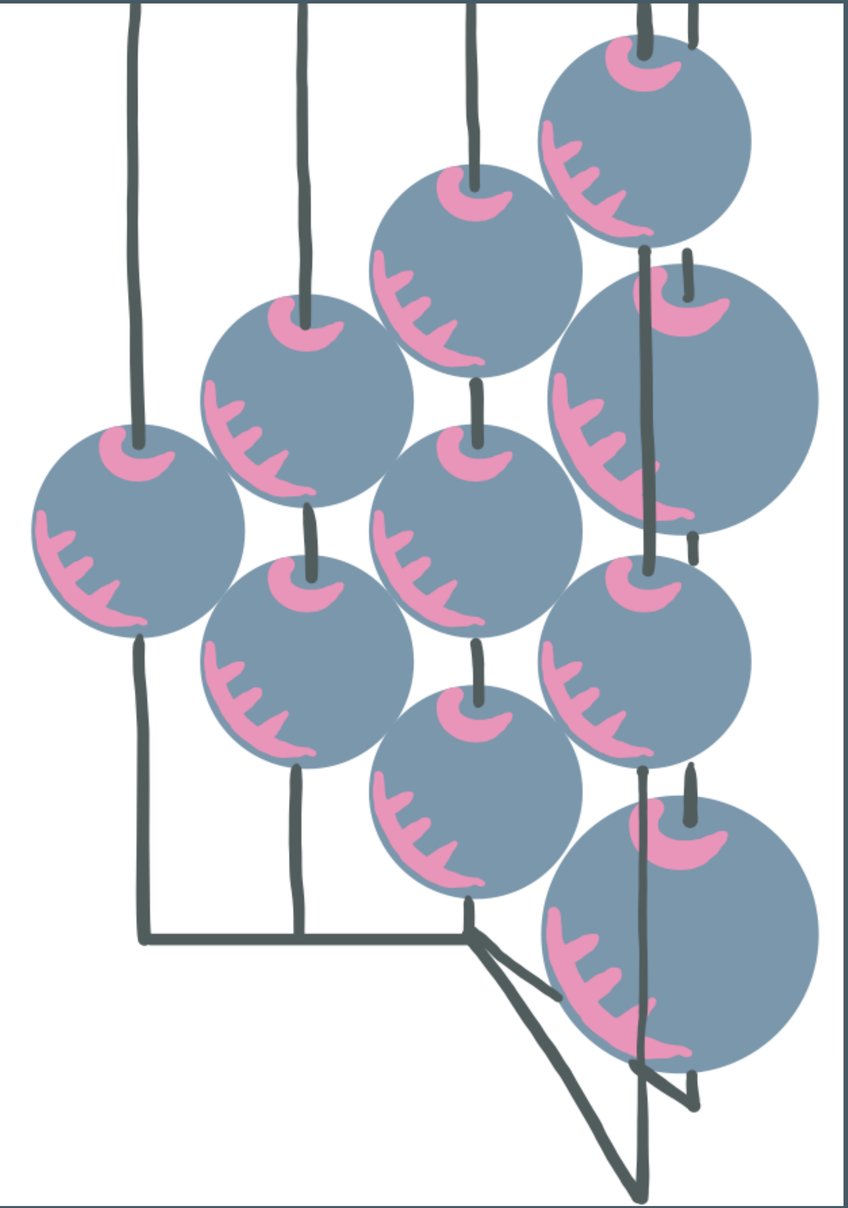
Let $\mathfrak{g}$ be semisimple, with Cartan $\mathfrak{h}$ and weight lattice Λ

- Def. $\lambda \in \Lambda^+$ is minuscule if W acts transitively on the weights of $V(\lambda)$
- Def. $\lambda \in \Lambda^+$ is a minuscule witness for $w \in W$ if
  - for some reduced word $(i_1, \ldots, i_l)$

$$w_k \lambda = \lambda - \alpha_{i_k} - \cdots - \alpha_{i_l} \qquad w_k := s_{i_k} \cdots s_{i_l}$$

- Def. w is (dominant) minuscule if it admits a (dominant) witness
  - E.g. $w = s_1 s_3 s_4 s_2$ is minuscule for $\lambda = \omega_2$ $(\Gamma = D_4)$ ⚠️
- Stembridge: If w is minuscule then it's *fully commutative* and the condition above holds for *any* reduced word

Heaps and the abacus model



- Heaps encode reduced words for minuscule w
- Let $\underline{w} = (i_1, \ldots, i_l)$ be a reduced word for w
- $H(\underline{w}) \subset \Gamma \times \mathbb{R}_{\geq 0}$ is the poset $\{1, 2, \ldots, l\}$ got by taking the transitive closure of the relation

$$s \prec t \iff s > t \text{ and } a_{i_s, i_t} < 0$$

- $\Gamma = D_5$ and $\underline{w} = (5, 3, 2, 4, 1, 3, 2, 5, 3, 4)$
- If w is minuscule then $H(w)$ is well-defined
  - Moreover $\{v \in W : v \leq_L w\} \cong J(H(w))$

Crystals

- Def. The set B is a $\mathfrak{g}$-crystal if the following maps satisfy some axioms
  - $\mathrm{wt} : B \to \Lambda$
  - $\varepsilon_i, \varphi_i : B \to \mathbb{N}$
  - $\tilde{e}_i, \tilde{f}_i : B \dashrightarrow B$
- We write $B(\lambda)$ for the crystal of $V(\lambda)$
- Def. For any $w \in W$ and $\underline{w} = (i_1, \ldots, i_l)$ reduced the Demazure crystal $B_w(\lambda) \subset B(\lambda)$ is the set

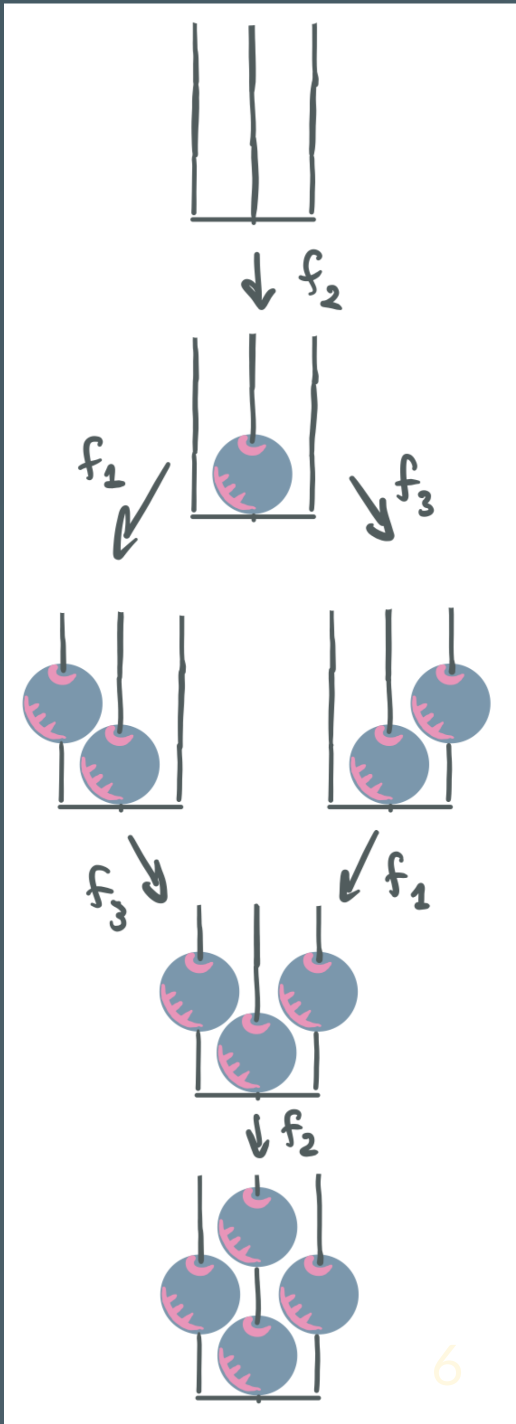
$$\bigcup_{m_s \geq 0} f_{i_1}^{m_1} \cdots f_{i_l}^{m_l} b_\lambda$$

Crystal heaps

- Prop. If w be λ-minuscule then $J(H(w)) \cong B_w(\lambda)$
  - $\mathrm{wt}(v) = v\lambda$
  - $\tilde{f}_i(v) = \begin{cases} s_i v & v < s_i v \leq_L w \\ 0 & \text{else} \end{cases}$
- $J \subset I; W_J = \langle s_j : j \in J \rangle$; the set of minimal length representatives

$$W^J = J(H(w_0^J)) \cong B(\lambda) \qquad \lambda = \sum_{j \notin J} \omega_j$$

- We can generalize this to *minuscule multiples* $B_w(n\lambda)$



Reverse plane partitions

- Def. Reverse plane partitions of shape $H(w)$ and height n are elements of the set

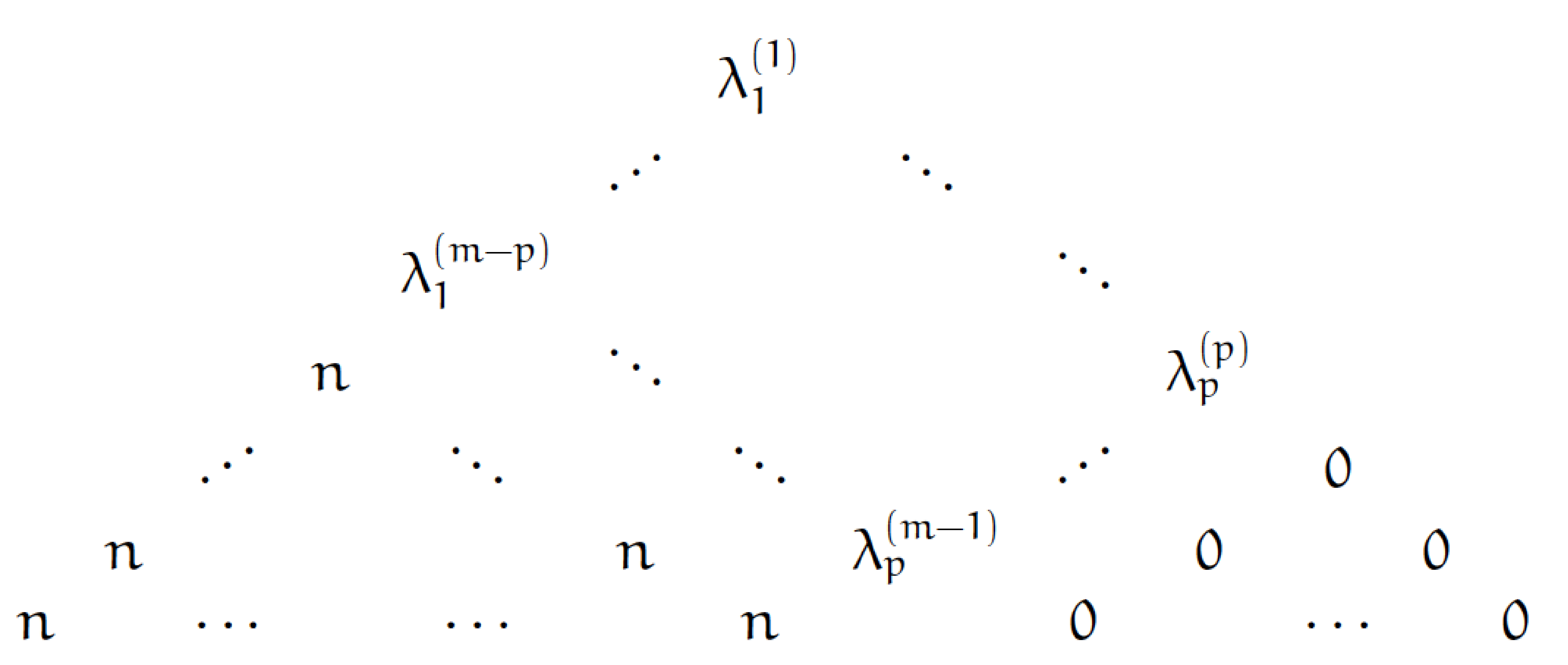
$$R(w,n) := \{H(w) \xrightarrow{\Phi} \{0,1,\ldots,n\} : \Phi(x) \geq \Phi(y) \text{ if } x \leq y\}$$

- $\Phi \mapsto (\phi_1, \ldots, \phi_n)$ defines $R(w,n) \hookrightarrow J(H(w))^n$
    - the layers $\phi_k := \Phi^{-1}(\{n-k+1, \ldots, n\})$ form an increasing chain
    - the tensor product rule preserves $\{\phi : \phi_k \subset \phi_{k+1}\}$
    - commutes with $B_w(n\lambda) \hookrightarrow B_w(\lambda)^{\otimes n}$
- E.g. if $\underline{w} = (2,1,3,2)$ then $\begin{matrix} & 1 & \\ 2 & & 3 \\ & 4 & \end{matrix} \mapsto \begin{matrix} & 0 & \\ 0 & & 0 \\ & 1 & \end{matrix} \otimes \begin{matrix} & 0 & \\ 0 & & 1 \\ & 1 & \end{matrix} \otimes \begin{matrix} & 0 & \\ 1 & & 1 \\ & 1 & \end{matrix} \otimes \begin{matrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{matrix}$ and the RHS can be viewed as an element of $B_w(\omega_2)^{\otimes 4}$

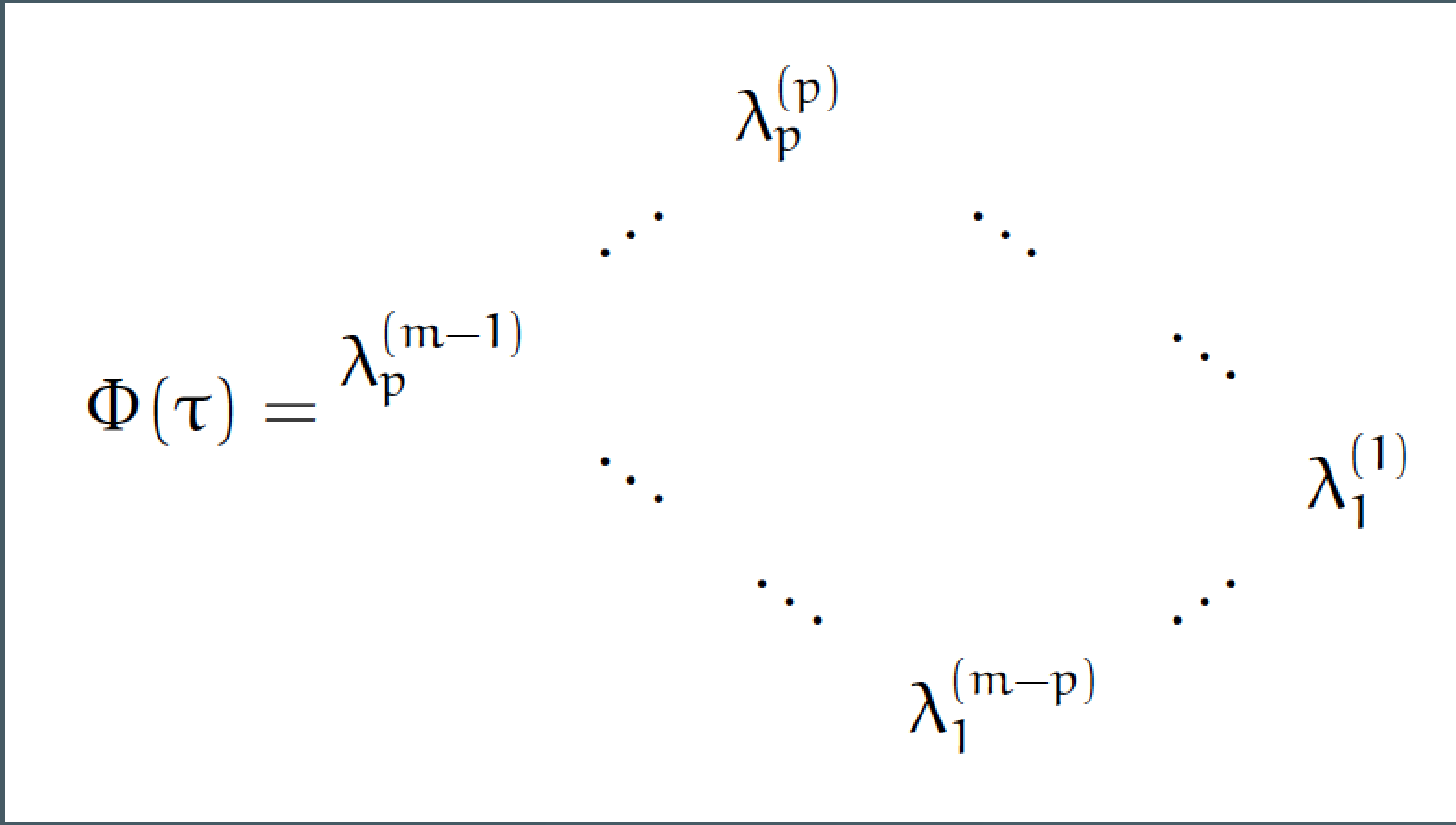
RPP's and tableaux via GT patterns

- In type A_{m-1} we can go from tableaux to RPP's via GT patterns
- The GT pattern of a tableau τ is the shape array $(\lambda^{(1)}, \ldots, \lambda^{(m)})$ made up of shapes $\lambda^{(i)}$ of tableaux $\tau^{(i)}$ got by deleting from τ any box with label exceeding i
    - When τ is a rectangular tableau having shape $\lambda = (n^p)$ its GT pattern can be identified with a $p \times (m-p)$ rectangular array
    - This array can be viewed as an RPP
    - This is a crystal isomorphism up to Schutzenberger involution and then the crystal structure is $B(n\omega_p)$ ⚠️

Reflecting in a vertical axis and rotating 90 degrees counter-clockwise we arrive at $\Phi(\tau) \in R(w_0^J, n)$



$\Phi(\tau)$ has shape $H(w_0^J)$ for $J = I \setminus \{p\}$, the heap of the *Grassmannian permutation* w_0^J that takes $12\ldots m$ to $m-p+1\ \ldots m\, 1 \ldots m-p$



Modules (for the preprojective algebra) from heaps

- Let $\lambda = \sum n_i \omega_i \in \Lambda^+$ and consider $Q(\lambda) = \bigoplus_i Q(i)^{\oplus n_i}$ where $Q(i)$ denotes the injective hull of $S(i)$
    - Th. (Nakajima, Savage-Tingley) $\mathrm{Gr}(Q(\lambda)) := \{M \subset Q(\lambda)\} \cong B(\lambda)$
- If λ is minuscule and $J = \{j : s_j \lambda = \lambda\}$ then $Q(\lambda) \cong \mathbb{C}H(w_0^J)$
    - Moreover $\phi \mapsto \mathbb{C}\phi$ is a bijection $J(H(w_0^J)) \to \mathrm{Irr}\mathrm{Gr}(Q(\lambda))$
- Th. With the help of certain ad hoc nilpotent endomorphisms of modules we upgrade this to a map $\mathrm{Irr}\mathrm{Gr}(Q(\lambda)^{\oplus n}) \to R(w_0^J, n)$
    - The case $\lambda = \omega_p$ or $J = I \setminus \{p\}$ recovers $F(A) \to Y(\lambda)$

Connections to cluster algebras

Consider

- $B^{\pm}_{I\setminus J} \subset G$ generated by $B^{\pm}$ and the 1-parameter root subgroups $\{x_i^{\pm}(t) : i \notin J\}$ respectively
- the unipotent radical $N_{I\setminus J}$ of $B_{I\setminus J}$
- the injective preprojective algebra module $Q_{I\setminus J} = \oplus_{i \notin J} Q(i)$
- Geiss, Leclerc, and Schroer constructed a cluster algebra $\mathcal{A}_J \subset \mathbb{C}[N_{I\setminus J}]$ and lifted it to a cluster algebra

$$\widetilde{\mathcal{A}}_J \subset \mathbb{C}[G/B^{-}_{I\setminus J}] = \bigoplus_{\lambda \in \Pi_{I\setminus J}} L(\lambda)$$

The fundamental example

- When $\Gamma = A_N$ and $I \setminus J = \{p\}$ this is the familiar Grassmannian cluster algebra

$$\widetilde{A}_J = \mathbb{C}[G/B_p^-] = \mathbb{C}[\mathrm{Gr}(p, N)] = \bigoplus_{n \geq 0} L(n\varpi_p)$$

- GLS: recipe for initial seeds from basic complete rigid modules parametrized by certain reduced words for w_0 in $\mathrm{Sub}\ Q_{I \setminus J}$
  - mutation (in the direction of an indecomposable nonprojective direct summand X of such a module) using short exact sequences
- Qu. no. 1: How do the RPP crystals for $L(n\varpi_p)$ interact with the cluster structure on this coordinate ring?

The open question

- Qu. no. 1': In particular, what does mutation look like for RPP's?
- GLS: $\mathbb{C}[N_{I\setminus J}] \subset \mathbb{C}[N]$ as $\mathrm{Span}(\{\phi_M : M \in \mathrm{Sub}Q_J\})$
- Wild conjecture: the various perfect bases of $\mathbb{C}[N]$ intersect in the set of cluster monomials
- With Bai and Kamnitzer, we checked that the MV basis in the $\mathfrak{sl}_4$ case contains the cluster monomials
  - We relied on the geometry of the affine Grassmannian and an MV isomorphism
  - Conceptually easy but computationally difficult

Fusion by toggling?

- In this type A check we could label our geometrically constructed basis elements by tableaux
  - In terms of tableau the exchange relations we witnessed were of the form

$$\tau \cdot \sigma = \tau \leftarrow \sigma + \tau \rightarrow \sigma$$

- Observation: Translating our equations to RPP's we notice that the mutation $\sigma = \mu_i(\tau)$ can be obtained by toggling $\Phi(\tau)$ at i

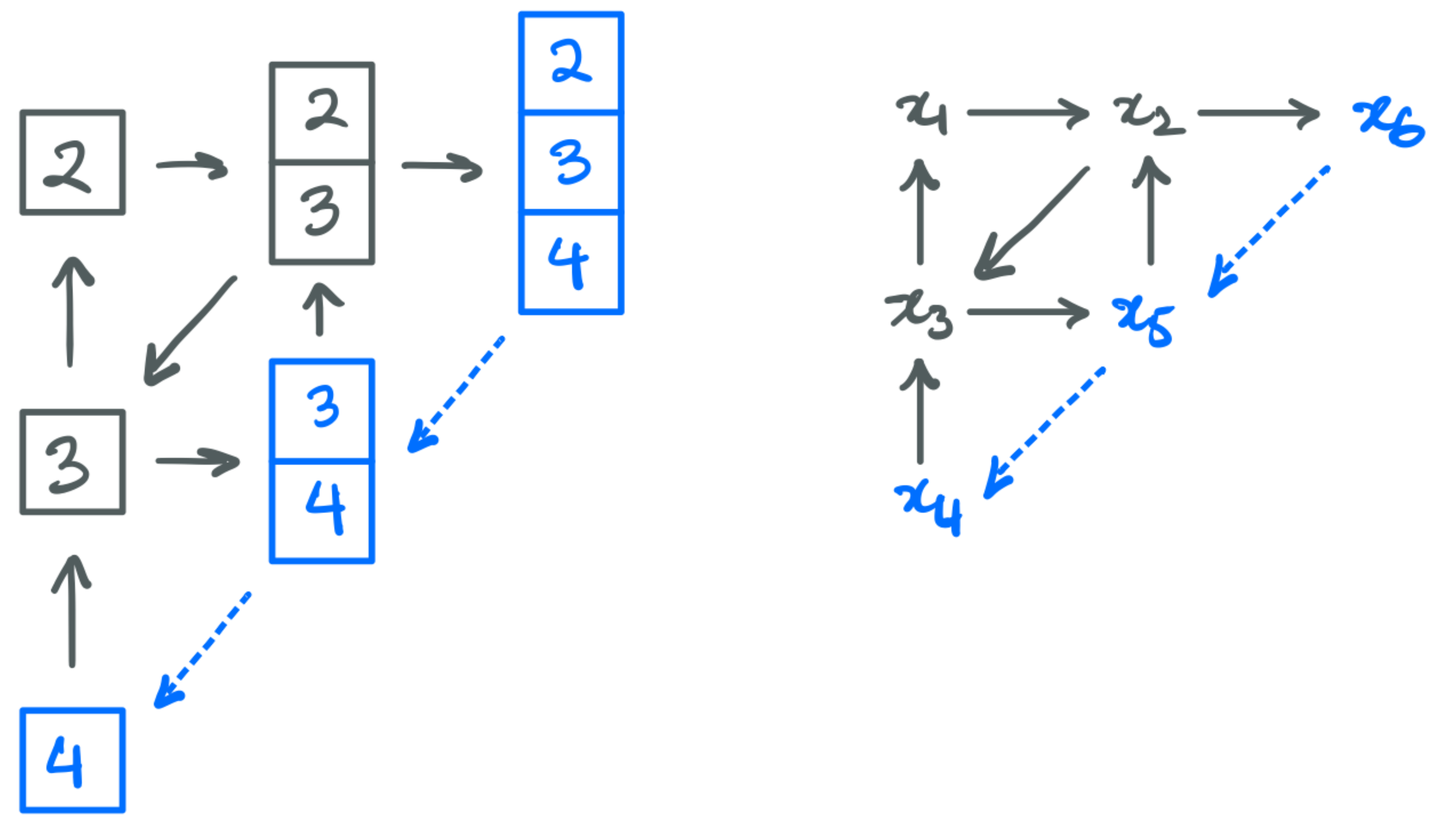
GPT toggling RPP's

- Garver, Patrias and Thomas extended the notion of toggle for a poset to the toggle of $\rho \in R(w,n)$ at $x \in H(w)$ by fixing $\rho(y)$ for any $y \neq x$ and replacing $\rho(x)$ by

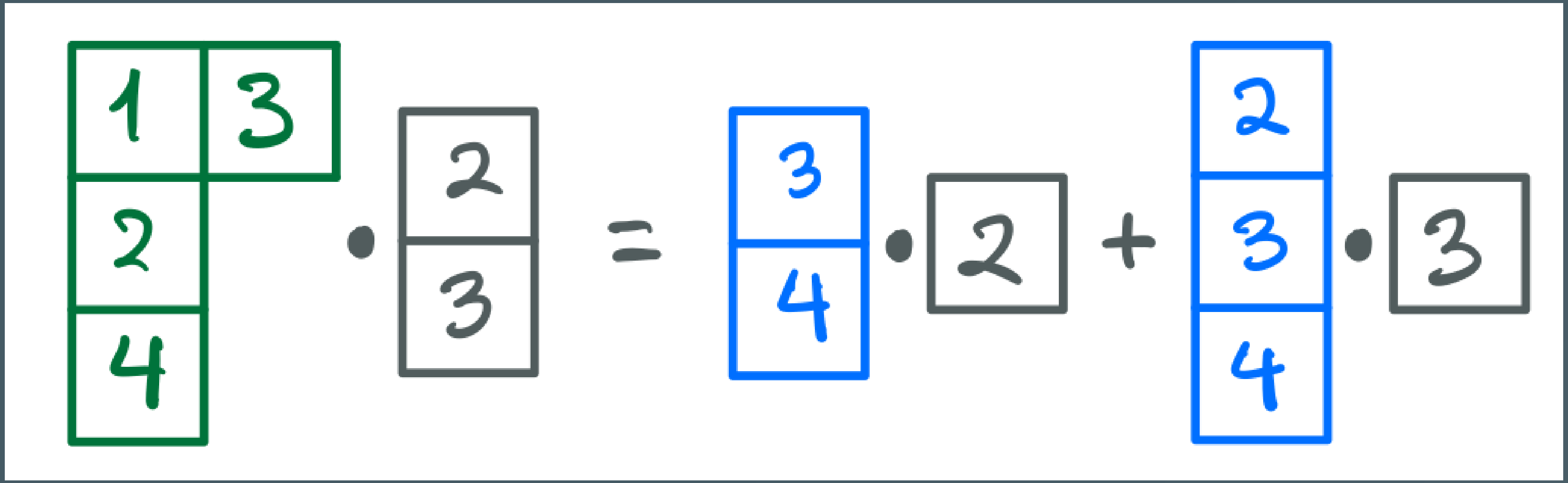
$$\max_{x \lessdot y_1} \rho(y_1) + \min_{y_2 \lessdot x} \rho(y_2) - \rho(x)$$

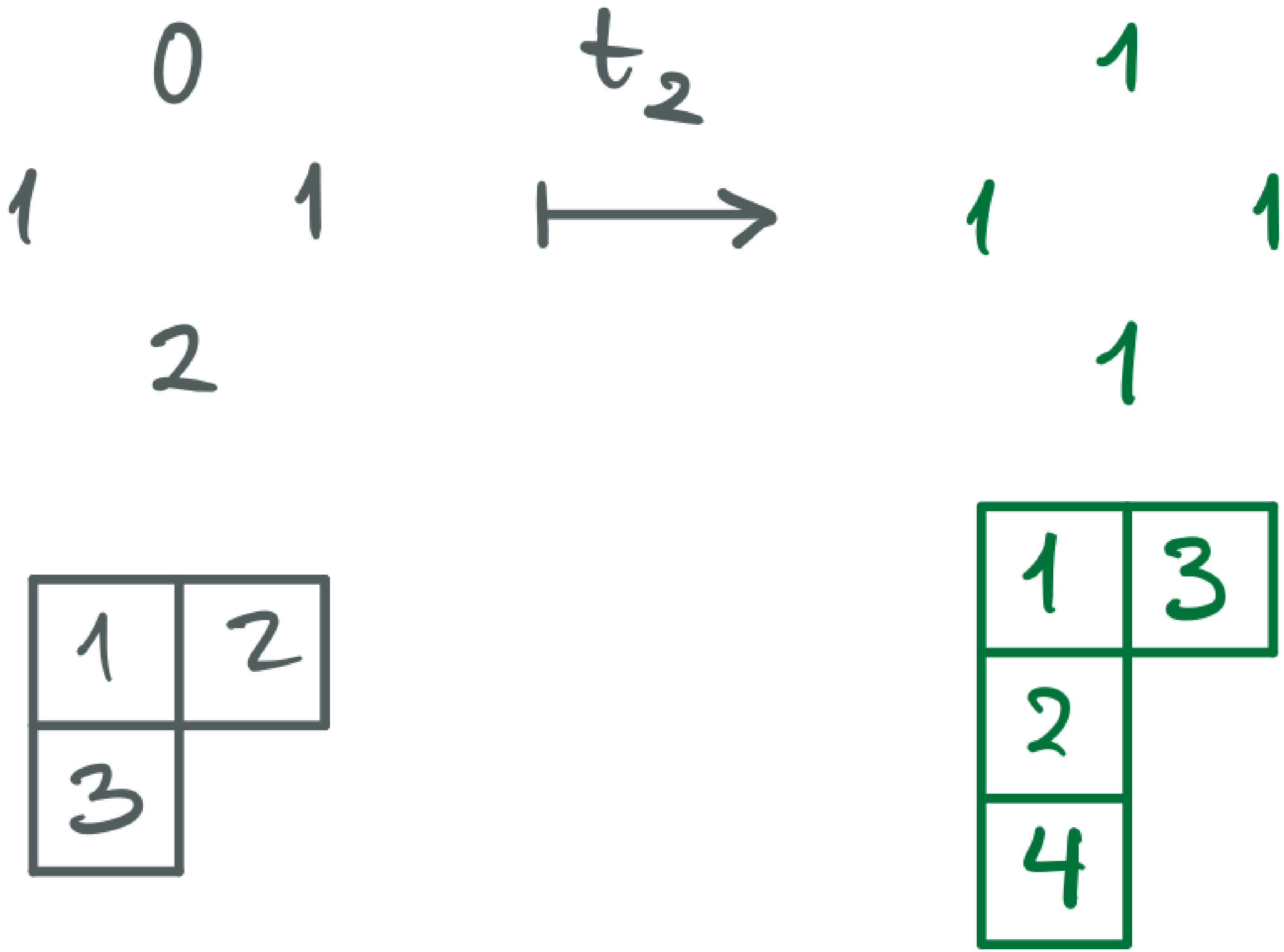
- The resulting RPP is denoted $t_x(\rho)$
- If $x, y \in \pi^{-1}(i)$ then $[t_x, t_y] = 0$ so the composition $\prod_{x \in \pi^{-1}(i)} t_x$ can be unambiguously referred to as t_i the (composite) toggle at i

$\mathfrak{sl}_4$ evidence



$$(1 \leftarrow 2 \rightarrow 3) * (1 \rightarrow 2) = P_2 \oplus S_1 + P_1 \oplus (1 \leftarrow 2)$$

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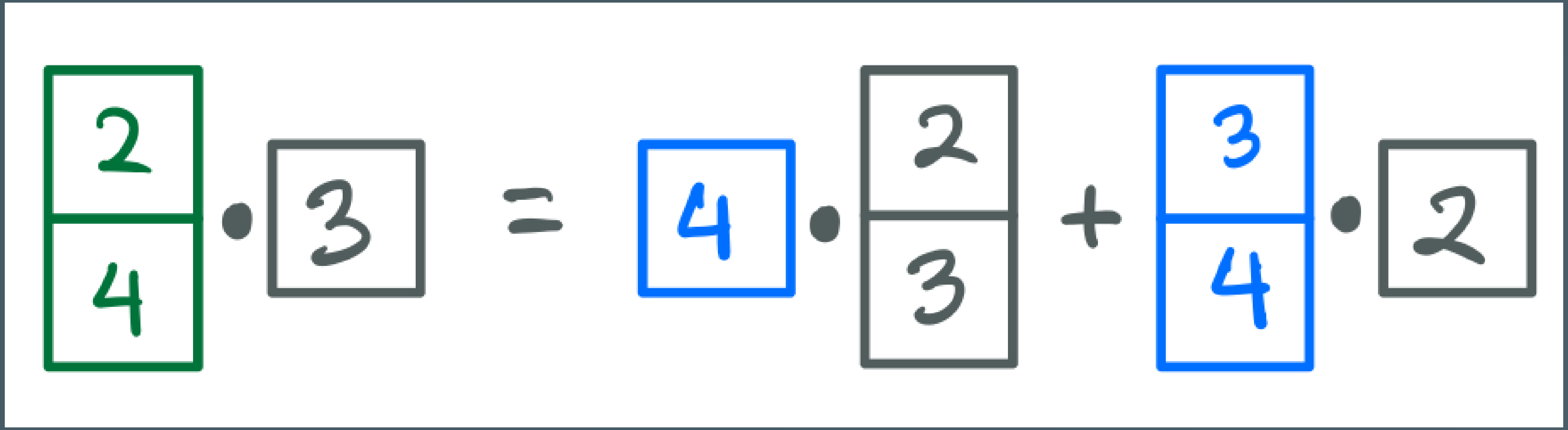
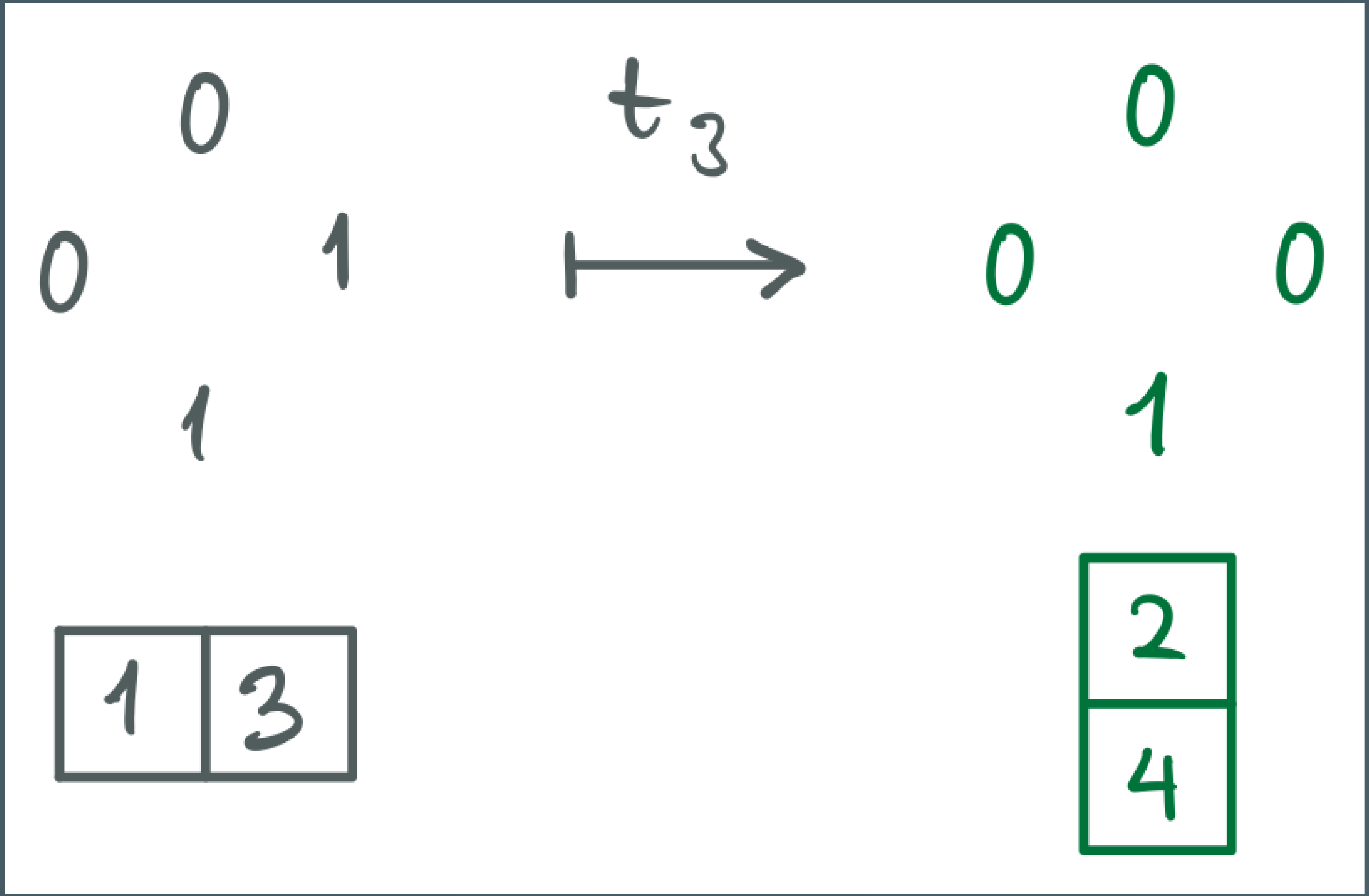
$$(1 \to 2 \leftarrow 3) * (1 \leftarrow 2) = P_3 \oplus (1 \to 2) + P_2 \oplus S_1$$





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Thank You 🎈