# The complexity of the isomorphism relation between oligomorphic groups

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# Oligomorphic groups

- $S_\infty$ denotes the group of permutations of $\mathbb{N}$.
- $S_\infty$ is a topological group: the subgroups $U_n$ of permutations fixing $0, \ldots, n-1$ form a basis of neighbourhoods of $1$.
- For any model $M$ with domain $\mathbb{N}$, the group $\mathrm{Aut}(M)$ is a closed subgroup of $S_\infty$.
- A closed subgroup $G$ of $S_\infty$ is called oligomorphic (Cameron, 1980s) if for each $k$, the action of $G$ on $\mathbb{N}^k$ has only finitely many orbits.
- whether $G$ is oligomorphic depends on the way $G$ is embedded into $S_\infty$.
- An oligomorphic group cannot be locally compact, let alone countable.

# Oligomorphic groups as automorphism groups

**Fact**

A closed subgroup $G$ of $S_\infty$ is oligomorphic $\iff$
$G$ is the automorphism group of an $\omega$-categorical structure $A$ (with domain $\mathbb{N}$).

**Proof.**

$\Leftarrow$: this follows from the Ryll-Nardzewski Theorem that for each $k$, the structure $A$ has only finitely many $k$-types.

$\Rightarrow$:

- Given a subgroup $U \leq S_\infty$ let $A_U$ be the structure with a $k$-ary relation symbol for each orbit of $U$ on $\mathbb{N}^k$.
- Then $\overline{U} = \mathrm{Aut}(A_U)$. So for closed $G$ we have $G = \mathrm{Aut}(A_G)$.
- If $G$ is oligomorphic then $A_G$ is $\omega$-categorical.

# Oligomorphic groups as automorphism groups

**Fact (Recall)**

A closed subgroup $G$ of $S_\infty$ is oligomorphic $\iff$
$G$ is the automorphism group of an $\omega$-categorical structure $A$.

For instance, the following automorphism groups are oligomorphic:

- $S_\infty$
- Aut(random graph)
- $\mathrm{Aut}(\mathbb{F}_p^{(\omega)})$ where $\mathbb{F}_p^{(\omega)})$ is vector space of dimension $\omega$ over the field with $p$ elements
- $\mathrm{Aut}(\mathbb{Q}, <)$, the group of order-preserving permutations of the rationals.

# Properties of the oligomorphic group $\mathrm{Aut}(\mathbb{Q},<)$

Let $G = \mathrm{Aut}(\mathbb{Q},<)$.

- $G$ is highly homogeneous: for each $k$, its action on $\mathbb{N}^{[k]}$ is transitive
- $G$ has three nontrivial normal subgroups
- $G$ has the small index property: any subgroup of G of index $< 2^{\aleph_0}$ is open
- $G$ has a dense subgroup isomorphic to the free group $F_2$ (Glass and McCleary 1990). In particular, $G$ is topologically finitely generated
- $G$ is extremely amenable: each continuous action on a compact space has a fixed point (Pestov).

# Borel reducibility $\leq_B$

A standard Borel space is a space of the form $(Z, \mathcal{B})$, where $\mathcal{B}$ is the $\sigma$-algebra generated by the open sets of a Polish topology on $Z$. Sets in $\mathcal{B}$ are called Borel sets of $Z$.

- Let $X, Y$ be standard Borel spaces. A function $g\colon X \to Y$ is Borel if the preimage of each Borel set in $Y$ is Borel in $X$.
- Let $E, F$ be equivalence relations on $X, Y$ respectively. We write $E \leq_B F$, and say that $E$ is Borel below $F$, if there is a Borel function $g\colon X \to Y$ such that

  $$uEv \leftrightarrow g(u)Fg(v)$$

  for each $u, v \in X$.

An equivalence relation is called smooth if it is Borel-below $\mathtt{id}_Y$, the identity relation on some uncountable Polish space $Y$ (say, $\mathbb{R}$).

# The Borel space of closed subgroups of $S_\infty$

For a 1-1 map $\sigma\colon \{0,\dots,n-1\} \to \mathbb{N}$ let

$$N_\sigma = \{\alpha \in S_\infty \colon \forall i < n\,[\sigma(i) = \alpha(i)]\}$$

The closed subgroups of $S_\infty$ can be seen as points in a standard Borel space. To define the Borel sets, we start with sets of the form

$$\{G \leq_c S_\infty \colon G \cap N_\sigma \neq \emptyset\},$$

where $G \leq_c S_\infty$ means that $G$ is a closed subgroup of $S_\infty$.

The Borel sets are generated from these basic sets by complementation and countable union.

Example: for every $\alpha \in S_\infty$, the set $\bigcap_k \{H \colon H \cap N_{\alpha\restriction k} \neq \emptyset\}$ is Borel. Its elements are the closed subgroup of $S_\infty$ that contain $\alpha$.

# Via Borel transformations, oligomorphic groups can be seen as countable structures

Easy fact: the oligomorphic groups form a Borel set in the space of closed subgroups of $S_\infty$.

**Theorem (N., Schlicht, Tent)**

Isomorphism of oligomorphic groups is Borel bi-reducible with the isomorphism relation $\cong_{\mathcal{B}}$ on an invariant Borel set $\mathcal{B}$ of structures with domain $\mathbb{N}$ for the language with one ternary relation symbol.

# Some basics towards proving the theorem

- Recall that $S_\infty$ is a topological group: the subgroups
$$U_n = \{g \in S_\infty \colon \forall i \le n\, g(i) = i\}$$
form a basis of neighbourhoods of $1$.
($S_\infty$ is totally disconnected.)
- For each closed subgroup $G$ of $S_\infty$ the open subgroups (of the form $G \cap U_n$ if you like) form a nbhd basis of $1$.
- So the open cosets form a basis of the topology.
- Note that each open left coset is also an open right coset, because $aU = (aUa^{-1})a$.
- in $S_\infty$ an open left coset is essentially the same as a nbhd $N_\sigma = \{\alpha \in S_\infty \colon \forall i < n\, [\sigma(i) = \alpha(i)]\}$.

A closed subgroup $G$ of $S_\infty$ is called Roelcke precompact (R.p.) if (∗) for each open subgroup $U$ of $G$

there is a finite set $F \subseteq G$ such that $UFU = G$.

**Fact**

(1) The class of R.p. closed subgroups of $S_\infty$ is Borel.

(2) From such a group $G$ we can in a Borel way determine a listing $A_0, A_1, \dots$ without repetition of all the open cosets.

Proof of (1). It suffices to check the condition (∗) for the basic open subgroups $G_n = G \cap U_n$, where $U_n$ the group of permutations of $\mathbb{N}$ fixing $0, \dots, n-1$. If $F$ exists for $U$, we can pick it from a countable dense set predetermined from $G$ in a Borel way.

Proof of (2). Each open subgroup is a union of finitely many double cosets $U_n a U_n$, for some $n$ depending on $U$ only.

# oligomorphic $\Rightarrow$ Roelcke precompact

Recall: A closed subgroup $G$ of $S_\infty$ is called Roelcke precompact if for each open subgroup $U$ there is finite $F \subseteq G$ such that $UFU = G$.

**Fact (Rosendal, Tsankov)**

Each oligomorphic group $G$ is Roelcke precompact, and hence has only countably many open subgroups.

Proof: It suffices to show the condition for subgroups $U = G \cap U_n$.

- Write $\overline{a} = (0, \dots, n-1)$, so $U$ is the stabilizer of $\overline{a}$.
- Let $g_1, \dots, g_k \in G$ be such that each orbit of $G$ on $G\overline{a} \times G\overline{a} \subseteq \mathbb{N}^{2n}$ contains an element of the form $(\overline{a}, g_i\overline{a})$.
- Then $G = UFU$ where $F = \{g_1, \dots, g_k\}$.

**Theorem (Kechris, N, Tent, JSL, 2018)**

Topological isomorphism of Roelcke precompact groups is Borel reducible to the isomorphism relation on the class of countable models with one ternary predicate.

Proof idea.

- For Roelcke precompact $G$, let $\mathcal{M}(G)$ be the structure with domain the open cosets. Via the listing $A_0, A_1, \ldots$ above, we can identify the domain of $\mathcal{M}(G)$ with $\omega$.
- The ternary predicate $R(A, B, C)$ holds in $\mathcal{M}(G)$ if $AB \subseteq C$.

The map $G \to \mathcal{M}(G)$ is Borel. The main work is to show that for Roelcke precompact $G, H$,

$$G \cong H \iff \mathcal{M}(G) \cong \mathcal{M}(H).$$

A similar argument works for the locally compact groups.
(Note that RP $\cap$ locally compact = compact.)

**Theorem (to prove)**

Isomorphism of oligomorphic groups is Borel bi-reducible with the isomorphism relation of on an isomorphism invariant Borel set $\mathcal{B}$ of structures with domain $\mathbb{N}$.

For Roelcke precompact $G$, we defined a structure $\mathcal{M}(G)$ with domain consisting of the cosets of open subgroups. We can in a Borel way find a bijection of these cosets with $\mathbb{N}$. Showed $G \cong H \iff \mathcal{M}(G) \cong \mathcal{M}(H)$.

We will define an "inverse" operation $\mathcal{G}$ of the operation $\mathcal{M}$ on a Borel set $\mathcal{B}$ of models. For oligomorphic $G$ and $M \in \mathcal{B}$ we will have

$$\mathcal{G}(\mathcal{M}(G)) \cong G \text{ and } \mathcal{M}(\mathcal{G}(M)) \cong M$$

This suffices because it implies the converse reduction: for $M, N \in \mathcal{B}$,

$$\mathcal{G}(M) \cong \mathcal{G}(N) \iff M \cong N.$$

# Axiomatizing the range of the map $\mathcal{M}$

- We first define the map $\mathcal{G}$ on an isomorphism invariant co-analytic set $\mathcal{B}$ of $L$-structures that contains range($\mathcal{M}$).
- Since $\mathcal{M}(\mathcal{G}(M)) \cong M$ for each $M \in \mathcal{B}$, actually $\mathcal{B}$ equals the closure of range($\mathcal{M}$) under isomorphism, so $\mathcal{B}$ is also analytic, and hence Borel.
- We will observe a number of properties, called axioms, of all the structures of the form $\mathcal{M}(G)$. They can be expressed either in $\Pi^1_1$ form or in $L_{\omega_1,\omega}$ form.

The set $\mathcal{B}$ of countable structures encoding all the oligomorphic groups will be the set of structures satisfying all the axioms.

Don't confuse the structure $\mathcal{M}(G)$ with the structure $A_G$ that has $G$ as automorphism group. These are totally different. Isomorphism of groups means bi-interpretability of those structures, not isomorphism.

# Definable relations in $\mathcal{M}(G)$

Recall that our language $L$ only has one ternary relation $R(A, B, C)$ (which is interpreted by $AB \subseteq C$ for cosets $A, B, C$).

- The property of $A$ to be a *subgroup* is definable in $\mathcal{M}(G)$ by the formula $AA \subseteq A$. That a subgroup $A$ is contained in a subgroup $B$ is definable by the formula $AB \subseteq B$.
- $A$ is a *left coset* of a subgroup $U$ if and only if $U$ is the maximum subgroup with $AU \subseteq A$; similarly for $A$ being a *right coset* of $U$.
- $A \subseteq B \iff AU \subseteq B$ in case $A$ is a left coset of $U$.

The first few axioms posit for a general $L$-structure $M$ that the formulas behave reasonably. E.g., $\subseteq$ is transitive. We use terms like "*subgroup*", "*left coset* of" to refer to elements satisfying them.

# The filter group $\mathcal{F}(M)$: domain and operations

Given a structure $M$, denote by $\mathcal{F}(M)$ the set of filters (for $\subseteq$) that contain both a left and a right coset for each subgroup.

These cosets are unique because axioms require that distinct left cosets are disjoint etc. We use letters $x, y, z$ for such filters.

**Definition (Operations on $\mathcal{F}(M)$)**

$$x \cdot y = \{C \in M \mid \exists A \in x \exists B \in y \; AB \subseteq C\}.$$

For $A$ a *right coset* of $V$ and $B$ a *left coset* of $V$, let $A^* = B$ if $AB \subseteq V$. Let $x^{-1} = \{A^* \colon A \in x\}$.

The filter of *subgroups* is in $\mathcal{F}(M)$. We view this as the identity $1$.

# The filter group $\mathcal{F}(M)$: topology, actions

We can express by $\Pi^1_1$ axioms that these operations behave as a group: the operation $\cdot$ is associative, and $\forall x\,[x \cdot x^{-1} = 1]$.

The sets $\{x \colon U \in x\}$, where $U \in M$ is a *subgroup*, are declared to be a basis of neighbourhoods for the identity. Positing the right axioms, we ensure that $\mathcal{F}(M)$ is a Polish group.

For a *subgroup* $V \in M$, $LC(V)$ denotes the set of *left cosets* of $V$.

There is an action $\mathcal{F}(M) \curvearrowright LC(V)$ given by

$$x \cdot A = B \text{ iff } \exists S \in x\,[SA \subseteq B].$$

# Faithful subgroups

- Let $G$ be a closed subgroup of $S_\infty$, and let $V \leq G$. The translation action $G \curvearrowright LC(V)$ is given by $g \cdot (aV) = (ga)V$
- Each oligomorphic $G$ has an open subgroup $V$ such that the action $G \curvearrowright LC(V)$ is faithful and oligomorphic.
- To show this, let $V$ be the pointwise stabiliser of $\{n_1, \ldots, n_k\}$, where the $n_i$ represent the $k$ many $1$-orbits.
- Call such a $V$ a faithful subgroup.
- By a further axiom for an abstract $L$-structure $M$, we require the existence of such $V$, and that the embedding of $\mathcal{F}(M)$ into $S_\infty$ given by the action $G \curvearrowright LC(V)$ is topological (these axioms are in $L_{\omega_1,\omega}$ but not first-order).
- Then $\mathcal{F}(M)$ is oligomorphic and hence Roelcke precompact.

# Showing that the coset structure of $\mathcal{F}(M)$ is isomorphic to $M$

Mainly, we have to show that each open subgroup $\mathcal{U}$ of $\mathcal{F}(M)$ has the form $\mathcal{U} = \{x \colon U \in x\}$ for some *subgroup* $U$ in $M$.

- By definition of the topology, $\mathcal{U}$ contains a basic open subgroup $\widehat{W} = \{x \colon W \in x\}$, for some *subgroup* $W \in M$.
- Since $\mathcal{F}(M)$ is Roelcke precompact, $\mathcal{U}$ is a finite union of double cosets of $\widehat{W}$.
- We require as an axiom for $M$ that each such finite union that is closed under the group operations corresponds to an actual *subgroup* in $M$.

# Turning $\mathcal{F}(M)$ into closed subgroup $\mathcal{G}(M)$ of $S_\infty$

- By $\Pi^1_1$ uniformization (Addison/Kondo), from $M \in \mathcal{B}$ we can in a Borel way determine a faithful *subgroup* $V$.
- Let $A_0, A_1, \ldots$ list $LC(V)$ in the natural order.
- Then the action $\mathcal{F}(M) \curvearrowright LC(V)$ yields a topological embedding of $\mathcal{F}(M)$ into $S_\infty$.
- Its range is the desired closed subgroup $\mathcal{G}(M)$.

By the arguments above we have $\mathcal{G}(\mathcal{M}(G)) \cong G$ for each oligomorphic $G$, and $\mathcal{M}(\mathcal{G}(M)) \cong M$ for each $M \in \mathcal{B}$.

**Theorem (Finished)**

Isomorphism of oligomorphic groups is Borel bi-reducible with the isomorphism relation on an invariant Borel set $\mathcal{B}$ of structures with domain $\mathbb{N}$.

# Complexity of isomorphism

# of oligomorphic subgroups of $S_\infty$

# Conjugacy of oligomorphic groups

Fact. For oligomorphic groups, being conjugate is Borel reducible to $id_{\mathbb{R}}$. (This fails for other classes, e.g. for profinite by KNS '18.)

Proof.

- Given a closed subgroup $G$ of $S_\infty$, let $V_G$ be the corresponding orbit equivalence structure: for each $k > 0$ introduce a $2k$-ary relation that holds for two $k$-tuples if they are in the same orbit of $\mathbb{N}^k$.
- $G$ is oligomorphic $\Rightarrow$ $V_G$ is $\omega$-categorical.
- One checks that for oligomorphic groups $G, H$

$$G \text{ and } H \text{ are conjugate in } S_\infty \iff V_G \cong V_H.$$

- Isomorphism of $\omega$-categorical structures $M$, $N$ for the same language is smooth, because $M \cong N \iff \mathtt{Th}(M) = \mathtt{Th}(N)$.

# Upper bound on complexity of isomorphism

An equivalence relation is essentially countable if it is Borel reducible to a Borel equivalence relation with all classes countable. (Things like $E_0$, or $\equiv_T$.) These are way below graph isomorphism.

**Theorem (N., Tent, Schlicht '18)**

Isomorphism of oligomorphic groups is essentially countable.

- We use a result by Hjorth/Kechris 1995 that characterizes essential countability of the isomorphism relation on a Borel class of structures by model theory in $L_{\omega_1,\omega}$.
- We have to adapt some of our axioms so that they can be expressed in $L_{\omega_1,\omega}$.

# Hjorth-Kechris result in infinitary model theory

$R$ is a ternary relation. One says that $F \subseteq L_{\omega_1,\omega}(R)$ is a fragment if $F$ is closed under subformulas, substitution, and first order operations such as finite Boolean combinations and quantification.

- Given: a Borel, isomorphism invariant class such as $\mathcal{B}$.
- By the Lopez Escobar theorem, $\mathcal{B}$ can be axiomatised by a sentence $\sigma$ in $L_{\omega_1,\omega}(R)$.
- Let $F$ be a countable fragment containing $\sigma$.

**Theorem (Hjorth and Kechris. 1995)**

The following are equivalent. (We will only use (i)$\rightarrow$(ii). )

(i) for each $M \in \mathcal{B}$ there is a tuple $\overline{a}$ in $M$ such that $\mathtt{Th}_F(M, \overline{a})$ is $\aleph_0$-categorical.

(ii) The isomorphism relation on $\mathcal{B}$ is essentially countable

# Upper bound on complexity of isomorphism: finish

$\mathcal{B}$ is Borel invariant class. Sentence $\sigma \in L_{\omega_1,\omega}$ describes it.

Recall Hjorth/Kechris (i)$\rightarrow$(ii): Suppose that for each $M \in \mathcal{B}$ there is a tuple $\overline{a}$ in $M$ such that $\mathtt{Th}_F(M, \overline{a})$ is $\aleph_0$-categorical.
Then the isomorphism relation on $\mathcal{B}$ is essentially countable.

- I our case let $F$ be a countable fragment containing $\sigma$ and the formula $\delta(W)$ describing a faithful subgroup $W$.
- Check that $\mathtt{Th}_F(M, W)$ is $\aleph_0$-categorical.

This shows that $\cong_{\mathcal{B}}$ is essentially countable.

# Extension to quasi-oligomorphic groups

A closed subgroup $G$ of $S_\infty$ is called quasi-oligomorphic if it is isomorphic to an oligomorphic group.

**Corollary**

This class is Borel.

Its isomorphism relation is also essentially countable.

Idea:

- $\mathcal{M}(G)$ is defined for any Roelcke precompact group.
- $\mathcal{G}(\mathcal{M}(G))$ is oligomorphic via its natural embeddeding into $S_\infty$.

# Some open problems

- How complex is isomorphism of arbitrary closed subgroups of $S_\infty$? Is it $\leq_B$-complete for analytic equivalence relations?
- What is a lower bound for the complexity of isomorphism for oligomorphic groups? Is $E_0$ Borel reducible to it?
- Find a good upper bound for the Scott rank of the structures $\mathcal{M}(G)$. (Their rank is bounded by a countable ordinal because the isom. relation is Borel.)

**References** Kechris, N. and Tent, The complexity of topological group isomorphism, The Journal of Symbolic Logic, 83(3), 1190-1203. arXiv: 1705.08081

N., Schlicht and Tent, Oligomorphic groups are essentially countable, submitted, on arXiv. Also Logic Blog 2018 (bi-interpretability).