Set theory of the reals (19w5064)

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1 General Overview of the Program

Set theory as a discipline dedicated to rigorous study of the infinite was born in 1874 when G. Cantor demonstrated the uncountability of the real numbers. His work originated in the study of sets of uniqueness of harmonic series and the relation between set theory and mathematical anlaysis has remained close ever since. Cantor's Continuum Hypothesis (CH), problem number one on Hilbert's influential 1900 list, stimulated much of the research in set theory for almost a century including two ground breaking discoveries: the work of K. Gödel who proved CH consistent with the rest of the axioms of set theory and, more importantly, uncovered the permanence of independence phenomena in axiomatically construed mathematics, complemented by the work of P. Cohen who proved the independence of CH, introducing the method of *forcing* which has since been developed into a powerful and versatile tool for producing consistency results, and showed that the phenomenon of independence applies also to *meaningful mathematical statements*. This has become more evident in recent decades with e.g.:

- Shelah's solution to the Whitehead problem in the theory of Abelian groups.
- Laver's consistency of the Borel conjecture in real analysis.
- Dales, Esterle, Solovay and Woodin's independence of Kaplansky's conjecture in harmonic analysis.
- Farah's consistency of all automorphisms of the Calkin algebra being inner in the theory of operator algebras.

Independence even touches on the frontiers of theoretical physics:

- Farah and Magidor's independence of the existence of Pitowsky spin models

As mathematics continues to grow both in depth and volume, it becomes increasingly more susceptible to the use of special set-theoretic axioms. It is one of the main tasks of logicians and set-theorists to identify those areas of mathematics likely to depend on extra axioms, and work in collaborations with the experts in given areas to delineate their use.

There are two extreme paradigms in the study of independence phenomena, Gödel's *Axiom of Constructibility* $V = L$ with strong *guessing principles* on one hand, and *forcing axioms* Martin's Axiom MA, the Proper Forcing Axiom PFA, and Martin's Maximum MM – strong forms of the Baire category theorem – on the other hand. With a combination of these, researchers have been able to establish the independence of many key problems including most of the ones mentioned above. However, not all problems are satisfactorily settled by the two. There are a number of problems the solution of which requires a search for a model of set theory which shares certain attributes of the *constructible universe* and certain attributes of a model of a forcing axiom:

- *Near Coherence of filters.*
- *Every compact space with perfectly normal square is metrizable.*
- *Every separable Fréchet topological group is metrizable.*
- *The normality of box products.*

An analysis of problems requiring these *ad hoc* forcing models usually depends on *cardinal invariants of the continuum*. These are combinatorially defined cardinal numbers bounded between $\aleph_1$ and 2^ω, but which are consistently different from both. These cardinal invariants (and corresponding $\diamondsuit$-principles) form a grid against which consistent combinatorial constructions can be measured with two possible outcomes: (typically) to obtain information facilitating search for a forcing model, or (seldom) to split the problem into cases which can be handled separately in order to arrive at a ZFC proof.

Together with basic combinatorial notions such as almost disjoint families and ultrafilters, the notion of a cardinal invariant is central to the proposed workshop.

1.1 Recent developments and objectives of the workshop

The goal of this workshop is to bring together researchers working in different fields of set theory and their applications; special focus will be put on the following recent and important developments:

1. new forcing techniques like multi-dimensional matrix iterations and, in particular, Boolean ultrapowers of forcing notions,
2. new aspects of cardinal invariants of the continuum, like their generalization to the uncountable context and their analogy to highness properties of oracles,

There have recently been surprising new developments concerning cardinal invariants of the continuum. Malliaris and Shelah have, using techniques originating from model theory, solved one of the main and oldest problems in the area by showing that the *pseudo-intersection number* $\mathfrak{p}$ and the *tower number* $\mathfrak{t}$ provably coincide, a result so highly valued that it earned them the 2017 Hausdorff medal.

On the side of consistency results, Mejía, partially in joint work, has developed and applied *matrix iterations* and generalized them to higher dimensions, and Shelah together with Raghavan, and Goldstern and Kellner, respectively, have invented a new method of forcing with *Boolean ultrapowers*. Both methods have a common goal of separating the values of several cardinal invariants simultaneously, a task which was considered notoriously difficult. With respect to results, the latter has been more powerful, but this comes at the expense of using large cardinal consistency strength. One of the main objectives of the workshop will be to present these new promising and deep techniques to a wider audience of experts and young researchers to fine tune these as tools to attack other longstanding problems.

In two exciting developments, the scope of the study of cardinal invariants has been vastly extended in recent years. On the one hand, replacing Cantor space 2^ω or Baire space ω^ω by their generalized counterparts 2^κ and κ^κ where κ is an uncountable regular cardinal, many of the classical cardinal invariants have been redefined in this more general context, and a number of the classical results, both ZFC inequalities and independence results showing one cardinal is not provably larger than another, have been reproved, often with quite novel arguments. The general tenet here is that ZFC results generalize more easily and, in fact, there are situations in which we have independence at ω and a ZFC inequality at larger κ: for example, Raghavan and Shelah recently proved $\mathfrak{s}_\kappa \leq \mathfrak{b}_\kappa$ while the consistency of $\mathfrak{b} < \mathfrak{s}$ is a classical result where $\mathfrak{b}$ and $\mathfrak{s}$ are the *bounding* and *splitting numbers*, and $\mathfrak{b}_\kappa$ and $\mathfrak{s}_\kappa$ their generalized counterparts. Furthermore, it is known that many forcing constructions for ω do not generalize to $\kappa > \omega$ and there is an array of open problems, though the work of Raghavan and Shelah on Boolean ultrapowers mentioned above has made an important breakthrough on some of them.

On the other hand, while the investigation of cardinal invariants in set theory and of highness properties of Turing oracles in computability theory have proceeded independently for several decades, it has become clear recently that there are strong analogies between the concepts and the methods of proofs in these two fields: ZFC inequalities between cardinal invariants correspond to implications between highness properties, while a consistency result corresponds to exhibiting a Turing degree having one highness property and failing another. There is now a fruitful interaction between the fields: the concepts of *evasion and prediction*, originally introduced by Blass in the cardinal invariant context for investigating homomorphisms from subgroups of the Baer-Specker group $\mathbb{Z}^\omega$ to $\mathbb{Z}$ have been investigated from the point of view of Turing degrees, while the Gamma question from computability theory has led to new cardinal invariants.

Finally, a large part of the workshop will be dedicated to strengthening the link between set theory of the reals and other areas of mathematics such as topology, analysis and algebra. Set theoretic phenomena are well known to have a significant impact on dual spaces in analysis and algebra. In algebra the set theoretic influence is observed in the study of the Baer-Specker group and the homomorphisms from its subgroups to $\mathbb{Z}$ (see above). In analysis, the resolution of the Kaplansky conjecture provided a seminal example of the use of set theory in the theory of Banach algebras (see introduction).

In both of these classical examples the theory of ultrafilters plays a key role by providing homomorphisms that cannot be constructed without the Axiom of Choice. On the other hand, from the purely set theoretic point of view ultrafilters have provided the impetus for many of the questions, as well as solutions, related to combinatorial cardinal invariants. The space of all ultrafilters provides an important example of a compact space for topologists and an important subspace of the second dual of ℓ_1 with the weak* topology. But while ultrafilters are conceptually attractive and decades of study have revealed several of their secrets, we should not lose sight of the fact that the unit ball of the second dual of ℓ_1 consists of more than just ultrafilters and the study of this more complicated structure, the space of finitely additive measures on $\mathbb{N}$, is likely to provide some surprises. One deep difference has long been known, atomless, finitely additive measures can exist even in the absence of non-principal ultrafilters.

The question of the continuity of multiplication in the double dual provides an other instance where the double dual is more complicated than the case for the space of ultrafilters. It was shown by Civin and Yood that for abelian groups G the only elements x of the double dual of $L_1(G)$ for which the mapping $y \mapsto xy$ is weak* continuous are those from $L_1(G)$ itself. A string of results by various researchers have since refined this result. In particular, the case restriction to $\beta\mathbb{N} \subseteq \ell^{**}$ is not quite well understood. The connection to set theory was discovered by van Douwen who showed that if G is a countably infinite, discrete group and q is a *P-point* then the mapping $(x, y) \mapsto xy$ from $\beta G \setminus G \times \beta G \setminus G$ to $\beta G \setminus G$ is continuous at (p, q). Work of Protasov and Zelenyuk now provides us with quite a complete picture for the continuity on βG. Continuity on dual spaces in general is less well understood. For example, a collection of problems in this area in which many open questions remain has to do with finding small (finite) test sets for continuity.

Work of Rosenblatt, Talagrand, Yang and Foreman on amenable group actions with unique invariant means also deserves further attention by set theorists. The only unique invariant means known without assuming the Continuum Hypothesis are ultrafilters. It is an unmet challenge to set theorists to determine why this is so. Indeed, we do not even know whether the existence of an amenable group with an action on $\mathbb{N}$ with a unique invariant mean requires any extra set theoretic hypotheses. Once again, the set theoretic focus on ultrafilters at the expense of finitely additive measures may be at fault. In this same vein, do we understand Talagrand's solution of Maharam's problem as well as we should?

2 Activities and progress made during the workshop

2.1 General structure

The workshop was organized with the intent of providing participants with up to five thematically related, 50 minute talks each day. Included in these talks were pairs of survey talks devoted to three of the main themes of the conference. (It was decided during the organization phase to leave out the forth theme, on dual spaces, because many of the key participants were not able to attend.) The first of these survey pairs, on recent developments in the iterated forcing theory and their application to the analysis of the Cichoń diagram, was presented by Diego Mejía and Martin Goldstern. The second of these, on higher cardinal invariants, was given by Dilip Raghavan, and the third, on the connections between Turing degrees and cardinal invariants, by Noam Greenberg. The other talks were research talks about recent developments in which the speakers had played a major role.

An hour before dinner on the Monday, Tuesday and Thursday evenings was devoted to problem sessions. Participants volunteered to present problems they found interesting at the blackboard and say a few words of background. In many cases audience members were able to provide further enlightening comments. Even though most of the problems presented were familiar to the more senior participants, this was an excellent opportunity for students and younger researchers to incorporate new ideas into their research programs.

2.2 Presentation highlights

2.2.1 Cichoń's diagram and new forcing techniques

Cardinal invariants appear throughout set theory and any attempt at cataloguing them all would be a major undertaking. However, there is a small family of invariants that warrants special attention because of its close connections with real analysis going back to the 19$^{\text{th}}$ century and, in particular, the discovery of the Baire Category Theorem. As it is usually stated, this theorem says that the union of countably many meagre sets does not cover the real line, but an alternative formulation is enlightening and reveals a number of implicit questions: The least number of meagre sets whose union is $\mathbb{R}$ is at least $\aleph_1$. So what is the least cardinal of a family of meagre sets whose union covers the reals? This cardinal is known a $\operatorname{cov}(\mathcal{M})$ and is one of the components of the Cichoń diagram.

Other components of the diagram include $\operatorname{non}(\mathcal{M})$, the least cardinality of a non-meagre set and $\operatorname{add}(\mathcal{M})$, the least cardinal of a family of meagre sets whose union is not meagre. Analogous cardinal invariants can obtained by replacing the ideal of meagre sets $\mathcal{M}$ with the ideal of Lebesgue null sets $\mathcal{N}$. Some inequalities between these invariants are immediate; for example, $\operatorname{cov}(\mathcal{N}) \leq \operatorname{non}(\mathcal{M})$. Others, though, are deep theorems. Once all the known information about these invariants has been collected, it can be summarized in a diagram of the following form:

$$
\begin{tikzcd}
& \operatorname{cov}(\mathcal{N}) \arrow[r] & \operatorname{non}(\mathcal{M}) \arrow[r] & \operatorname{cof}(\mathcal{M}) \arrow[r] & \operatorname{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
& & \mathfrak{b} \arrow[r] \arrow[u] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \operatorname{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \operatorname{add}(\mathcal{M}) \arrow[r] \arrow[u] & \operatorname{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \operatorname{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}
$$

Here arrows indicate a provable inequality between the corresponding cardinal invariants.

An intense period of focussed study by various researchers throughout the 70's and 80's showed that none of the arrows in this diagram can be collapsed. In other words, for each arrow, there is a model of set theory where the corresponding inequality is strict. The natural next question then concerns the global structure of the diagram. Are there hidden inequalities among triples of invariants or, perhaps, among large groups? This is not an idle question since results of this type are known for other combinations of invariants.

The focus of the lecture series by Mejía and Goldstern was on describing recent results that provide answers to these questions. Mejía explained how many different values can be obtained simultaneously on the left side of the diagram by a finite support iteration, the main issue being the combinatorial ideas necessary for forcing $\mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathfrak{c}$ [GMS]. He also provided an introduction to the related technique of matrix iteration. A key idea described by Goldstern in his work with Jakob Kellner and Saharon Shelah [GKS] was the notion of a Boolean ultrapower of complete Boolean algebras that are used in forcing constructions. By modifying the notion of an ultrapower embedding to their context, they were able to also make the cardinal invariants on the right side in Cichoń's diagram take on a broad spectrum of values simultaneously. He also explained a very recent technique (developed in the last months only) which uses restrictions of partial orders to appropriately chosen elementary submodels to achieve the same results [GKMS]. The advantage of this new approach is that it does not need large cardinals and is on the basis of the standard axiom system set theory. However, the delineating line between the upper and lower half of the diagram cannot be arbitrary and there is still much work to be done on discovering what sorts of divisions are possible.

In a related talk, Guzmán presented a breakthrough result with Kalajdzievski, saying that consistently (on the basis of ZFC) $\mathfrak{u} = \aleph_1 < \mathfrak{a} = \aleph_2$ [GK], where the ultrafilter number $\mathfrak{u}$ is the least size of a base of a nonprincipal ultrafilter on ω and the almost disjointness number $\mathfrak{a}$ is the least size of a maximal almost disjoint family of subsets of ω. This solved an old question of Shelah.

Asperó and Yorioka contributed talks on forcing axioms, Asperó's presenting his joint work with Schindler of another breakthrough result linking forcing axioms to the $\mathbb{P}_{max}$ theory of Woodin.

2.2.2 Higher cardinal invariants

While Dilip Raghavan has been involved in the Boolean ultrapower constructions described by Goldstern, he chose instead to talk about new work with Shelah on cardinal invariants beyond the continuum. Recall that the cardinal invariants $\mathfrak{b}$ and $\mathfrak{d}$ that appear in the Cichoń diagram are defined by growth rates of functions from $\mathbb{N}$ to $\mathbb{N}$. One can, however, define similar invariants for larger cardinals and, indeed, many of the combinatorial invariants have generalizations to this setting in larger cardinals. Raghavan's talk described new results in this area, focussing on phenomena that do not appear in the countable case. For example, while $\mathfrak{b} < \mathfrak{s}$ is consistent, for regular uncountable κ, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ holds in ZFC [RS1] (here $\mathfrak{s}$ is the splitting number). Similarly, $\mathfrak{b} = \aleph_1 < \mathfrak{a} = \aleph_2$ is consistent while $\mathfrak{b}(\kappa) = \kappa^+$ implies $\mathfrak{a}(\kappa) = \kappa^+$ [RS2]. He also sketched the construction (using large cardinals) of a model in which the ultrafilter number on $\aleph_{\omega+1}$ is strictly less than $2^{\aleph_{\omega+1}}$ [RS3]. Closely related to this is a long standing open question of Kunen on whether ultrafilters on ω_1 can have generating sets of cardinality less than the trivial upper bound $2^{\aleph_1}$.

There were further talks on iterated forcing theory and (higher) cardinal invariants on the same day, e.g. Schlicht's talk on ideal topologies on the higher Cantor space 2^κ.

2.2.3 Turing degrees and cardinal invariants

Recently, a close similarity between proofs about highness properties of oracles in computability theory, i.e. properties saying that an oracle is far from being computable, and proofs about cardinal invariants has been noticed by several researchers, though the areas have developed independently for many years. An oracle is *high* if it computes a function eventually dominating all computable functions. It is *bi-immune* if it computes a set a such that neither a nor its complement have an infinite r.e. subset. If a is high then it is bi-immune, and the proof is the same as the proof of the cardinal invariant inequality $\mathfrak{b} \leq \mathfrak{r}$. On the other hand, a sufficiently Cohen real is bi-immune but not high while the Cohen model witnesses the consistency of $\mathfrak{b} < \mathfrak{r}$. Thus inequalities between cardinal invariants translate to implications between highness properties, and independence proofs, to witnesses if non-implication.

The focus of Greenberg's lecture series (based on his recent joint work with Rutger Kuyper and Dan Turetsky [GKT]), apart from having been an introduction to certain aspects of computability theory for set theorists, was on presenting a general framework, using Weihrauch reducibility from computability theory, for proving results strong enough to entail such classical results from both set theory and computability theory. One of the main novel aspects of this approach is the constructive treatment of the sequential composition of relations from set theory.

In a related talk on Muchnik degrees and cardinal invariants, Nies explained recent work in computability theory which, when translated to the set-theoretic context, gives rise to new cardinals between cov$(\mathcal{M})$ and non$(\mathcal{N})$.

2.2.4 Applications to topology and analysis

The talks by Bergfalk and Lambie-Hanson dealt with a new and exciting line of research linking set theory to algebraic topology and homological algebra. Lambie-Henson outlined the proof of their recent result that consistently, assuming the consistency of a measurable cardinal $\lim \mathbb{A}^n = 0$ for all $n \in \mathbb{N}$. This answers a long-standing open problem and clears the only known obstacle to a possible consistency proof of additivity of strong homology in the class of separable metric spaces. Bergfalk then presented a new descriptive set theoretic approach to (co)-homology with very promising features. The talks by Plebanek, Sobota and Dow exhibit the breadth of applications of combinatorial set theory of the reals, dealing with useful applications of the subject to the theory of Boolean algebras (Sobota), functional analysis (Plebanek) and modal logic (Dow). Foreman and Solecki presented deep results connecting set theory to ergodic theory and dynamics: Foreman presented an ingenious coding technique which exhibits non-classification of measure preserving diffeomorphisms on the 2-torus, and Solecki introduced a new concentration of measure with looming applications to the structural theory of both Polish groups and analytic P-ideals.

3 Diversity and impact on the Mexican set theory community

Historically set theory has seen very few female researchers working in the subject. Similarly, while Mexicans have long been well represented in the set-theoretic topology community, the same can not be said about the set theory community. In both cases, this has changed substantially in the last 10-15 years: set theory has seen many strong female researchers start working in the subject, and the growing community of young Mexican set theorists played an integral role in the workshop. The workshop served to expose the students and young researchers in Mexico's fledgling set theory group to a broad cross section of established experts.

4 Schedule of the Workshop: Set theory of the Reals

Monday, August 5

09:00 - 09:50 Osvaldo Guzmán: The ultrafilter and almost disjointness numbers

10:00 - 10:50 Diego Alejandro Mejía: Preservation theorems for finite support iterations I.

11:30 - 12:20 Diego Alejandro Mejía: Preservation theorems for finite support iterations II.

15:00 - 15:50 Martin Goldstern: Two proofs of Cichoń's maximum I.

16:30 - 17:20 Martin Goldstern: Two proofs of Cichoń's maximum II.

17:30 - 18:00 Problem session

Tuesday, August 6

09:00 - 09:50 David Aspero: Forcing axioms vs $(*)$

10:00 - 10:50 Dilip Raghavan: Higher cardinal invariants I.

11:30 - 12:20 Dilip Raghavan: Higher cardinal invariants II.

15:00 - 15:50 Teruyuki Yorioka: YPFA implies MRP

16:30 - 17:20 Philipp Schlicht: Ideal topologies on 2^κ

17:30 - 18:00 Problem session

Wednesday, August 7

09:00 - 09:50 Matthew Foreman: Independence results for diffeomorphisms of the 2-torus

10:00 - 10:50 Slawomir Solecki: Dynamics of Polish groups, submeasures, and a new concentration of measure

11:30 - 12:20 Alan Dow: Mad families and the modal logic of $\mathbb{N}^*$

Thursday, August 8

09:00 - 09:15 Andre Nies: Muchnik degrees and cardinal characteristics (joint with Monin and Miller)

09:20 - 09:50 Andre Nies: The complexity of the isomorphism relation between oligomorphic groups (joint with Schlicht and Tent)

10:00 - 10:50 Noam Greenberg: Weihrauch reducibility, highness classes, cardinal characteristics, and forcing I.

11:30 - 12:20 Noam Greenberg: Weihrauch reducibility, highness classes, cardinal characteristics, and forcing II.

15:00 - 15:50 Chris Lambie-Hanson: Simultaneously vanishing higher derived limits

16:30 - 17:20 Jeffrey Bergfalk: Definable (co)homology, the rigidity of solenoids, and classification by (co)cycles

17:30 - 18:00 Problem session

Friday, August 9

08:30 - 09:20 Grzegorz Plebanek: Small almost disjoint families with applications to Banach spaces

09:30 - 10:20 Damian Sobota: Convergence of measures on minimally generated boolean algebras

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[RS3] Dilip Raghavan, Saharon Shelah. *A small ultrafilter number at smaller cardinals.* Arch. Math. Logic (to appear).