M-theory, symmetries and geometry

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Based on

- 1908.08027 with B. Heidenreich and D. Regalado,
- 2005.12831 with F. Albertini, M. Del Zotto and S. Hosseini,
- upcoming work with F. Apruzzi, F. Bonetti, S. Hosseini and S. Schäfer-Nameki.

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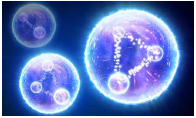

Simons Collaboration on Global Categorical Symmetries

Last Monday

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Today

All the global symmetries (of SCFTs) are in string theory.

What are symmetries?

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- We might not have a Lagrangian!

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The symmetries and anomalies of d-dimensional theories are encoded in a $(d+1)$-dimensional topological field theory.

In this talk I would like to:

- Motivate this answer.
- Identify these TFTs in some simple M-theory examples.

Anomalies

What are anomalies?

The textbook view on anomalies is that anomalies arise whenever we have a symmetry of the classical Lagrangian that is not a symmetry of the full quantum theory.

The problem is particularly serious whenever we are talking about gauge transformations: if a gauge transformation is anomalous then the theory is inconsistent.

The canonical example is the theory of a Weyl fermion in four dimensions charged under a $U(1)$ gauge symmetry

$$\mathcal{L} = \frac{1}{2g} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \psi^\dagger (i\partial_\mu - A_\mu) \sigma^\mu \psi$$

which looks fine classically, but is inconsistent quantum-mechanically.

A new approach to anomalies

One concise way to state the problem is that it might not be possible to define the phase of the partition function in a well defined way, as a function of the background fields modulo gauge invariance:

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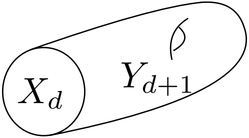
Recent developments [Dai, Freed '94], [Witten '15] have shed new light on this old topic.

These recent developments are geared towards condensed matter, but there are also interesting consequences for high energy physics.

The Dai-Freed viewpoint on anomalies

Consider the case that your space-time X_d is the boundary of some manifold Y_{d+1}, over which all the relevant structures on X_d extend.



We define the path integral of a fermion ψ on X_d as [Dai, Freed '04]

$$Z_\psi = |Z_\psi| e^{-2\pi i\, \eta(\mathcal{D}_{Y_{d+1}})}$$

with

$$\eta(\mathcal{D}_{Y_{d+1}}) = \frac{\dim\ker \mathcal{D}_{Y_{d+1}} + \sum_{\lambda\neq 0} \mathrm{sign}(\lambda)}{2}\,.$$

[*] For the experts, this is the same η that appears in the APS index theorem.

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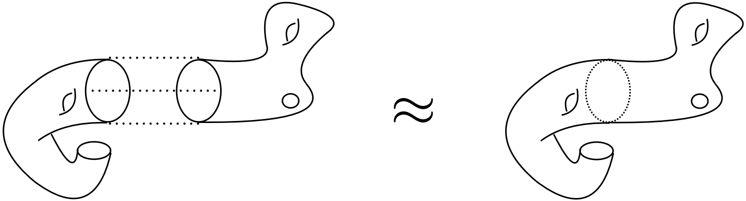
Nevertheless, it has very nice properties: if we change the orientation of the manifold the phase of the partition function changes sign:

$$e^{2\pi i\,\eta(\mathcal{D}_A)} = e^{-2\pi i\,\eta(\mathcal{D}_{\overline{A}})}$$

and it is "local", in the sense that η behaves nicely under gluing:

$$e^{2\pi i \eta(\mathcal{D}_A)} e^{2\pi i\,\eta(\mathcal{D}_B)} = e^{2\pi i\,\eta(\mathcal{D}_{A+B})}$$



The Dai-Freed viewpoint on anomalies

Anomalies, in this language, come from situations in which the phase of the partition function depends on the choice of Y_{d+1}:

$$e^{-2\pi i\,\eta(\mathcal{D}_{Y_{d+1}})} \neq e^{-2\pi i\,\eta(\mathcal{D}_{Y'_{d+1}})} \tag{1}$$

even if $\partial Y_{d+1} = \partial Y'_{d+1} = X_d$.

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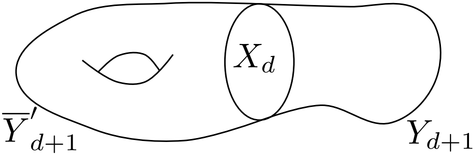
Gluing Y_{d+1} and $\overline{Y}'_{d+1}$ over X_d to form the closed manifold W_{d+1}, we find that the partition function is well defined as a function of the fields on X_d only if on every such W_{d+1}

$$e^{-2\pi i\,\eta(\mathcal{D}_{W_{d+1}})} = e^{-2\pi i\,\eta(\mathcal{D}_{Y_{d+1}})}/e^{-2\pi i\,\eta(\mathcal{D}_{Y'_{d+1}})} = 1 \tag{2}$$



The Dai-Freed viewpoint on anomalies

The theory with partition function

$$Z^{\mathcal{A}}(Y_{d+1}, A) = e^{2\pi i\, \eta(\mathcal{D}_A)}$$

is an example of a topological field theory in $(d+1)$-dimensions, known in this context as the *anomaly theory*.

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So when talking about anomalies, it is very natural to consider topological theories in one dimension higher. Later on I will give examples of anomaly theories for 1-form symmetries.

AdS/CFT

Classifying $\mathcal{N} = 4$ theories

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One can distinguish the different global forms by studying the partition function on four-manifolds $\mathcal{M}_4$ with $H^2(\mathcal{M}_4, C) \neq 0$, or by studying the properties and correlators of extended operators.

Holography and global structure

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Answered in [Witten '98]. The key insight is that we view the possible 4-dimensional theories as states in the Hilbert space of a 5-dimensional topological "bulk" theory, taking the radial direction as "time". [Friedan, Shenker '87], [Verlinde '88], [Moore, Seiberg '88], [Witten '89], . . . , [Witten '98], . . . , [Belov, Moore], . . .

Quantization of the bulk TQFT

(Following [Witten '98])

The reduction of IIB on S^5 gives an effective action

$$L_{\mathsf{CS}} = \frac{N}{2\pi i}\int_{X_5} B_2 \wedge dC_2\,. \tag{3}$$

The equations of motion are

$$dB_2 = dC_2 = 0 \tag{4}$$

and B_2, C_2 are canonically conjugate ($B_2 = C_2 = 0$ is disallowed!):

$$[B_{ij}(x), C_{kl}(y)] = -\frac{2\pi i}{N}\epsilon_{ijkl}\delta^4(x-y)\,. \tag{5}$$

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In order to specify the boundary conditions, in addition to specifying the vevs of local gauge invariant operators, we need to specify

$$\alpha = \int_S B_2 \quad ; \quad \beta = \int_S C_2 \tag{6}$$

for any $S \subset \mathcal{M}_4$ near the boundary, $X_5 \approx \mathbb{R} \times \mathcal{M}_4$.

Quantization of the bulk TQFT

(Following [Witten '98])

Define operators measuring the flux

$$\Phi_{\mathsf{RR}}(S) = \exp\left(i\int_S C_2\right) \quad ; \quad \Phi_{\mathsf{NS}}(T) = \exp\left(i\int_T B_2\right) . \quad (7)$$

They do not commute:

$$\Phi_{\mathsf{RR}}(S)\Phi_{\mathsf{NS}}(T) = \Phi_{\mathsf{NS}}(T)\Phi_{\mathsf{RR}}(S)\exp\left(\frac{2\pi i}{N} S\cdot T\right) . \quad (8)$$

(**Note:** Commutativity $\equiv$ non-intersection mod N.)

The inequivalent operators are parameterized by classes in $H_2(\mathcal{M}_4, \mathbb{Z}_N)$, so the group of operators acting on the Hilbert space is the finite Heisenberg group W in

$$0 \to \mathbb{Z}_N \to W \to H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{NS}} \times H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{RR}} \to 0 \, . \quad (9)$$

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Up to redefinitions W has a single representation. It can be constructed starting from a maximal isotropic subspace $\mathcal{I}$, i.e. a maximal commuting set of operators $\Phi(w)$.

Define $\mathbf{a} = (a_{\mathsf{NS}}, a_{\mathsf{RR}}) \in \mathfrak{H} := H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{NS}} \times H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{RR}}$, and similarly $\mathbf{b} = (\mathbf{b}_{\mathsf{NS}}, \mathbf{b}_{\mathsf{RR}})$. Introduce

$$\mathbf{a} \cdot \mathbf{b} = a_{\mathsf{NS}} \cdot b_{\mathsf{RR}} - a_{\mathsf{RR}} \cdot b_{\mathsf{NS}} \,. \tag{10}$$

Define $\mathcal{I}$ to be a maximal subgroup of $\mathfrak{H}$ such that

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$\mathcal{I}$ fixes the familiar notions, such as the gauge *group*, but the choice of $\mathcal{I}$ is more fundamental: it applies to 6d and class-$\mathcal{S}$ too.

Reproducing the AST classification

The classification of [Aharony, Seiberg, Tachikawa '13] can be reproduced from this viewpoint [Tachikawa '13]:

Consider a choice $\mathcal{I}_{T^2} \otimes H^2(\mathcal{M}_4, \mathbb{Z}_N)$ of $\mathcal{I}$, with $\mathcal{I}_{T^2}$ a maximal isotropic subgroup of $H^1(T^2, \mathbb{Z}_N) = \mathbb{Z}_N \oplus \mathbb{Z}_N$. The conditions of maximality and Dirac quantization in AST map to maximality and isotropy of $\mathcal{I}_{T^2}$. (I.e. this class of polarizations agrees precisely with the AST classification.)

Examples:

- $\mathcal{I}_{T^2} = \{(1,0), (2,0), \ldots, (N-1,0)\} \mapsto SU(N)$
- $\mathcal{I}_{T^2} = \{(0,1), (0,2), \ldots, (0,N-1)\} \mapsto (SU(N)/\mathbb{Z}_N)_0$
- $\mathcal{I}_{T^2} = \{a(N,0) + b(0,N)\} \mapsto (SU(N^2)/\mathbb{Z}_N)_0$

Higher form symmetries

We can understand these choices of global form as the choice of 1-form symmetry in the theory [Kapustin, Seiberg '14], [Gaiotto, Kapustin, Seiberg, Willett '14]:

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This picture generalises, see for instance [Aharony, Tachikawa '16] for applications to discrete 0-form symmetries of $\mathcal{N} = 3$ S-folds in $d = 4$, [Bergman, Tachikawa, Zafrir '20] for applications to generalised symmetries of ABJM ($\mathcal{N} = 6$ in $d = 3$), and [Apruzzi, van Beest, Gould, Schäfer-Nameki '21] for non-conformal cases.

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- Even theories that do are subtle. For example, the case of $\mathcal{N} = 4$ with algebra $\mathfrak{so}(N)$ has not been worked out. Because of the orientifold projection the B_2 and C_2 supergravity fields are projected out, so reformulating Witten's argument verbatim seems to require some version of differential real K-theory. I don't know what the right differential generalised cohomology theory is for $\mathcal{N} = 3$ S-folds.

M-theory

Back to geometric engineering

Consider, as an example of a theory that cannot be understood holographically, M-theory on $\mathbb{C}^2/\Gamma$. This gives rise to 7d SYM with gauge algebra $\mathfrak{g}_\Gamma$. The 1-form symmetry group of G_Γ (the simply connected form) is its centre:

$\Gamma \subset SU(2)$	$\mathfrak{g}_\Gamma$	G_Γ	$Z(G_\Gamma)$
$\mathbb{Z}_N$	$\mathfrak{su}(N)$	$SU(N)$	$\mathbb{Z}_N$
Binary dihedral $\mathrm{Dic}_{(2k-2)}$	$\mathfrak{so}(4k)$	$Spin(4k)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
Binary dihedral $\mathrm{Dic}_{(2k-1)}$	$\mathfrak{so}(4k+2)$	$Spin(4k+2)$	$\mathbb{Z}_4$
Binary tetrahedral $2T$	$\mathfrak{e}_6$	E_6	$\mathbb{Z}_3$
Binary octahedral $2O$	$\mathfrak{e}_7$	E_7	$\mathbb{Z}_2$
Binary icosahedral $2I$	$\mathfrak{e}_8$	E_8	1

Other global forms are possible, for instance $SU(N)/\mathbb{Z}_N$, which has a magnetic 4-form symmetry.

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In [IGE, Heidenreich, Regalado '19] we argued that (like in holography) it is the second option that is realised: the choice of global form for the gauge group is encoded in a choice of boundary conditions (at infinity) for the supergravity fields, and all possible global forms can be obtained in this way. (Related work: [Del Zotto, Heckman, Park, Rudelius '15], [Morrison, Schäfer-Nameki, Willett '20], [Albertini, Del Zotto, IGE, Hosseini '20], [Closset, Schäfer-Nameki, Wang '20], [Del Zotto, IGE, Hosseini '20], . . .)

Non-commutativity of fluxes in M-theory

Let us put M-theory on $\mathcal{M}_{11} = \mathcal{N}_{10} \times \mathbb{R}$. We will try to understand the Hilbert space $\mathcal{H}(\mathcal{N}_{10})$, or more precisely its grading by flux. This was done in [Freed, Moore, Segal '06].

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M-theory contains 3-form gauge fields C_3. The magnetic charge is measured by the topological class of C_3. To measure the electric charge, recall that in the Hamiltonian formulation of the theory the canonical momentum Π_{C_3} conjugate to C_3 is $\star G_4$. This is what we integrate to measure the electric charge.

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M-theory contains 3-form gauge fields C_3. The magnetic charge is measured by the topological class of C_3. To measure the electric charge, recall that in the Hamiltonian formulation of the theory the canonical momentum Π_{C_3} conjugate to C_3 is $\star G_4$. This is what we integrate to measure the electric charge. If we express states in $\mathcal{H}(\mathcal{N}_{10})$ in terms of their wavefunctions $\psi(C_3)$, then a state of definite electric charge is an eigenstate of momentum:

$$\psi(C_3 + \lambda) = e^{2\pi i \int_{\mathcal{N}_{10}} Q_e \lambda} \psi(C_3)$$

for all flat λ. Here $Q_e \in H^7(\mathcal{N}_{10})$ is the electric charge.

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So we cannot simultaneously measure electric and magnetic charges, if there are flat topologically non-trivial λ. This is the case iff $\operatorname{Tor} H^4(\mathcal{N}_{10}) \neq 0$.

Non-commutativity of fluxes in M-theory

This can be restated in terms of the flux operators, as follows: for every $\sigma \in \operatorname{Tor} H_6(\mathcal{N}_{10};\mathbb{Z}) = \operatorname{Tor} H^4(\mathcal{N}_{10};\mathbb{Z})$ there is a unitary flux operator Φ_σ. Similarly for any $\sigma' \in \operatorname{Tor}(H_3(\mathcal{N}_{10};\mathbb{Z})) = \operatorname{Tor} H^7(\mathcal{N}_{10};\mathbb{Z})$.

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These operators in general do not commute:

$$\Phi_\sigma \Phi_{\sigma'} = e^{2\pi i\, \mathsf{L}(\sigma,\sigma')} \Phi_{\sigma'} \Phi_\sigma$$

where $\mathsf{L}(\sigma,\sigma')$ is the linking pairing on $\mathcal{N}_{10}$: choose $n \in \mathbb{Z}$ such that $n\sigma = \partial D$. Then

$$\mathsf{L}(\sigma,\sigma') = \frac{1}{n} D \cdot \sigma' \mod 1\,.$$

Non-commutativity of fluxes in M-theory

The pairing $\mathsf{L}(\cdot,\cdot)$ is *perfect*, which implies that if $\mathrm{Tor}(H_3(\mathcal{N}_{10};\mathbb{Z})) = \mathrm{Tor}(H_6(\mathcal{N}_{10};\mathbb{Z})) \neq 0$, then for each $\sigma \neq 0$ there is some σ' such that $\mathsf{L}(\sigma,\sigma') \neq 0$, and thus

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$$\Phi_\sigma \Phi_{\sigma'} = e^{2\pi i\, \mathsf{L}(\sigma,\sigma')} \Phi_{\sigma'} \Phi_\sigma \neq \Phi_{\sigma'} \Phi_\sigma \,.$$

What this all implies, it that whenever $\mathrm{Tor}(H_3(\mathcal{N}_{10};\mathbb{Z})) \neq 0$ it is not possible to simultaneously diagonalize all Φ_σ. In particular, it is not consistent to take the simple "fluxless" choice $\Phi_\sigma = 1$ for all σ. We need to turn on *some* flux at infinity!

Maximal isotropic subspaces

Despite the perhaps unfamiliar setting, the final algebraic structure is the same as in holography: we have a Hilbert space, and a set of non-commuting operators acting on it.

We can specify a state in the Hilbert space as usual: by choosing a maximal subspace $\mathcal{I} \subset \mathrm{Tor}(H_3(\mathcal{N}_{10}); \mathbb{Z})$ such that the corresponding group of operators $\{\Phi_x\}$ for $x \in \mathcal{I}$ is abelian, and imposing that

$$\Phi_x \left|0; L\right\rangle = \left|0; L\right\rangle \qquad \forall x \in \mathcal{I}$$

In our M-theory setting, this corresponds to setting to zero on the boundary as many fluxes as possible.

Back to M-theory on $\mathbb{C}^2/\Gamma$

We want to consider M-theory on a space $\mathcal{M}_{11} = \mathbb{C}^2/\Gamma \times \mathcal{M}_7$ with Γ a discrete subgroup of $SU(2)$. Let us apply our methods to classify the space of possible theories for a fixed $\mathfrak{g}$.

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We have that $\mathbb{C}^2/\Gamma$ is a cone over S^3/Γ, so in order to understand the boundary conditions at infinity we want to quantize the flux sector of M-theory on $\mathbb{R} \times S^3/\Gamma \times \mathcal{M}_7$.

Back to M-theory on $\mathbb{C}^2/\Gamma$

Γ acts freely on S^3, so $\pi_1(S^3/\Gamma) = \Gamma$. By Hurewicz's theorem

$$H_1(S^3/\Gamma) = \frac{\pi_1(S^3/\Gamma)}{[\pi_1(S^3/\Gamma), \pi_1(S^3/\Gamma)]} = \Gamma^{\mathsf{ab}}\,.$$

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The group Γ^{ab} is easy to determine:

$\Gamma \subset SU(2)$	$\mathfrak{g}_\Gamma$	Γ^{ab}
$\mathbb{Z}_N$	A_{N-1}	$\mathbb{Z}_N$
Binary dihedral $\mathrm{Dic}_{(2k-2)}$	D_{2k}	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
Binary dihedral $\mathrm{Dic}_{(2k-1)}$	D_{2k+1}	$\mathbb{Z}_4$
Binary tetrahedral $2T$	E_6	$\mathbb{Z}_3$
Binary octahedral $2O$	E_7	$\mathbb{Z}_2$
Binary icosahedral $2I$	E_8	1

(Notice that $\Gamma^{\mathsf{ab}} = Z(G_\Gamma)$, with G_Γ the simply connected Lie group with algebra g_Γ.)

Back to M-theory on $\mathbb{C}^2/\Gamma$

From here

$$H_*(S^3/\Gamma) = \{\mathbb{Z}, \Gamma^{\mathsf{ab}}, 0, \mathbb{Z}\}\,.$$

To make my life easier I will assume that $\mathcal{M}_7$ is closed and has no torsion in homology. Then Künneth's formula implies

$$\begin{aligned}\mathrm{Tor}(H_3(\mathcal{M}_7 \times S^3/\Gamma)) &= H_2(\mathcal{M}_7) \otimes H_1(S^3/\Gamma) = H_2(\mathcal{M}_7) \otimes \Gamma^{\mathsf{ab}} \\ &= H_2(\mathcal{M}_7; \Gamma^{\mathsf{ab}})\,.\end{aligned}$$

and similarly

$$\mathrm{Tor}(H_6(\mathcal{M}_7 \times S^3/\Gamma)) = H_5(\mathcal{M}_7; \Gamma^{\mathsf{ab}})\,.$$

Given elements $\sigma_a = a \otimes \ell_a$, $\sigma_b = b \otimes \ell_b$, we have the linking form

$$\mathsf{L}(\sigma_a, \sigma_b) = (a \cdot b)\, \mathsf{L}_{S^3/\Gamma}(\ell_a, \ell_b)\,.$$

Back to M-theory on $\mathbb{C}^2/\Gamma$

It is not difficult to compute the linking form on S^3/Γ, we find:

Γ	G_Γ	Γ^{ab}	L_Γ
$\mathbb{Z}_N$	$SU(N)$	$\mathbb{Z}_N$	$\frac{1}{N}$
$\mathrm{Dic}_{(4N-2)}$	$\mathrm{Spin}(8N)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathrm{Dic}_{(4N-1)}$	$\mathrm{Spin}(8N+2)$	$\mathbb{Z}_4$	$\frac{3}{4}$
$\mathrm{Dic}_{(4N)}$	$\mathrm{Spin}(8N+4)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathrm{Dic}_{(4N+1)}$	$\mathrm{Spin}(8N+6)$	$\mathbb{Z}_4$	$\frac{1}{4}$
$2T$	E_6	$\mathbb{Z}_3$	$\frac{2}{3}$
$2O$	E_7	$\mathbb{Z}_2$	$\frac{1}{2}$
$2I$	E_8	0	0

Back to M-theory on $\mathbb{C}^2/\Gamma$

Classification

The possible global forms of the $d = 7$ theories on $\mathcal{M}_7$ are given by maximal commuting subspaces of $H_2(\mathcal{M}_7; \Gamma^{\mathsf{ab}}) \times H_5(\mathcal{M}_7; \Gamma^{\mathsf{ab}})$, with commutators as above.

This result agrees with what one obtains from applying the ideas in [Gaiotto, Moore, Neitzke '10], [Aharony, Seiberg, Tachikawa '13].

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An alternative derivation of this result can be obtained by thinking about screening of line operators, closely following [Aharony, Seiberg, Tachikawa '13]. This was done in geometric language in [Del Zotto, Heckman, Park, Rudelius '15], where they introduce the *defect group*, which in this case is

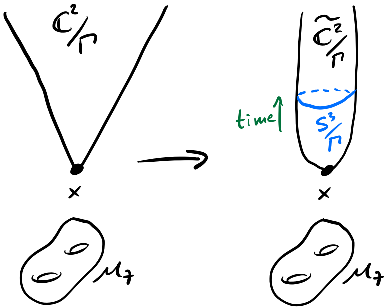
$$\mathbb{D} = \frac{H_2(\mathbb{C}^2/\Gamma, S^3/\Gamma)}{H_2(\mathbb{C}^2/\Gamma)} \times \frac{H_2(\mathbb{C}^2/\Gamma, S^3/\Gamma)}{H_2(\mathbb{C}^2/\Gamma)}$$

It is easy to show that $H_2(\mathbb{C}^2/\Gamma, S^3/\Gamma)/H_2(\mathbb{C}^2/\Gamma) = H_1(S^3/\Gamma) = \Gamma^{\mathrm{ab}}$.

The symmetry theory

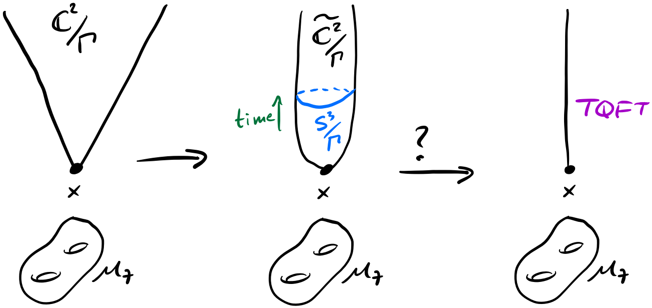
An effective 8d TQFT

The previous derivation was really looking to a modified asymptotic structure.



An effective 8d TQFT

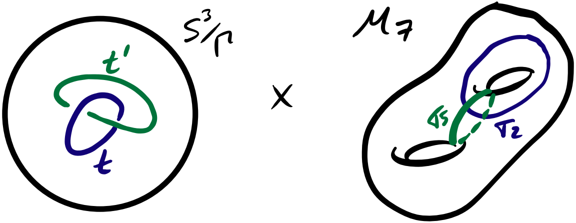
The previous derivation was really looking to a modified asymptotic structure. This suggests a strategy for deriving the TQFT associated to the field theory: dimensional reduction on the link of the singularity:



The BF theory

We have already obtained part of the structure of this TQFT: we know that in the full theory on $S^3/\Gamma \times X^8$ there are non-commuting operators wrapping $t \times \sigma_2$ and $t' \times \sigma_5$, with $t, t' \in H_1(S^3/\Gamma) = \Gamma^{\text{ab}}$ and $\sigma_i \in H_i(X^8)$. Their commutation relations (on a spatial slice $\mathcal{M}_7$ of X^8) are

$$\Phi(t \times \sigma_2)\Phi(t' \times \sigma_5) = e^{2\pi i L(t,t')\sigma_2 \cdot \sigma_5}\Phi(t' \times \sigma_5)\Phi(t \times \sigma_2)\,.$$



The BF theory (continued)

Fix $\Gamma = \mathbb{Z}_N$ for concreteness. Then from the point of view of X_8 we have $\mathbb{Z}_N$ 2-surface operators and 5-surface operators whose relative phase goes with the intersection number divided by N. This can be represented as a

$$S_{\text{top}} = N \int_{X_8} B_2 \wedge dC_5$$

topological action (as in [Witten '98]).

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The choice of global structure is a choice of gapped boundary conditions for this TFT at "infinity".

Mixed anomalies

(Work in progress with F. Apruzzi, F. Bonetti, S. Hosseini and S. Schäfer-Nameki)

The 7d theory, in addition to the 1-form and/or 4-form symmetries acting on Wilson lines / 't Hooft surfaces, has a $U(1)_I$ continuous 2-form symmetry acting on instanton surfaces.

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There is a mixed 't Hooft anomaly between the $U(1)_I$ symmetry and the 1-form symmetry, of the form

$$S_{\text{anomaly}} = \int_{X_8} dC_I^{(3)} \wedge r_{\mathfrak{g}} \frac{\mathcal{P}(B_2)}{2}$$

with $r_{\mathfrak{g}}\mathcal{P}(B_2)/2$ the fractional instanton number in the presence of a background for the 1-form symmetry, and $C_I^{(3)}$ the background for the instanton 1-form symmetry.

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This can be derived by "reducing" $\int_{\mathcal{M}_{11}} C_3 G_4 G_4 + C_3 X_8$ on S^3/Γ, *keeping track of the torsion sector*. (See also recent work by [Cvetič, Dierigl, Lin, Zhang '21] for a different approach.)

Differential cohomology

KK reductions beyond de Rham

Mathematically, we want to extract a (discrete) cohomology invariant on $d+1$ dimensions from $\int_{\mathsf{Link}^{10-d}}(C_3 G_4 G_4 + C_3 X_8)$.

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By means of this formalism we can derive the 7d result in the previous slide and (for example) the much more subtle anomaly theory in 5d for $SU(p)_q$ [Gukov, Pei, Hsin '20]

$$S^{(5d)}_{\text{anomaly}} = \int_{X_6} dC^{(1)}_I \wedge \frac{p(p-1)}{2\,\text{gcd}(p,q)} \mathcal{P}(B_2) + \frac{qp(p-1)(p-2)}{6\,\text{gcd}(p,q)^3} B_2^3 \,.$$

Conclusions

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String theory provides a beautiful geometrisation of these developments. In some simple examples in 7d and 5d we could derive systematically the symmetry theory from doing dimensional reduction of the M-theory Chern-Simons sector on the link of the singularity. We did not need any Lagrangian information about the theory, only the geometry!

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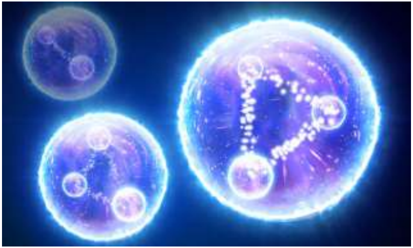
  - A good probe of the structure of the field theories, without requiring any Lagrangian description (for example, for probing duality proposals).
  - Perhaps even better: a generalised form of the Landau paradigm.

If this sounds fun... we're hiring!



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Review of anomalies (I)

Consider a (Lagrangian) theory $\mathcal{T}$ with some global symmetry G. We can introduce a background connection A_G for G, and compute the path integral

$$Z(A_G) = \int [D\psi] e^{-S(A_G, \psi)} \tag{12}$$

where ψ are some fundamental fields. (Only the fermionic fields, and the connection they couple to, matter for my discussion.)

Denote by $\mathcal{M}$ the space of all A_G. We have an anomaly whenever $Z(A_G)$ is not well defined as a function on the manifold $\mathcal{M}/G$:

- Non-invariance under small loops (curvature) in $\mathcal{M}/G$: *local anomaly*.
- Non-invariance under parallel transport for non-trivial loops in $\mathcal{M}/G$: *global anomalies*.

Review of anomalies (II)

Ungappable fields only

If a field can get a mass without breaking the symmetry G (it is *gappable*), then it can be integrated out without breaking the symmetry, and can be ignored for the purposes of determining anomalies.

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This means that for Lagrangian theories anomalies are at most phases: for any field ψ in a representation R, we can include an extra field $\tilde{\psi}$ in a rep $\overline{R}$ (and with an action which is the conjugate of that for ψ), and then the full matter content can be made massive without breaking any symmetries. So

$$Z(A_G) = Z_\psi(A_G) Z_{\tilde{\psi}}(A_G) = Z_\psi(A_G)\overline{Z_\psi(A_G)} = \left|Z_\psi(A_G)\right|^2 . \tag{13}$$

Since the $\psi + \tilde{\psi}$ theory is gappable, we have that $\left|Z_\psi(A_G)\right|$ is a well defined function on $\mathcal{M}$.

Review of anomalies (III)

In general, $Z(A_G)$ is a section of some bundle over $\mathcal{M}/G$. If the bundle is non-trivial the theory is still consistent; we say that we have a 't Hooft anomaly, which may be local or global.

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What an anomaly means is that the symmetry G cannot be gauged, since gauging involves integration of $Z(A_G)$ over $\mathcal{M}/G$.

Review of anomalies (IV)

The local anomaly

Local anomalies are easy to describe: the object that encodes the curvature of $Z(A)$ on $\mathcal{M}/G$ is the "anomaly polynomial"

$$\mathcal{I}_{d+2} = \mathrm{ch}(F)\hat{A}(R)|_{d+2} \tag{14}$$

Review of anomalies (IV)

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$$\mathcal{I}_{d+2} = \mathrm{ch}(F)\hat{A}(R)|_{d+2} \tag{14}$$

We will consider the case in which there are no local anomalies, so that

$$\mathcal{I}_{d+2} = 0\,. \tag{15}$$

Geometrically, $Z(A)$ is a section of a *flat* line bundle on $\mathcal{M}/G$. How do we detect a possible global anomaly?

The "traditional" global anomaly

Consider a symmetry transformation $g\colon \mathbb{R}^d \to G$. We impose that $g \to 1$ at infinity, so if gauged it leads to a proper gauge transformation. The resulting set of transformations are topologically classified by maps $S^d \to G$ up to continuous deformations, i.e. by $\pi_d(G)$.

Now, for any choice of $[g] \in \pi_d(G)$, pick a representative g and consider the family of (not pure gauge) connections

$$A_G(g;t) = f(t) g^{-1} dg \tag{16}$$

for some smooth $f(t)$ such that $f(0) = 0$ and $f(1) = 1$. This defines a loop in the space of connections (modulo gauge transformations). So there is a global anomaly if

$$\frac{Z(A_G(g;1))}{Z(A_G(g;0))} = \frac{Z(0^g)}{Z(0)} = e^{i\mathcal{A}} \neq 1\,. \tag{17}$$

The "traditional" global anomaly: example

Consider for example the case in which the fermions are *real*. This means that the mass coupling

$$m\psi\psi = 0 \tag{18}$$

does not break G, but it identically vanishes.
We can add an extra copy of the fermions, and introduce a mass coupling

$$m\psi_1\psi_2 \neq 0 \tag{19}$$

This implies that $Z(A_G)^2$ is well defined, so the anomaly is $\mathbb{Z}_2$-valued (i.e. $e^{i\mathcal{A}} = \pm 1$ at most).

The "traditional" global anomaly: example

An example of real fermions are 4d Weyl fermion ψ_1 in the fundamental of $SU(2)$. This is a real fermion (the mass term is allowed, but it identically vanishes), since the fundamental of $SU(2)$ is pseudoreal, and the Weyl spinor of $\mathrm{Spin}(4) = SU(2) \times SU(2)$ is pseudoreal.

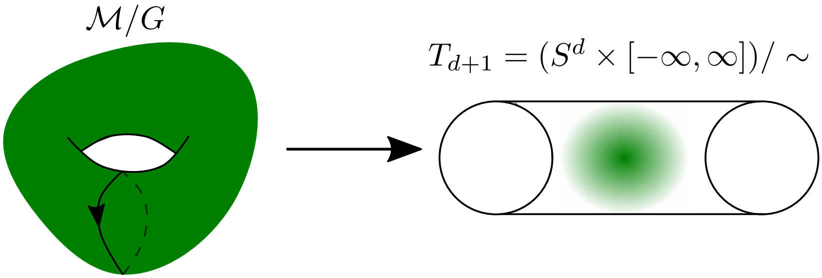
Famously [Witten '82], this system has a global anomaly:

$$Z(0) = -Z(0^g) \tag{20}$$

for $[g]$ the non-trivial generator of $\pi_4(SU(2)) = \mathbb{Z}_2$. This implies that the theory becomes ill-defined when we try to gauge the $SU(2)$ group:

$$\int_{\mathcal{M}} [DA] Z(A) e^{-\operatorname{Tr}(F^2)} = \int_{\mathcal{M}/G} [DA] K(1 + (-1)) Z(A) e^{-\operatorname{Tr}(F^2)} = 0\,. \tag{21}$$

The mapping torus



$$\frac{Z(A_G(g;1))}{Z(A_G(g;0))} = \mathrm{ind}(\mathcal{D}_{T_{d+1}}) \text{ mod } 2 \quad (22)$$

Other groups, other spacetimes

Note that from this point of view, we are only looking to what happens to S^d, or equivalently a neighbourhood of a point. (We are looking to anomalies which are "local" in spacetime.)

There can only be such anomalies when $\pi_d(G) \neq 0$, and for $d = 4$ this is only the case for $G = USp(n)$.

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Could we have new anomalies once we consider more general spacetime topologies? (These would be anomalies which are "global" in spacetime.)

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The expression on the right hand side is the local anomaly polynomial, so in the absence of local anomalies (easily checked, I'll assume it) we have that

$$\exp(2\pi i\, \eta(\mathcal{D}_{W_{d+1}})) = 1 \qquad (25)$$

whenever W_{d+1} is a boundary.

Anomalies and bordism

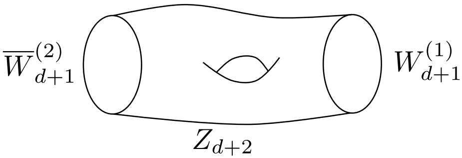
What this means is that if we have some manifold Z_{d+2} such that

$$\partial Z_{d+2} = W_{d+1}^{(1)} - W_{d+1}^{(2)} \tag{26}$$

then

$$\exp(2\pi i\,\eta(\mathcal{D}_{W_{d+1}^{(1)}})) = \exp(2\pi i\,\eta(\mathcal{D}_{W_{d+1}^{(2)}})) \tag{27}$$

This is a huge simplification! For the purposes of anomalies any two manifolds which can be connected via a third manifold are then equivalent: $W_{d+1}^{(1)} \sim W_{d+1}^{(2)}$



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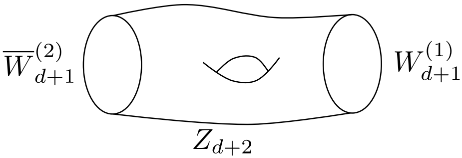
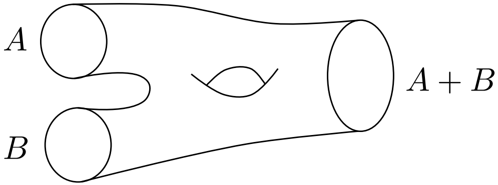


This equivalence relation is known as **bordism**, and the resulting equivalence class of manifolds is denoted Ω_{d+1}.

Some basic properties of bordism and η

The equivalence class Ω_{d+1} is an abelian group, under disjoint union of manifolds:



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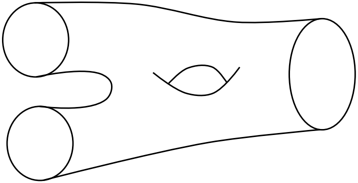

We have that

$$e^{2\pi i\eta(\mathcal{D}_A)}e^{2\pi i\,\eta(\mathcal{D}_B)} = e^{2\pi i\,\eta(\mathcal{D}_{A+B})} \tag{28}$$

so the global anomaly is a *homomorphism*

$$e^{e\pi i\eta}\colon \Omega_{d+1} \to U(1) \tag{29}$$

So, for example, if $\Omega_{d+1} = 0$, the anomaly necessarily vanishes.

Decorating bordism

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We are interested in gauge theories. That is, in understanding the partition function as a function of the connection on a principal bundle $\mathcal{P}_G$ on the manifold, for some Lie group G. This can be probed by decorating the manifolds with maps $W_{d+1} \to BG$, with BG the "classifying space of G". Some examples

G	BG
$\mathbb{Z}_2$	$\mathbb{RP}^\infty$
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In general, bordism groups of Spin manifolds W_{d+1} decorated with a map to $\mathcal{M}$ are denoted by

$$\Omega_{d+1}^{\mathrm{Spin}}(\mathcal{M})\,. \tag{30}$$

The strategy

The beauty of the Dai-Freed approach is that we can formulate necessary and sufficient conditions for quantum consistency on any manifold X_d for a theory with group G:

- Construct all the bordism groups in one dimension higher with the right structure. For instance $\Omega_{d+1}^{\mathrm{Spin}}(BG)$.
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Otherwise, we need to find some generators of $\Omega_{d+1}^{\mathrm{Spin}}(BG)$ on which we can compute η. Not an easy task!

Is this really an inconsistency?

What we really have if we have a “Dai-Freed anomaly” is that the partition function depends on a choice of bulk.

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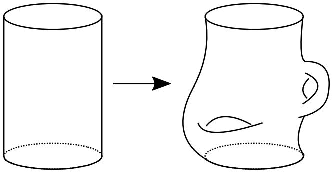
It seems nevertheless natural to assume that Dai-Freed anomaly-cancellation is the right prescription once we couple to gravity. **Conjecturally:**

$$\Omega_d(BG) = \overline{\Omega}_d(BG) \qquad (31)$$

with $\overline{\Omega}$ the bordism group generated by *generalised mapping tori*.



Generalised mapping tori and bordism

The following somewhat heuristic reasoning indicates that

$$\Omega_d(BG) = \overline{\Omega}_d(BG) \tag{32}$$

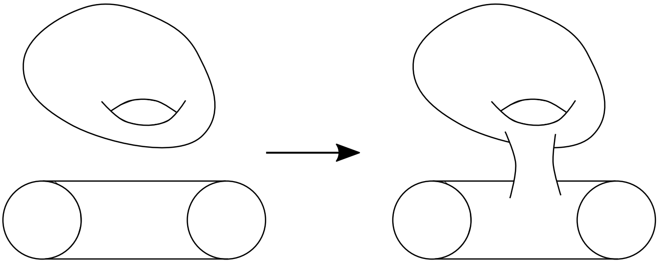
holds.

Note that in bordism

$$[X \# Y] = [X] + [Y] \tag{33}$$

so we can obtain generalised mapping tori by gluing arbitrary generators to mapping tori.

Generalised mapping tori and bordism



The $(0, 2)$ viewpoint

It is straightforward to extend the previous discussion to the 6d $(0, 2)$ A_{N-1} theory. [Witten '98] Holographically, the key term is

$$\mathcal{L} = N \int_{AdS_7} C_3 \wedge dC_3 + \ldots \tag{34}$$

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The same arguments as before work basically unmodified. We end up with the requirement of choosing a maximal isotropic subgroup of $H^3(\mathcal{M}_6, \mathbb{Z}_N)$.

When $\mathcal{M}_6 = \mathcal{M}_4 \times T^2$ there is a trivial map to the previous discussion, choosing one of the generators of $H^1(T^2, \mathbb{Z})$ as the "NS" direction and one as the "RR" direction.