Geodesic complexity for nongeodesic spaces

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Definition. (Recio-Mitter) If X is a metric space and E is a subspace of $X \times X$, a *geodesic motion planning rule* (GMPR) on E is a continuous map $s : E \to P(X)$ such that $s(x_0, x_1)$ is a (minimal) geodesic from x_0 to x_1.

Definition. (Recio-Mitter) $\mathrm{GC}(X)$ is the smallest k such that $X \times X = E_0 \sqcup \cdots \sqcup E_k$ with a GMPR on each E_i.

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$F(X,2)=\{(x,x')\in X\times X:\ x'\neq x\}$.

$F(\mathbf{R}^n,2)$ is not geodesic. (Linear path (a,a') to (b,b') might "collide.")

$F_\epsilon(\mathbf{R}^n,2)=\{(x,x'):d(x,x')\geq\epsilon\}$ is geodesic and has same homotopy type as $F(\mathbf{R}^n,2)$. Found explicit geodesics and showed $\mathrm{GC}=\mathrm{TC}$.

We introduce "near geodesics" and use it to define and compute GC for non-geodesic spaces, many with GC=TC, but one with $GC=TC+1$.

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Definition. Let X be a metric space such that the completion $\overline{X}$ is geodesic. The set of points (x_0, x_1) of $X \times X$ for which there is no geodesic from x_0 to x_1 is called the *nogeo* set of X. If x_0 and x_1 are in the nogeo set of X, a *near geodesic* from x_0 to x_1 is a map $\phi : I \to P(\overline{X}; x_0, x_1)$ satisfying

i. $\phi(0)$ is a geodesic in $\overline{X}$ from x_0 to x_1;
ii. $\phi((0, 1]) \subset P(X; x_0, x_1)$;
iii. if $s_n \to 0$, then $\text{length}(\phi(s_n)) \to \text{length}(\phi(0))$.

Definition. If E is contained in the nogeo set of X, a *near geodesic motion planning rule* (NGMPR) on E is a continuous map Φ from E to $P(\overline{X})^I$ such that, for all $(x_0, x_1) \in E$, $\Phi(x_0, x_1)$ is a near geodesic from x_0 to x_1. The *geodesic complexity* $\mathrm{GC}(X)$ is defined as the smallest k such that $X \times X$ can be partitioned into ENRs $E_0, \ldots, E_k$ such that each E_i has either a GMPR or NGMPR. It is also allowed that E_i be the union of topologically disjoint sets, of which one has a GMPR and the other a NGMPR.

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Examples with explicit near geodesics and $GC = TC$.

1. $\mathbf{R}^n - Q$, Q finite.
2. $F(\mathbf{R}^n, 2)$.
3. $F(\mathbf{R}^n - \{x_0\}, 2)$.
4. $C(\mathbf{R}^n, 2)$.
5. $F(Y, 2)$, Y the Y-graph.

Theorem. If $X = F(\mathbf{R}^n - Q, 2)$ with n even and Q a finite subset containing points q_1, q_2, q_3, q_4 such that the segments q_1q_2 and q_3q_4 intersect, and no other points of Q are in an expanded disk determined by these two segments, then $\mathrm{GC}(X) = 5$, while $\mathrm{TC}(X) = 4$.

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Theorem. Let Q be a finite subset of $\mathbf{R}^n$ with $|Q| \geq 2$, and $X = \mathbf{R}^n - Q$. Then $\mathrm{GC}(X) = \mathrm{TC}(X) = 2$.

Proof. (n even). Let

$$E_i = \{(a,b) : \ |ab \cap Q| = 0, 1, \geq 2\}, \ i = 0, 1, 2.$$

Use linear geodesic on E_0. On E_1 and E_2, use

$$\Phi(a,b)(s)(t) = (1-t)a + tb + \delta(a,b) \cdot s \cdot g(t) \cdot v(\tfrac{a-b}{\|a-b\|}),$$

v a unit vector field on S^{n-1},
$g(t) = \sin(\pi t)$, $0 \leq t \leq 1$,
$\delta(a,b) = \frac{1}{2}\min(d(ab, Q - ab), 1)$.

Therefore $\mathrm{GC}(X) \leq 2$. We have $\mathrm{GC}(X) \geq \mathrm{TC}(X) = 2$ by cup products.

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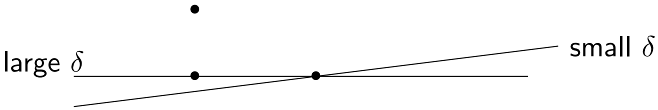

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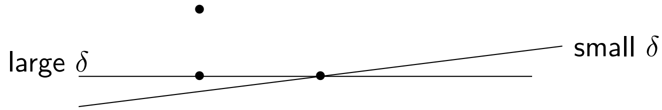

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Theorem. $X = F(\mathbf{R}^n - Q, 2)$, n even, $|Q| \geq 4$ implies $\mathrm{GC}(X) \leq 5$.
Proof. Points of $X \times X$ are $((a, a'), (b, b'))$.

Subsets E_0, E_1, E_2, $E_{1,1}$, $E_{1,2}$, $E_{2,2}$. ab and $a'b'$ don't collide. Subscripts are number of points of Q on the two segments.

Subsets C_0, C_1, C_2, $C_{1,1}$, $C_{1,2}$, $C_{2,2}$. Segments collide, but not at a point of Q. Not collinear.

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L_0, L_1, L_2. Collinear. aa' and bb' have opposite orientation.

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Sketch of proof. One domain with GMPR is the geoset E_0. It cannot be combined with any of our nogeo sets. Will show that $C_0 \cup C_1 \cup C_2 \cup C_{1,2} \cup C_{2,2}$ cannot be partitioned $S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4$ with NGMPR on each.

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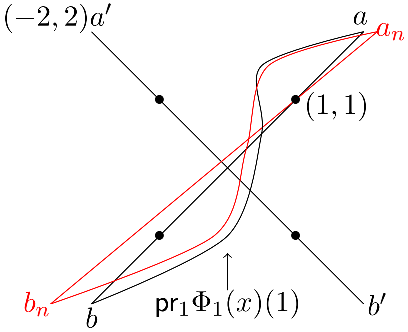
Let $x = ((a, a'), (b, b')) \in S_1$ with $\mathrm{pr}_1\Phi_1(x)(1)$ as shown.



Choose $x_n = ((a_n, a'), (b_n, b')) \to x$ with $a_n b_n$ on side of $(-1, 1)$ opposite to $\mathrm{pr}_1\Phi_1(x)(1)$. If $x_n \in S_1$, then $\mathrm{pr}_1\Phi_1(x_n)(1) \to \mathrm{pr}_1\Phi_1(x)(1)$, so passes on right of $(-1, 1)$. But homotopy to $\mathrm{pr}_1\Phi_1(x_n)(0)$ can't pass through $(-1, 1)$. Hence $x_n \notin S_1$.

Infinitely many x_n in some S_j. So may say $x_{n_1} \in S_2$.

Consider $x_{n_1,n_2} = ((a_{n_1,n_2}, a'), (b_{n_1,n_2}, b')) \to x_{n_1}$ with $a_{n_1,n_2}b_{n_1,n_2}$ parallel to $a_{n_1}b_{n_2}$. May assume all $x_{n_1,n_2} \in S_j$. Will show $j \neq 2$ and $j \neq 1$. Then may say $x_{n_1,n_2} \in S_3$.

Assume $j = 2$. If $a_{n_1,n_2}b_{n_1,n_2}$ passes $(1,1)$ on side opposite to $\text{pr}_1\Phi_2(x_{n_1})(1)$, then homotopy $\text{pr}_1\Phi_2(x_{n_1,n_2})$ must pass through $(1,1)$.

If $j = 1$, get contradiction as on previous page, since $a_{n_1,n_2}b_{n_1,n_2}$ is very close to $a_{n_1}b_{n_1}$.

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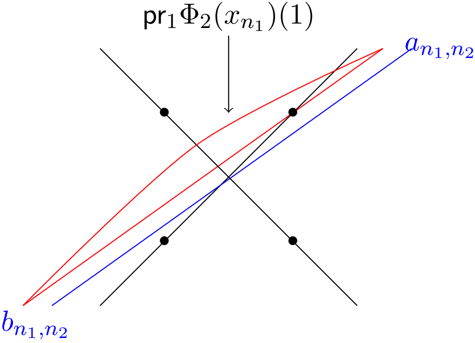

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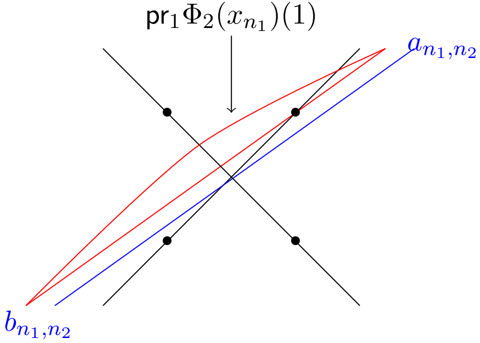

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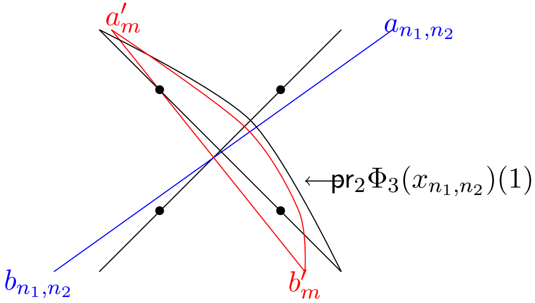
Choose points $x_{n_1,n_2,m} = ((a_{n_1,n_2}, a'_m), (b_{n_1,n_2}, b'_m)) \to x_{n_1,n_2}$ with $a'_m b'_m$ passing through $(-1,1)$ and passing $(1,-1)$ on the side opposite to $\mathrm{pr}_2\Phi_3(x_{n_1,n_2})(1)$, and all in the same S_j. If $j = 3$, since $\mathrm{pr}_2\Phi_3(x_{n_1,n_2,m})(1)$ converges uniformly to this, we obtain a contradiction since the homotopy cannot pass through $(-1,1)$



If $x_{n_1,n_2,m} \in S_2$, then $x_{n_1,n_2,n_2} \to x_{n_1}$, and we can get the same contradiction as before, using $\mathrm{pr}_1\Phi_2$, and similarly we can show $j \neq 1$. Therefore, $x_{n_1,n_2,m} \in S_4$.

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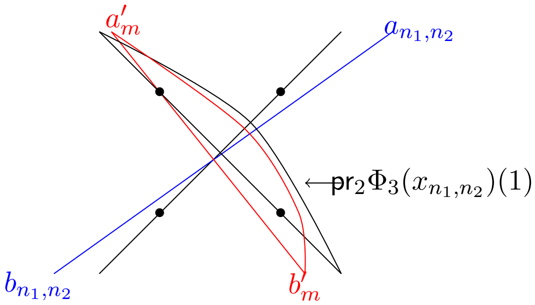

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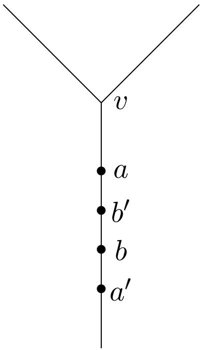
By similar methods, we obtain a sequence $x_{n_1,n_2,m,m'}$ not in $S_1 \cup S_2 \cup S_3 \cup S_4$, completing the proof.

For $X = F(Y,2)$, $\overline{X} = Y \times Y$ is geodesic, but some geodesics in $\overline{X}$ cannot be approximated by paths in X; e.g., from (a, a') to (b, b') below. If instead we use the intrinsic metric $d_I(x_0, x_1)$ defined as the infimum of lengths of paths from x_0 to x_1, then the completion is $F(Y,2) \cup \{(v,v)\}$, path lengths and topologies are preserved, and we have near geodesics. In this case, the geodesic goes from (a, a') to (v, v), and then back to (b, b'), and the near geodesics go slightly beyond (v, v).



Thank you!