INDEPENDENCE RESULTS FOR THE 2-TORUS

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A CLASSICAL EARLY 20TH CENTURY QUESTION

Can you tell the difference between

“time running forwards”

and

“time running backwards”?

MATHEMATICALLY

Let M be a compact smooth manifold and let

$$\phi : \mathbb{R} \times M \to M$$

be a dynamical system (say solving some ODE).

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Since $\mathbb{R}$ is commutative we can define $\psi : \mathbb{R} \times M \to M$ by setting

$$\psi(t, \vec{x}) = \phi(-t, \vec{x})$$

and get another dynamical system with "Time running backwards."

Is $\phi \cong \psi$?

DOES YOUR BEST PHYSICAL THEORY PROVE THAT TIME RUNS FORWARDS?

HOW MANY BAD SCIENCE FICTION BOOKS ABOUT TIME TRAVEL ARE THERE?

WHAT DOES ISOMORPHISM MEAN?

Since M is a compact manifold it carries a smooth volume form λ that is absolutely continuous with respect to Lebesgue measure.

Is there an invertible measure preserving transformation θ that conjugates ϕ to ψ:

$$\theta^{-1}\phi\theta = \psi?$$

BY THE ERGODIC THEOREM, MEASURE PRESERVING TRANSFORMATIONS PRESERVE STATISTICAL MEASUREMENTS.

Z VS. R

If we let $T : M \to M$ be defined by $T = \phi(1)$, then we get a $\mathbb{Z}$-action where the forward vs. backwards question is whether

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We can go back to an $\mathbb{R}$ action from a $\mathbb{Z}$-action by interpolating using the method of *suspensions*. So everything I say applies to $\mathbb{R}$-actions.

A QUESTION OF VON NEUMANN

Let $(X, \mathcal{B}, \mu)$ be a standard measure space. Is there *any* invertible measure preserving transformation where

$$T \not\cong T^{-1}?$$

FIRST EXAMPLE

It was not until 1951 that Anzai gave an example of a $T \not\cong T^{-1}$ by inventing the method of *skew-product.*

THIS TALK IS GOING TO EXPLAIN WHY THIS IS A HARD PROBLEM

Theorem 1 (Main Theorem) *There is a computable function*

$$F : \{\textit{Codes for } \Pi^0_1\textit{-sentences}\} \to \{\textit{Codes for computable diffeomorphisms of } \mathbb{T}^2\}$$

such that:

1. *m is the code for a true statement if and only if $F(m)$ is the code for a computable T, where T is measure theoretically isomorphic to T^{-1};*

   and

2. *For $m \neq n$, $F(m)$ is not isomorphic to $F(n)$.*

The diffeomorphisms in the range of F are ergodic.

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To appear in a joint paper with J. Gaebler

FOR THOSE OF YOU WHO FORGOT YOUR FIRST YEAR LOGIC COURSE

A sentence ϕ in the language $\mathcal{L}_{PA} = \{+, *, 0, 1, <\}$ is Π^0_1 if it can be written in the form $(\forall x_0)(\forall x_1)\ldots(\forall x_n)\psi$, where ψ is a Boolean combination of equalities and inequalities of polynomials in the variables $x_0, \ldots x_n$ and the constants $0, 1$.

These sentences have Goedel numbers: "codes"

WHAT IS AN (EFFECTIVELY) COMPUTABLE DIFFEOMORPHISM?

We can code a modulus of continuity for a uniformly continuous function $f : \mathbb{T}^2 \to \mathbb{T}^2$ by a $g : \mathbb{N} \to \mathbb{N}$ such that:

To know $f(x, y)$ up to n-digits it suffices to supply me with the first $g(n)$ digits of (x, y). Moreover the computation of the digits of $f(x, y)$ is recursive.

A diffeomorphism is computably C^∞ if all of its differentials are computably continuous.

WHAT IS AN (EFFECTIVELY) COMPUTABLE DIFFEOMORPHISM?

A function $T : \mathbb{T}^2 \to \mathbb{T}^2$ is said to be a *computable diffeomorphism* if there exist computable functions $d : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $f : \mathbb{N} \times (\{0,1\} \times \{0,1\})^{<\mathbb{N}} \to \mathbb{N}$ such that $d(k,-)$ and $f(k,-)$ are the modulus of continuity and approximation of the k-th differential of T, respectively.

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Let's try to see what the theorem is saying?

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Goldstern: "ZFC + $\omega_1 < \mathfrak{b} < \mathfrak{s} < \mathfrak{d} < 2^\omega$" is consistent.

Hrusak: "ZFC + no p-points + no q-points" is consistent.

Other audiences:

1. Riemann Hypothesis
2. Goldbach's Conjecture

WHAT THE THEOREM SAYS:

There are diffeomorphisms of the torus

$$T_{Dow},\ T_{Goldstern},\ T_{Hrusak},\ T_{Riemann},\ T_{Goldbach}$$

such that the question of being isomorphic to their inverses is equivalent to the corresponding statement.

INDEPENDENCE RESULTS

1. "ZFC is consistent"
2. "ZFC + there is a supercompact cardinal" is consistent

The question of whether $T_\phi \cong T_\phi^{-1}$ is (presumably) independent of ZFC.

HOW TO CHEAT:

Take two diffeomorphisms of the torus, S_0 and S_1 with $S_0 \cong S_0^{-1}$ and $S_1 \not\cong S_1^{-1}$.

The *choosing the right* i, $T = S_i$ works for the Riemann Hypothesis.

BUT WE DIDN'T CHEAT.

Take two diffeomorphisms of the torus, S_0 and S_1
with $S \cong S^{-1}$ and $T \cong T^{-1}$.

The *choosing the right* i, $T = S_i$
works for the Riemann Hypothesis.

The same i doesn't work for all examples!
Let's look at the statement of the theorem again.

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Primitive Recursive

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Time forwards and backwards

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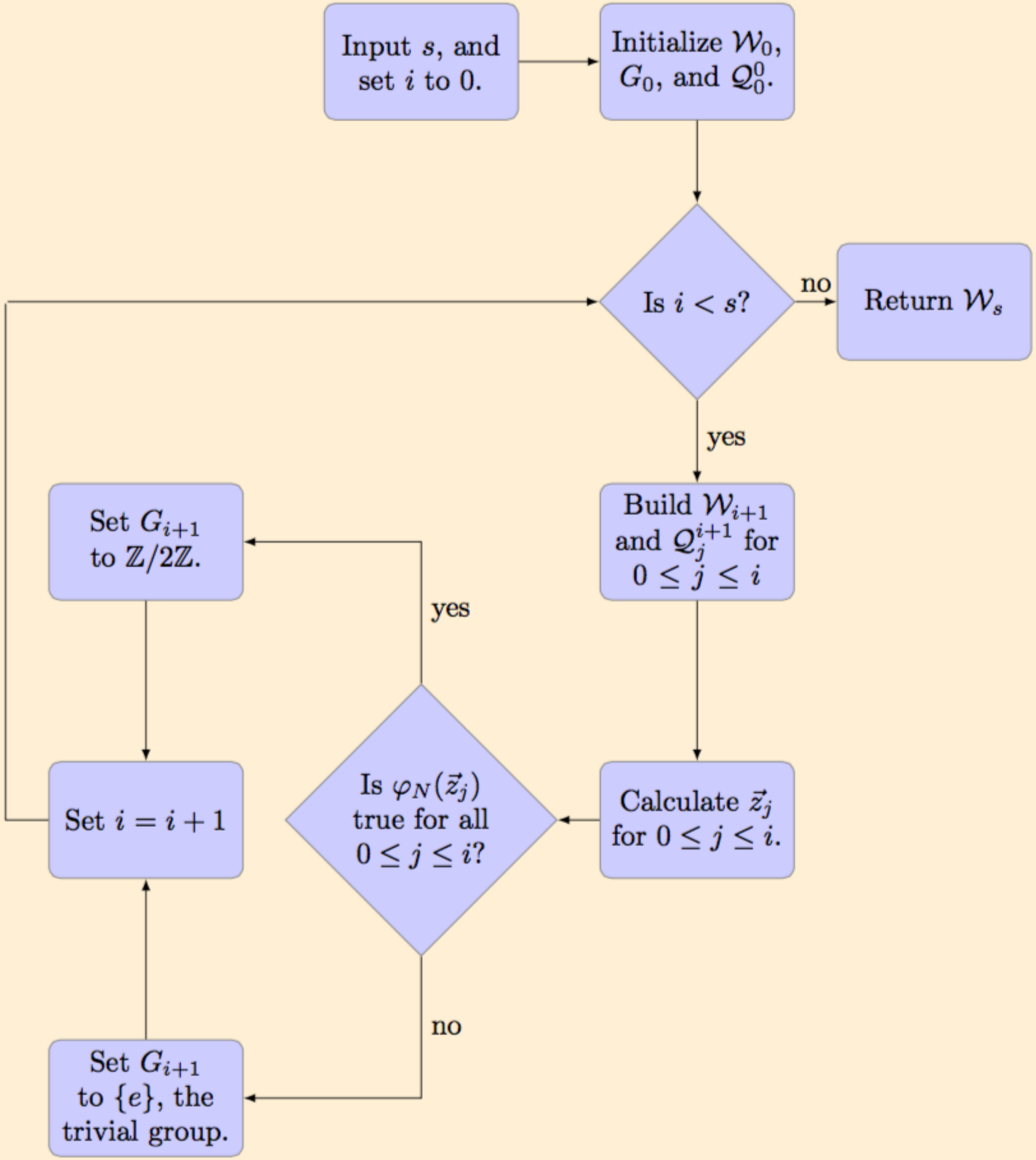
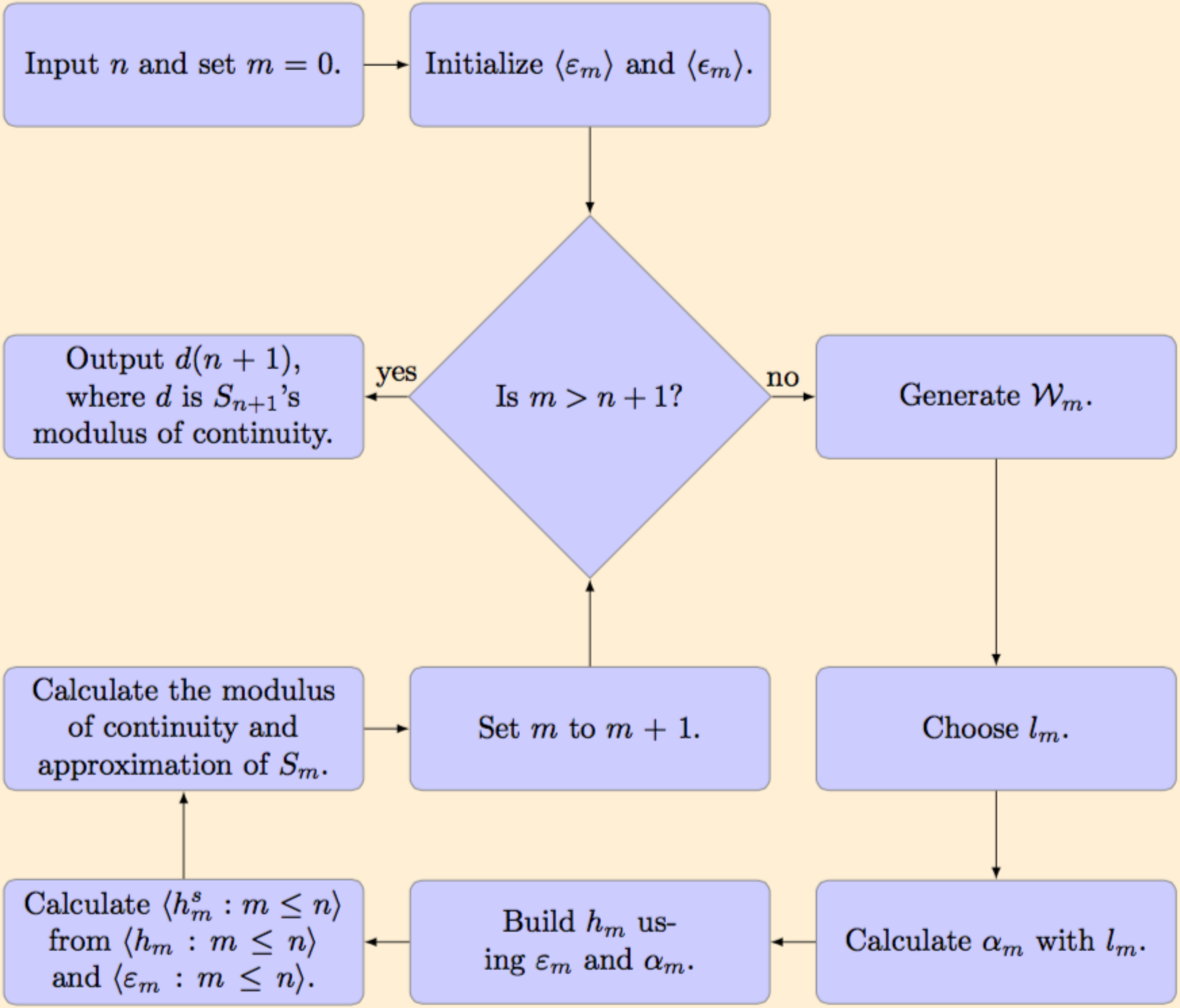
REVERSE MATH

<= ACA_0

WHAT ABOUT THE PROOF?


Input s, and set i to 0.
Initialize $\mathcal{W}_0$, G_0, and $\mathcal{Q}_0^0$.
Is $i < s$?
no
Return $\mathcal{W}_s$
yes
Build $\mathcal{W}_{i+1}$ and $\mathcal{Q}_j^{i+1}$ for $0 \leq j \leq i$
Calculate $\vec{z}_j$ for $0 \leq j \leq i$.
Is $\varphi_N(\vec{z}_j)$ true for all $0 \leq j \leq i$?
yes
Set G_{i+1} to $\mathbb{Z}/2\mathbb{Z}$.
no
Set G_{i+1} to $\{e\}$, the trivial group.
Set $i = i + 1$



Input n and set $m = 0$.
Initialize $\langle \varepsilon_m \rangle$ and $\langle \epsilon_m \rangle$.
Is $m > n + 1$?
yes
Output $d(n + 1)$, where d is S_{n+1}'s modulus of continuity.
no
Generate $\mathcal{W}_m$.
Choose l_m.
Calculate α_m with l_m.
Build h_m using ε_m and α_m.
Calculate $\langle h^s_m : m \leq n \rangle$ from $\langle h_m : m \leq n \rangle$ and $\langle \varepsilon_m : m \leq n \rangle$.
Calculate the modulus of continuity and approximation of S_m.
Set m to $m + 1$.

IN ENGLISH (SORT OF)

The proof is an adaptation of a previous result of Benjy Weiss and I:

Theorem 1 *In the space of C^∞ measure preserving diffeomorphisms:*

$$\{T : T \cong T^{-1}\}$$

is complete Σ^1_1.

Corollary 2

$$\{(S,T) : S,T \text{ ergodic MP diffeos and } S \cong T\}$$

is not Borel.

This impossibility result answered another question asked by von Neumann in a 1931 paper. He proposed classifying the "statistical behavior" of smooth systems. Our result shows that this is not possible.

At least with countable resources

IN PROVING THAT THEOREM

We built a continuous reduction F from the space $TREES$ to $\text{Diff}^\infty(\mathbb{T}^2, \lambda)$ such that

- $\mathcal{T}$ is ill-founded

  iff

- $F(\mathcal{T}) \cong F(\mathcal{T})^{-1}$

How do you adapt this to Π^0_1?

Given a Π^0_1 statement $\forall n \psi$ you check:

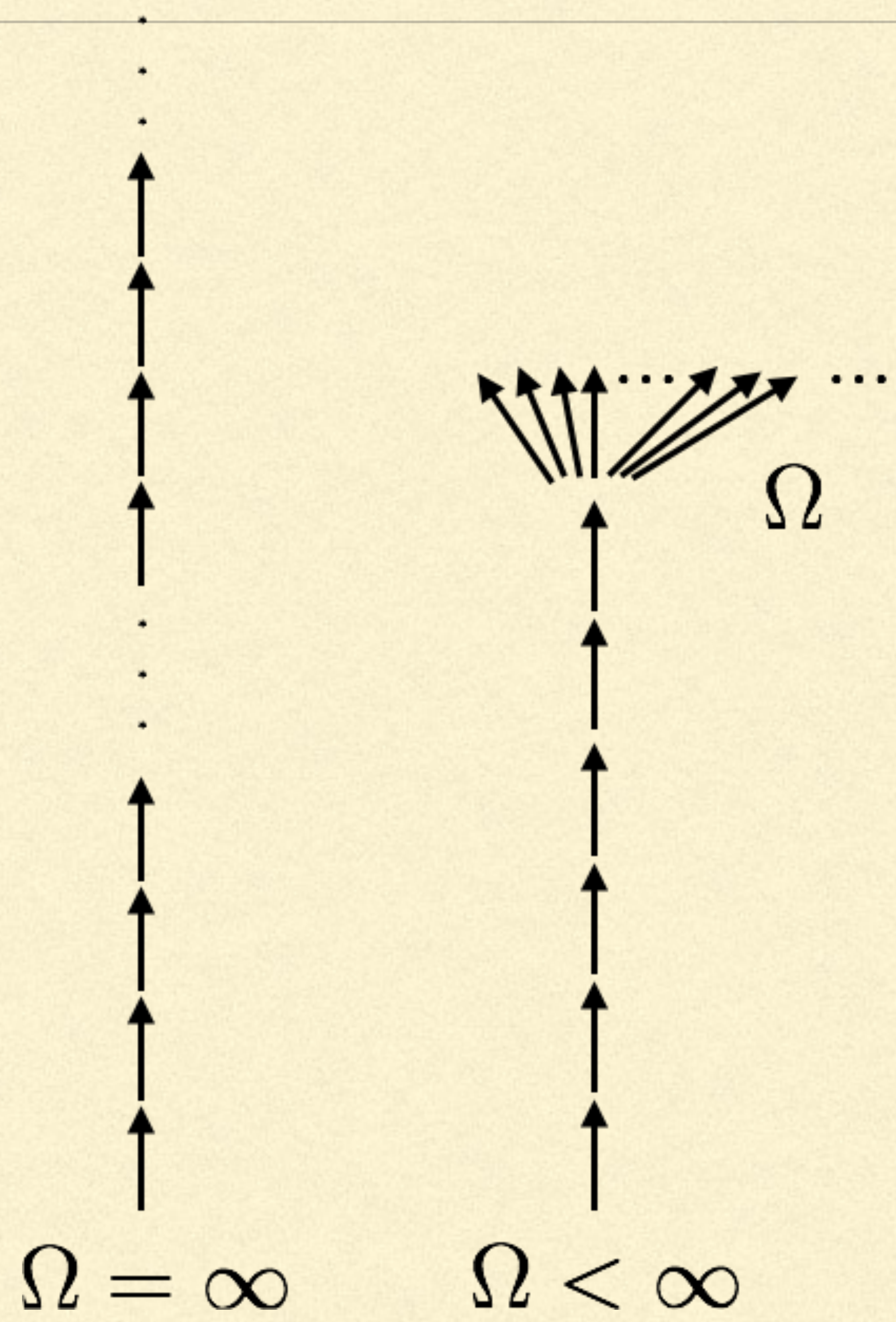
$$\psi(0), \psi(1), \psi(2) \ldots \psi(n) \ldots$$

You either hit a counterexample Ω
or you don't.

1. As long you don't hit a counterexample you keep trying to make $T \cong T^{-1}$

2. If you do hit a counterexample you start taking countermeasures.

THE RELEVANT TREES LOOK LIKE THIS:

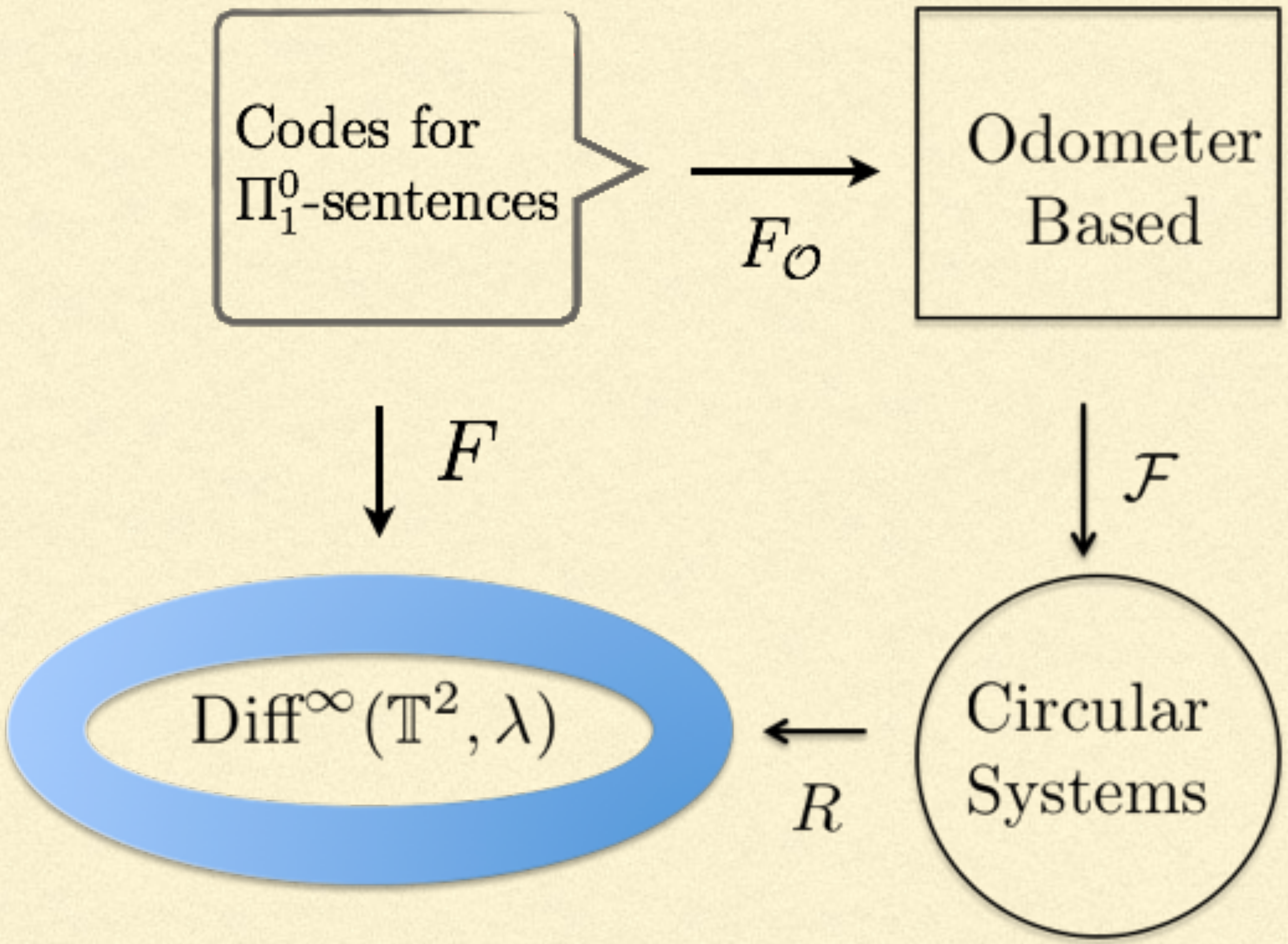


THE ACTUAL PROOF

LOTS OF TECHNICALITIES (350 PAGES)

1. Symbolic Shifts
2. Canonical way of building symbolic shifts: Construction sequences
3. 3 classes
   (a) Odometer based systems: easiest to understand
   (b) Circular Systems: Symbolic transformations of odometer based systems
   (c) Diffeomorphisms: realizable from Circular Systems

BIG PICTURE



Codes for Π^0_1-sentences
$F_{\mathcal{O}}$
Odometer Based
F
$\mathcal{F}$
$\text{Diff}^\infty(\mathbb{T}^2, \lambda)$
R
Circular Systems

GLOBAL STRUCTURE THEOREM

The classes of *odometer based systems* and *circular systems* naturally form categories. The morphisms are (roughly) factor maps and isomorphisms.

1. The odometer based systems are ubiquitous: they form a cone in the space of finite entropy MPTS

2. The circular systems are a tiny slice of MPT but they are all realizable as diffeomorphisms.

GLOBAL STRUCTURE THEOREM

Theorem 1 *(Modulo Details) The category of odometer based systems is isomorphic (as a category) to the category of circular systems.*

Corollary 2 *(modulo vagueness)* **ALL** *ergodic behavior of finite entropy systems is realized on the diffeomorphisms of the torus. For example*

1. *Distal Height*
2. *Simplices of invariant measures*
3. *etc.*

MAKING T ISOMORPHIC TO ITS INVERSE

This is not so bad:

1. if you make a word w occur frequently in a typical element of the symbolic system then a good approximation of the reverse word should appear frequently.

2. it *can't* be too good an approximation or you can't make $T \ncong T^{-1}$

COUNTERMEASURES

1. To prevent a symbolic shift from being isomorphic to its inverse you need to make the words that appear frequently statistically independent from their reverses.

2. Building large collections of such words directly is difficult (if known)

3. Instead use a finite version of the *Law of Large Numbers* to build the words probabilistically: for long words *most* of the collections of words will have the relevant property. (Even if you can't build such a collection explicitly.)

Tim Carlson suggested generalizing this for *lightface* Σ^1_1 subsets of $\mathbb{N}$. This works.

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THE METHOD OF OBTAINING INDEPENDENCE IS GENERAL (SORT OF)

Any sufficiently concrete continuous reduction between concrete Polish spaces is can be adapted to prove such a result.

THANK YOU!