Computational Mixed-Integer Programming

Ambros Gleixner and the SCIP team

Zuse Institute Berlin · gleixner@zib.de
SCIP Optimization Suite · http://scip.zib.de

Theory and Practice of Satisfiability Solving

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Zuse Institute Berlin – Fast Algorithms, Fast Computers

A research institute and computing center of the State of Berlin with research units:

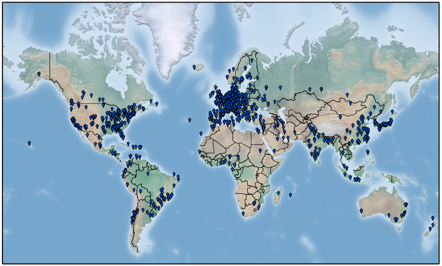
- Numerical Analysis and Modeling
- Visualization and Data Analysis
- Optimization:
  Energy – Transportation – Health – Mathematical Optimization Methods
- Scientific Information Systems
- Computer Science and High Performance Computing

SCIP: Solving Constraint Integer Programs

An open branch-cut-and-price framework with techniques from MIP, CP, SAT, and GO.

35+ active developers

- 10+ running Bachelor & Master projects
- 14+ running PhD projects
- 11 postdocs and professors



5 active development centers

- ZIB: SCIP, SoPlex, UG, ZIMPL
- TU Darmstadt: SCIP and SCIP-SDP
- FAU Erlangen-Nürnberg: SCIP
- RWTH Aachen & Uni. Lancaster: GCG

Many international contributors and users

- more than 14 000 downloads per year from 100+ countries

Careers

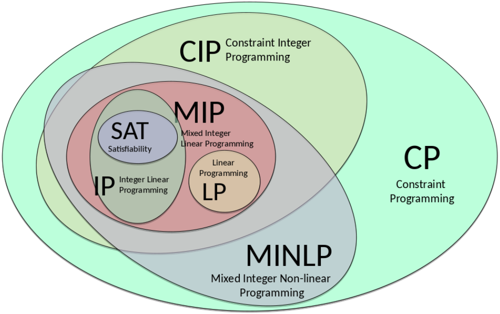
- 7 former developers are now building commercial optimization software at CPLEX, FICO Xpress, Gurobi, MOSEK, and GAMS
- 10 awards for Masters and PhD theses: MOS, EURO, GOR, DMV

Mixed-Integer Programming

A generalization of SAT:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}^I_{\geq 0} \times \mathbb{R}^C_{\geq 0} \end{aligned}$$

1. general linear constraints
2. general integer variables
3. continuous variables
4. objective function



Appealing for similar and different reasons than SAT:

- black-box solvers exist ⇝ separates modeling and algorithm design
- mostly open-box solvers ⇝ allows for problem-specific improvements
- global optimality guarantees ⇝ allows to stop early at near-optimal solutions
- ...

Outline

Essentials

- Linear programming relaxation
- LP-based branch-and-bound
- Cutting planes
- Simplex hot starts

Supplementary techniques

- Presolving & propagation
- Conflict analysis
- Branching heuristics
- Node selection
- Primal heuristics
- Symmetry handling

Numerics & exact certificates

- Numerics
- Verifying MIP results

Conclusion

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Relaxations and bounds

A common approach for hard nonconvex optimization problems like MIP: compute bounds on the optimal value

$$\begin{aligned} z^* = \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}^I_{\geq 0} \times \mathbb{R}^C_{\geq 0} \end{aligned}$$

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1. Lower bound $L \leq z^*$: relaxation
   - in MIP: **LP relaxation**, $\mathbb{Z}^I \rightsquigarrow \mathbb{R}^I$
   - convex and “fast” to solve $\rightsquigarrow x^{\text{LP}}$

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2. Upper bound $U \geq z^*$: feasible solutions
   - if LP relaxation is "accidentally" feasible $\rightsquigarrow$ optimal solution
   - later: primal heuristics

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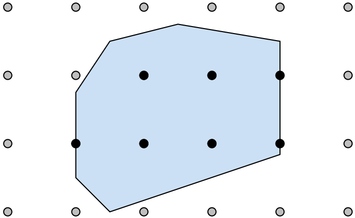
LP-based branch-and-bound

Systematic reduction of $U - L$ by divide-and-conquer [LD60, Dak65]

Branch-and-bound tree

Solution space



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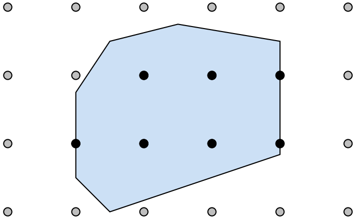

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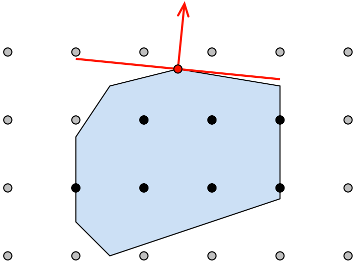

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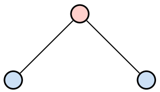

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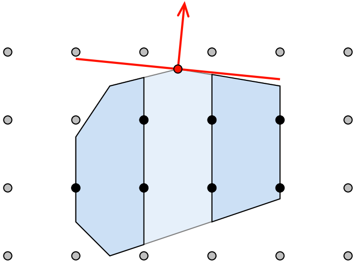

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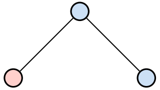

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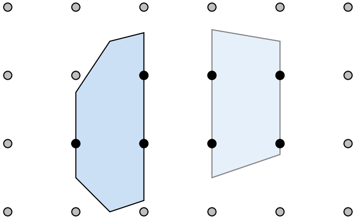

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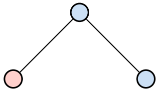

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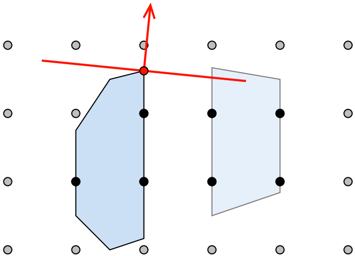

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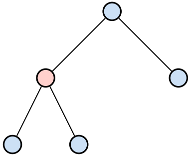

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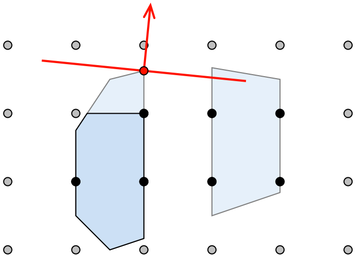

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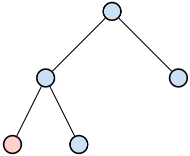

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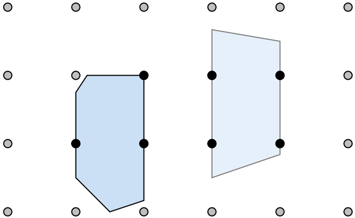

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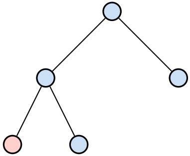

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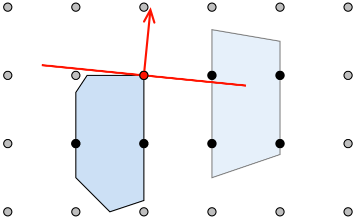

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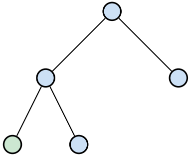

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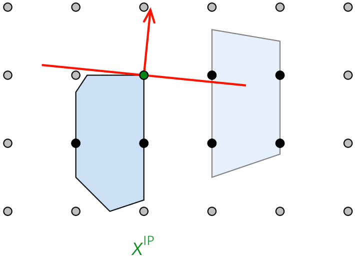

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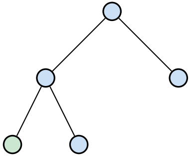

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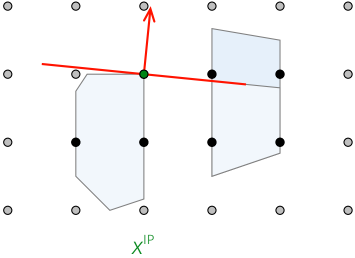

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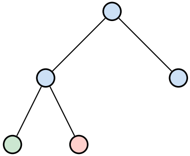

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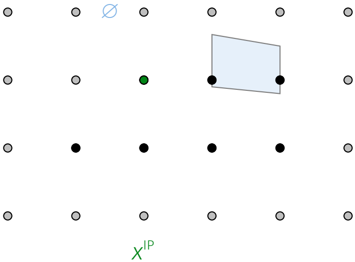

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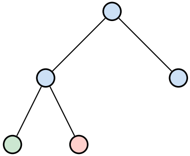

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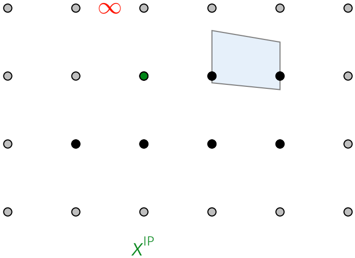

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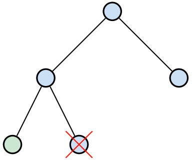

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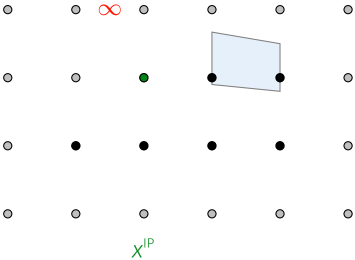

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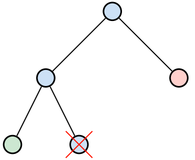

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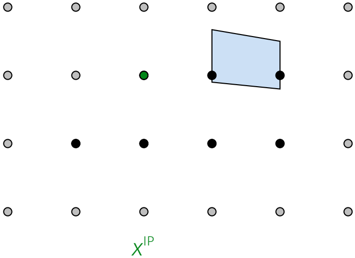

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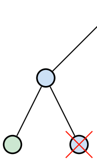

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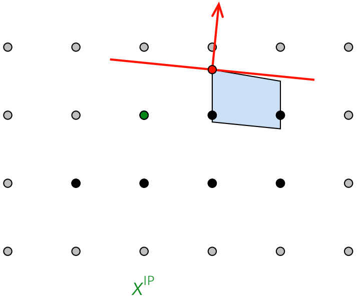

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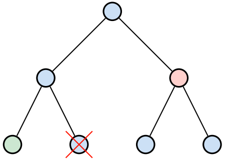

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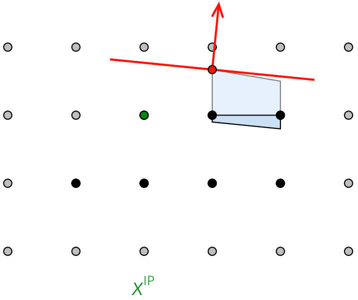

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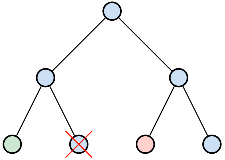

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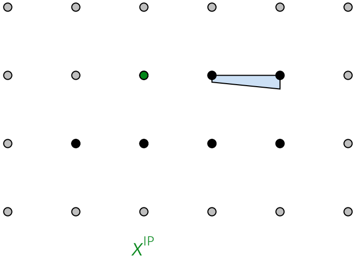

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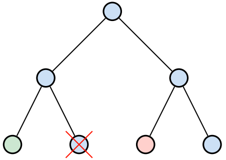

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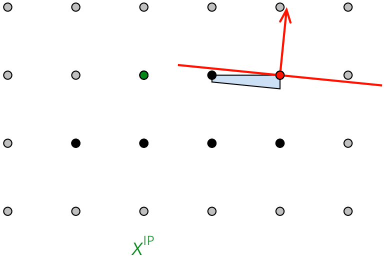

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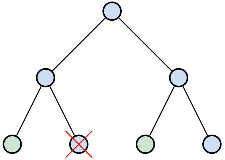

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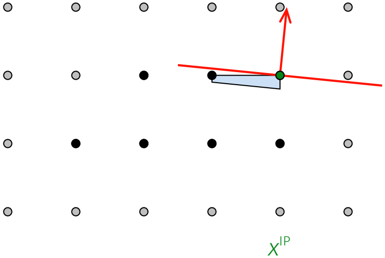

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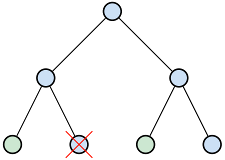

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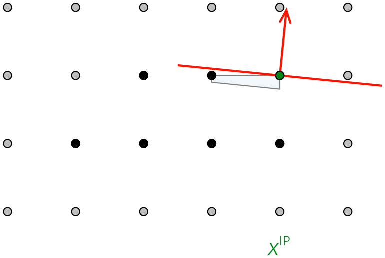

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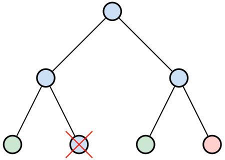

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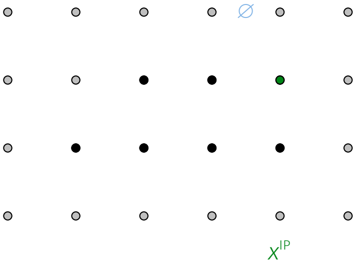

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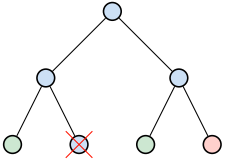

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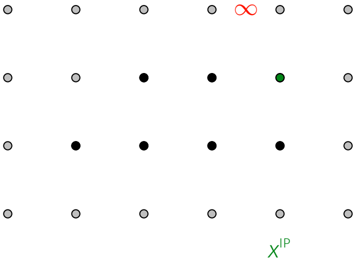

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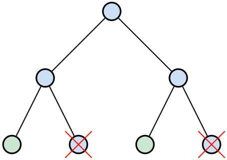

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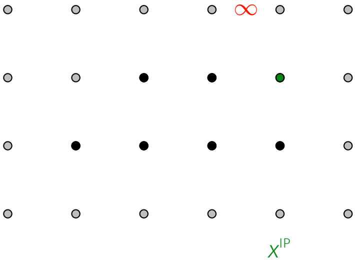

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5. branching on general constraints can yield smaller trees, but
    - increases and slows down the LP, and
    - the high degree of freedom makes designing branching heuristics challenging,

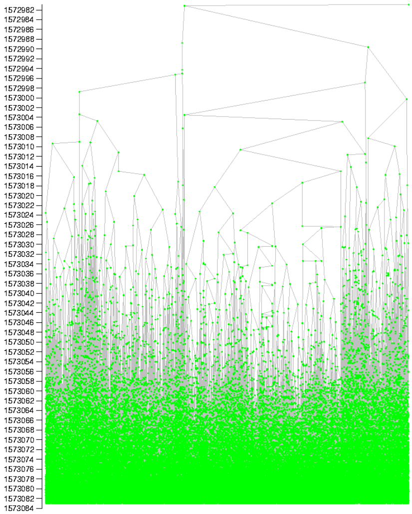
   so by default MIP solvers rely on variable-based branching, see [GMB+15] and references therein.

Example: the 15,112 cities TSP

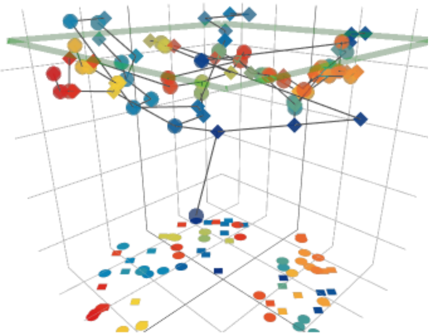
Schematic tree for a 15,112 cities traveling salesman problem

(World record 2001 by Applegate, Bixby, Chvátal, Cook)

`http://www.math.uwaterloo.ca/tsp/d15sol/`



Example: MIPLIB instance lseu



Spatial visualization of the branch-and-cut tree using multi-dimensional scaling, due to Matthias Miltenberger

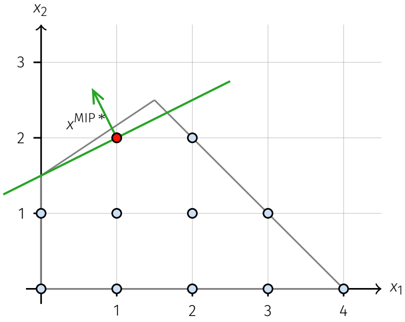
`http://www.zib.de/miltenberger/plotly`

Outline

Cutting planes

$$\mathcal{X}_{\text{MIP}} := \{x \in \mathbb{Z}^I \times \mathbb{R}^C : Ax \leq b\}$$

$$\mathcal{X}_{\text{LP}} := \{x \in \mathbb{R}^I \times \mathbb{R}^C : Ax \leq b\}$$



$$\min\{c^T x : x \in \mathcal{X}_{\text{MIP}}\}$$

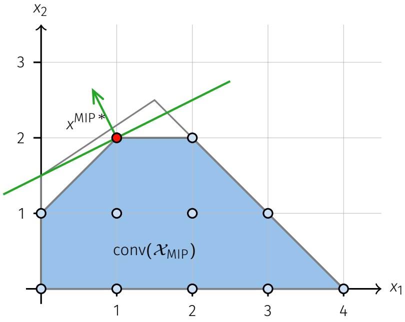
Cutting planes

Observation

- $\text{conv}(\mathcal{X}_{\text{MIP}})$ is a polyhedron
- IP could be formulated as LP

Problems with $\text{conv}(\mathcal{X}_{\text{MIP}})$:

- linear description not known
- large no. of constraints



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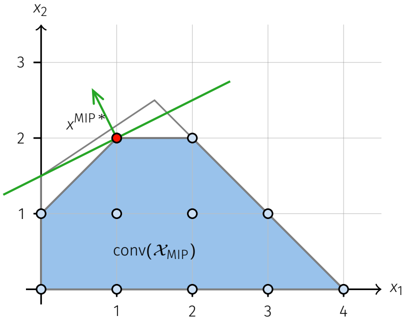

$\min\{c^T x : x \in \text{conv}(\mathcal{X}_{\text{MIP}})\}$

$$\mathcal{X}_{\text{LP}} \supseteq \mathcal{X} \supseteq \text{conv}(\mathcal{X}_{\text{MIP}})$$

$$\min\{c^\top x : x \in \mathcal{X}_{\text{LP}}\} \leq \min\{c^\top x : x \in \mathcal{X}\} = \min\{c^\top x : x \in \text{conv}(\mathcal{X}_{\text{MIP}})\}$$

General cutting plane method

Algorithm

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2. Solve
$$\begin{array}{rr} \min & c^\top x \\ s.t. & x \in \mathcal{X} \end{array}$$
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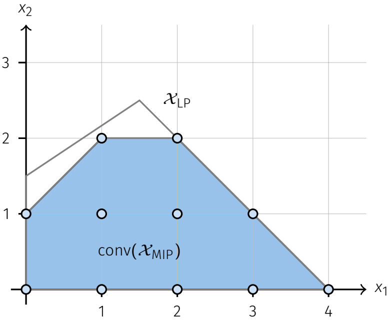

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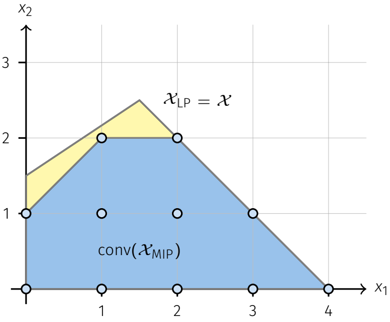

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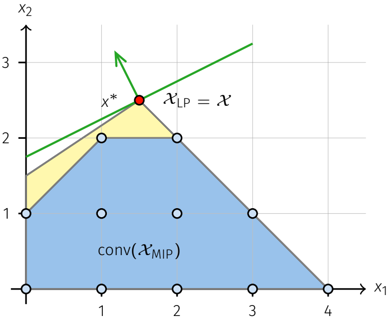

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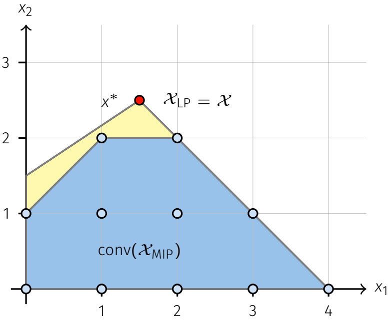

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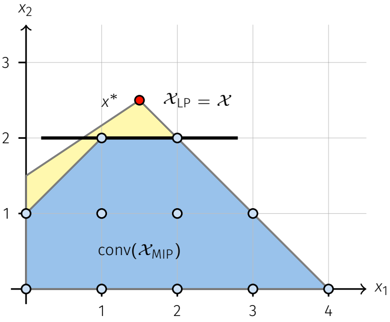

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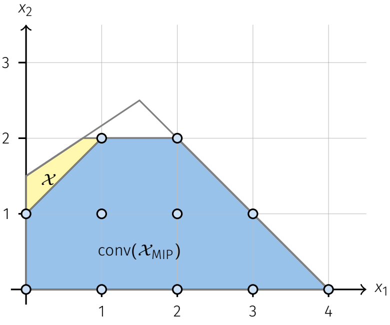

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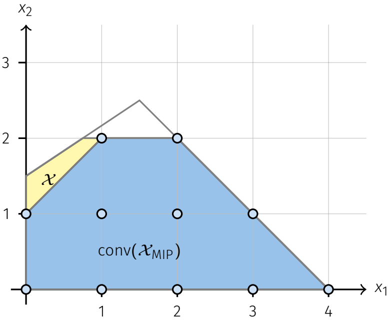

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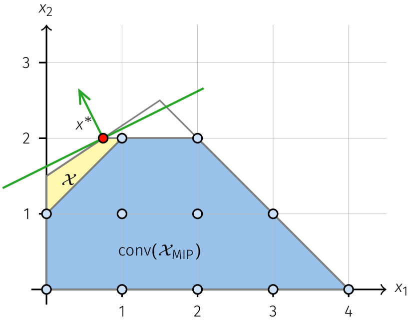

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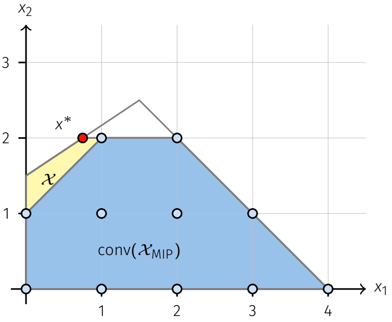

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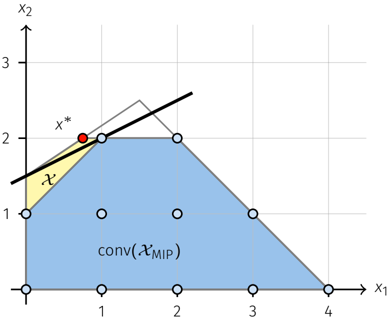

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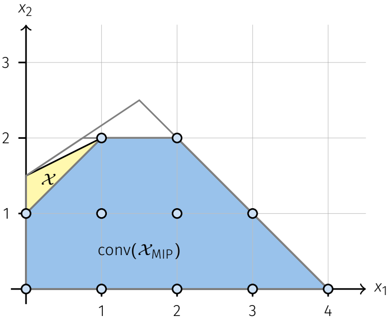

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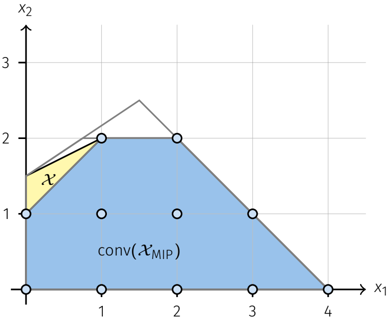

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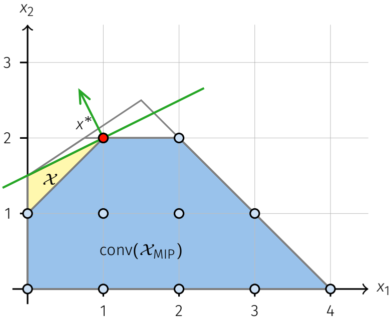

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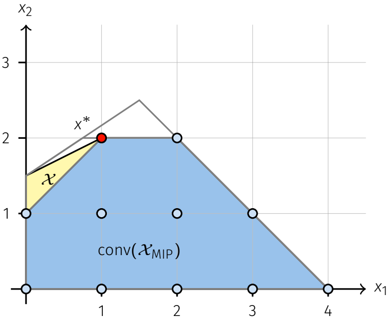

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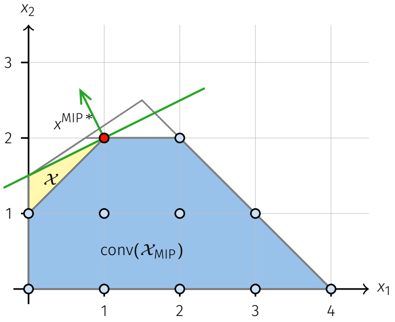

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- Gomory cuts yield a finitely convergent, pure cutting plane algorithm [Gom58].

Example: knapsack cover cuts

Start from single-row relaxation

$$X := \{\, x \in \{0,1\}^n \;:\; \sum_{j \in N} a_j x_j \leq b \,\}$$

for $b \in \mathbb{Z}_{>0}$, $a_j \in \mathbb{Z}_{>0}\;\; \forall j \in N$.

Minimal cover $C \subseteq N \Leftrightarrow \sum_{j \in C} a_j > b$ and $\sum_{j \in C \setminus \{i\}} a_j \leq b \quad \forall i \in C$.

$$\implies \sum_{j \in C} x_j \leq |C| - 1$$

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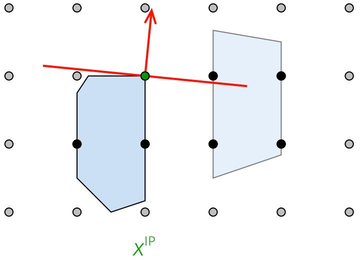
$$\Longrightarrow \sum_{j\in C} x_j \le |C| - 1$$

Separation problem

$$\min \Big\{ \sum_{j\in N} (1 - x_j^*) y_j : \sum_{j\in N} a_j y_j \ge b + 1, y_j \in \{0,1\} \Big\}$$

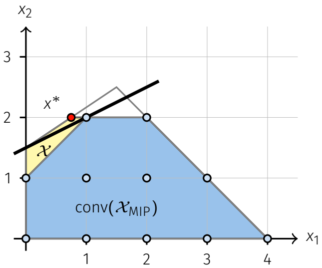
yields the most violated cut for given x^* (typically solved heuristically).

Branch-and-cut



branch-and-bound

+



cutting planes

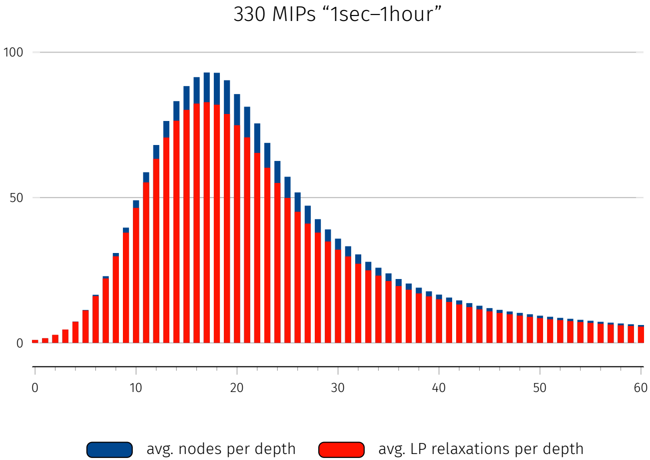
- limited cutting plane generation at root and nodes of a branch-and-bound tree
- cut selection and aging crucial

Outline

Dual simplex iterations during tree search

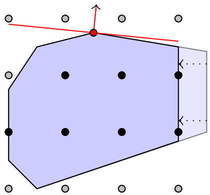
383.7/3.3 ≈ 116x speedup

depth	avg. iters
root	**383.7**
1	17.0
2	14.9
3	12.2
4	10.3
5	9.0
6	8.2
⋮	⋮
13	4.2
14	4.1
15	3.8
16	3.4
17	3.3
18	3.1
19	2.8
20	2.7
21	2.5
22	2.4
⋮	⋮



Outline

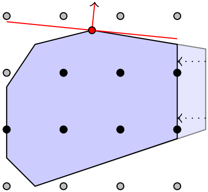
Presolving and propagation



Task

- reduce size of model by removing irrelevant information
- strengthen LP relaxation by exploiting integrality information
- make the LP relaxation numerically more stable
- extract useful information

Presolving and propagation



Task

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Primal Reductions:
- based on feasibility reasoning
- no feasible solution is cut off

Weak/strong dual reductions:
- consider objective function
- all/at least one optimal solution remains

Trivial presolving

Fast and useful:

- remove empty rows, columns
  - e.g., $0^T x \leq b_i,\ b_i < 0 \Rightarrow$ infeasible
- tighten fractional bounds of integer variables
- substitute fixed variables
- replace singleton rows
  - e.g., $a_{ij}x_j \leq b_i,\ a_{ij} < 0 \Rightarrow x_j \geq \frac{b_i}{a_{ij}} \Rightarrow$ new lower bound on x_j
- normalize constraints
  - e.g., if all coefficients are integral, divide by greatest common divisor and round rhs
- detect constraint types (knapsack, setppc, …)
- …

Linear presolving

Let a linear constraint $a^T x \le b$, and bounds $\ell \le x \le u$ be given.

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are the minimal and maximal activity of the linear constraint.

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are the minimal and maximal activity of the linear constraint.

First observation

- $\alpha_{\min} > b \Rightarrow$ problem infeasible
- $\alpha_{\max} \leq b \Rightarrow$ constraint redundant

Bound strengthening

Let $a_k > 0$. For all feasible solutions x, it holds that:

$$a^T x - a_k x_k + a_k x_k \leq b \Leftrightarrow x_k \leq \frac{b-(a^T x - a_k x_k)}{a_k} \Rightarrow x_k \leq \frac{b-\alpha_{\min}+a_k \ell_k}{a_k}$$

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$$\Rightarrow x_k \leq \min\left\{u_k, \frac{b-\alpha_{\min}+a_k \ell_k}{a_k}\right\}$$

Variants:

- $k \in I \Rightarrow x_k \leq \lfloor \frac{b-\alpha_{\min}+a_k \ell_k}{a_k} \rfloor$
- $a_k < 0 \Rightarrow x_k \geq \frac{b-\alpha_{\max}+a_k u_k}{a_k}$

Global information: the conflict graph [ANS00]

For the set of binary variables $\mathcal{B}$ in a MIP M, the conflict or clique graph is the undirected $G = (V, E)$ with nodes

$$V := \mathcal{B} \times \{0, 1\} = \{j_\kappa, j \in \mathcal{B}, \kappa \in \{0, 1\}\}.$$

and edges

$$\begin{aligned} E := \{\{v, w\} :& \kappa_v x_v + \kappa_w x_w + \\ & (1 - \kappa_v)(1 - x_v) + (1 - \kappa_w)(1 - x_w) \leq 1 \text{ valid for } M\} \end{aligned}$$

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Example

Let $V = \{1, 2, 3\} \times \{0, 1\}$, and let E consist of the following edges:

1. $\{1_1, 2_1\} \Rightarrow x_1 + x_2 \leq 1$
2. $\{2_1, 3_0\} \Rightarrow x_2 + (1 - x_3) \leq 1$
3. $\{3_1, 1_1\} \Rightarrow x_3 + x_1 \leq 1$

The conflict graph is first populated during presolving and used for separation and propagation during the solving process.

More presolving

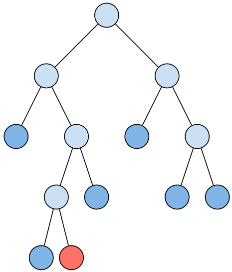
- probing: tentatively fix binary variables and propagate
- dominance test: pairwise comparison of rows/columns
- aggregation of equations with only two variables
  $a_k x_k + a_j x_j = b \Rightarrow x_k = \frac{b}{ai} - \frac{a_j}{ak} x_j$
- dual fixing: If $a_{ik} \geq 0$ for all i and $c_k \geq 0$, then x_k can be fixed to its lower bound
- dual aggregation: If $c_k \geq 0$ and there is exactly one i for which $a_{ik} < 0$, we can aggregate $x_k = \frac{b_i}{a_j} - \frac{1}{a_k} \sum a_j x_j$.
- dual bound reduction: Strengthen bounds of variables to the tightest value for which all its constraints are redundant
- clique detection (e.g., for knapsack constraints)
- variable lifting
- ...

Outline

Introduction: Conflict Analysis

Goal: When branch-and-bound reaches an infeasible subproblem, analyze the infeasibility to

- extract a shorter reason



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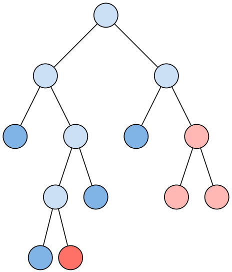

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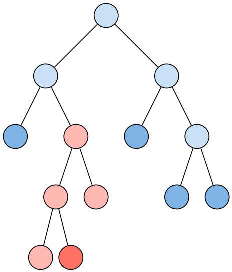

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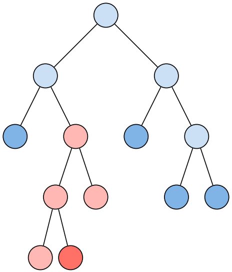

Example: contradicting bound changes after propagation

$$x_1 + x_2 + 2x_3 \leq 2 \qquad (1)$$

$$x_1 + x_2 - 2x_3 \leq 0 \qquad (2)$$

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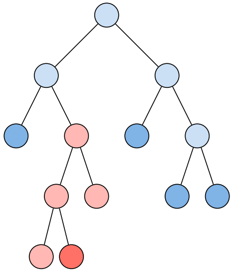

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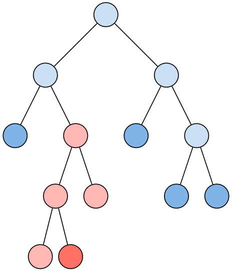

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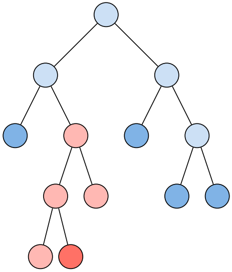

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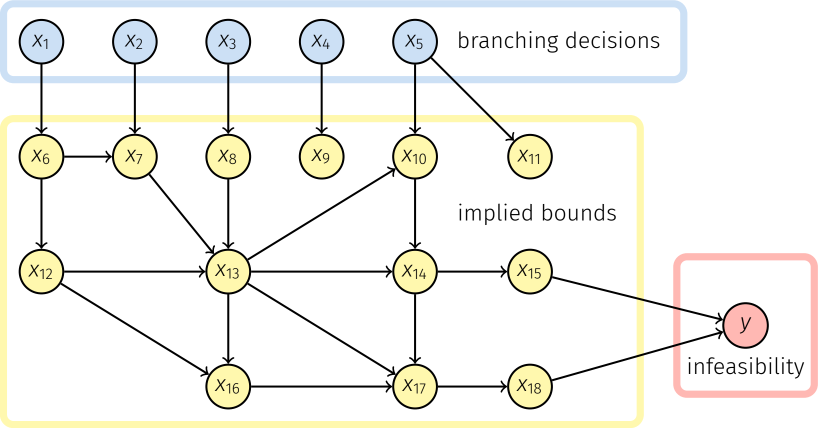
$\rightsquigarrow$ $x_2 \geq 1$ is a sufficient reason

Conflict Graph Analysis [MSS99]

- Consider implications that led to the local bounds



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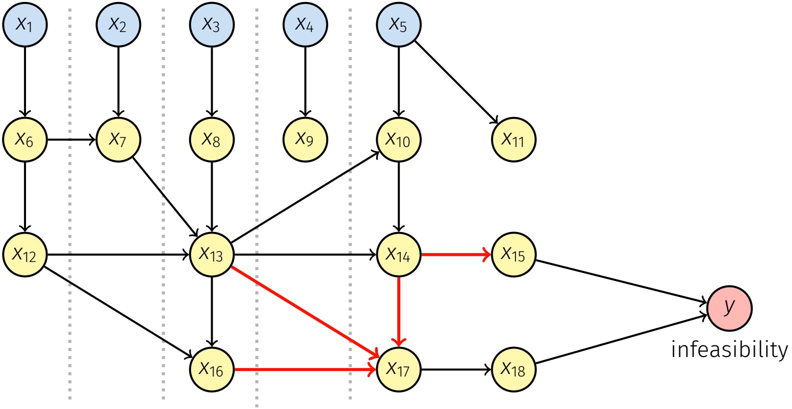

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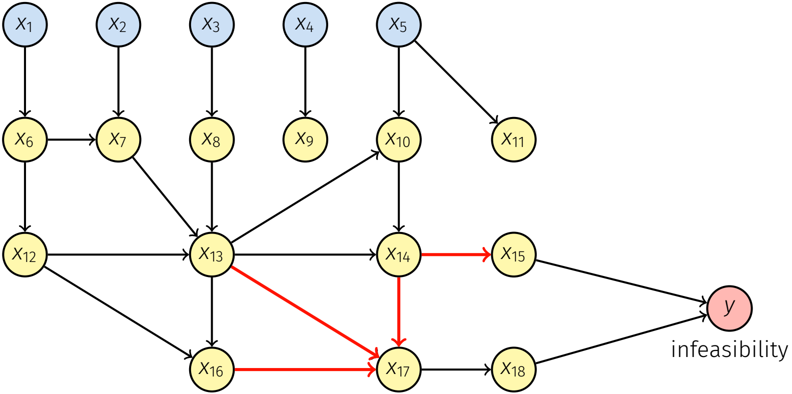
- Special: graph is not maintained, but constructed when needed (in SCIP)



Explaining LP infeasibility [SS06, Ach07]

- Assume a subproblem with bounds $\ell \leq \ell' \leq u' \leq u$

$$\min\{c^T x \mid Ax \geq b,\ \ell' \leq x \leq u',\ x_i \in \mathbb{Z}\ \forall i \in \mathcal{I}\} \tag{3}$$

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- i.e. a ray (y, s)

$$\begin{aligned} y^t A + s^t &= 0 \\ y^t b + s^t\{\ell', u'\} &> 0 \end{aligned}$$

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- Farkas constraint globally valid: propagate during tree search, strengthen via MIR rounding, ... [WBH17]
- analogous extension for bound exceeding LPs

Outline

Basic definitions for variable branching

Notation	Description
P	LP relaxation feasible region $$P := \{x \in \mathbb{R}^n_{\geq 0} : Ax \leq b\}$$
$\mathcal{F}$	Set of fractional variables (fractionality $\neq 0$) $$\mathcal{F} := \{j \in I : x_j^{\mathrm{LP}} \notin \mathbb{Z}\}$$
+,-	Branching directions

Fractionality: distance between LP solution value and branching bound

$$f_j^+ := \lceil x_j^{\mathrm{LP}} \rceil - x_j^{\mathrm{LP}}, \quad f_j^- := x_j^{\mathrm{LP}} - \lfloor x_j^{\mathrm{LP}} \rfloor$$

Most/least infeasible branching

Idea: Select fractional variable with highest fractionality

$$j \in \underset{j' \in \mathcal{F}}{\operatorname{argmax}} \{\min\{f_{j'}^-, f_{j'}^+\}\}$$

or lowest fractionality

$$j \in \underset{j' \in \mathcal{F}}{\operatorname{argmin}} \{\min\{f_{j'}^-, f_{j'}^+\}\}.$$

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Problem

LP degeneracy (multiple optimal node LP solutions) on many problems makes fractionality a weak variable attribute.

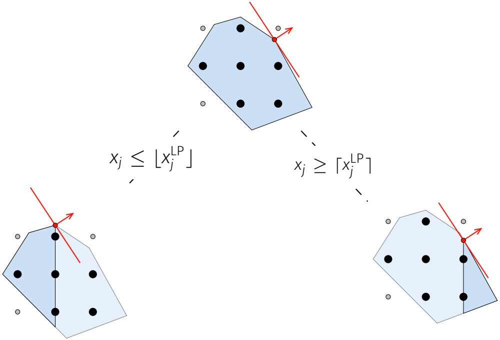
Poor branching performance yielding large trees, sometimes worse than randomized variable selection.

Dual gain

Branching children/descendants:

$$P_j^- := P \cap \{x_j \leq \lfloor x_j^{\mathrm{LP}} \rfloor\}, P_j^+ := P \cap \{x_j \geq \lceil x_j^{\mathrm{LP}} \rceil\}$$

Dual gain: LP objective between a descendant and its parent node P:



$$\Delta c_j^* := \min\{c^T x : x \in P_j^*\}\} - \min\{c^T x : x \in P\} \geq 0, \quad * \in \{-, +\}$$

Scoring function

Selecting fractional candidates based on scores for individual directions

$$s^- := \Delta c_j^-, s^+ := \Delta c_j^+ \forall j \in \mathcal{F}$$

requires scoring function:
$s(s^-, s^+)\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$

Possibilities:

- Weighted sum for $\lambda \in [0, 1]$:

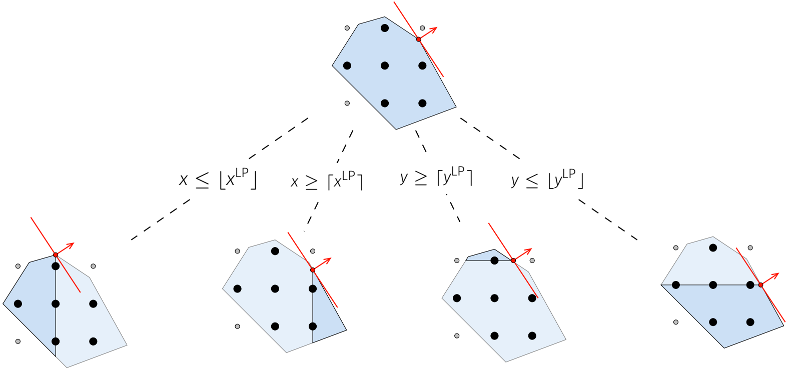
$$s(s^-, s^+) := \lambda \max\{s^-, s^+\} + (1 - \lambda) \min\{s^-, s^+\}$$

- Product for small $\epsilon > 0$:

$$s(s^-, s^+) := \max\{s^-, \epsilon\} \cdot \max\{s^-, \epsilon\}$$

Lookahead: strong branching

1. Perform an explicit look-ahead by solving all possible descendants of the current node.



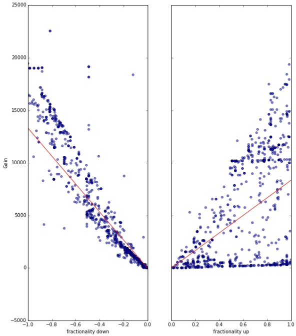
2. Select a fractional variable $j \in \operatorname{argmax}_{j' \in \mathcal{F}} \{ s\{\Delta c_{j'}^{-}, \Delta c_{j'}^{+}\}\}$.

Lookback: pseudocosts [BGG+71]

Estimate for objective gain based on past branching observations.

- unit gain:
  computed from fractionalities f_j^* and LP gains
- pseudocosts Ψ_j^*:
  average unit gain of branching history
- branching decision based on estimated gains:

$$s(f_j^- \Psi_j^-, f_j^+ \Psi_j^+)$$



Select a fractional variable $j \in \underset{j' \in \mathcal{F}}{\operatorname{argmax}} \{s(f_j^- \Psi_j^-, f_j^+ \Psi_j^+)\}$.

Reliability branching [AKM04]

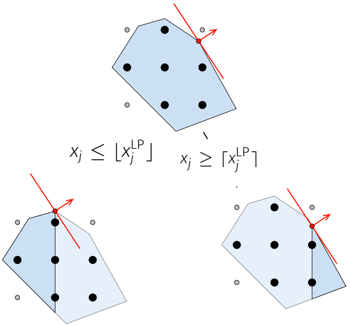
Pseudocosts are uninitialized at the beginning of the search.

Reliability branching

1. Determine the set of fractional variables $\mathcal{F} \neq \emptyset$.
2. Split $\mathcal{F}$ into reliable subset $\mathcal{F}^{\text{rel}}$ and unreliable subset $\mathcal{F}^{\text{url}}$.
3. Perform strong branching for all $j \in \mathcal{F}^{\text{url}}$.
4. Record unit gains and update pseudocosts.
5. Compare the best strong branching result with the best pseudocost prediction for the branching decision.

Variable branching information: other types

- cutoff information:
  average number of branchings yielding an infeasible node
- inference information:
  average number of domain reductions after branching
- conflict information:
  occurrence in recently learned conflict clauses
- conflict length information:
  occurrence in short conflicts



Hybrid branching [AB09]

Combine all types of variable branching history in a single, weighted score.

- scaling: divide each value by average over all variables
- normalize by $f\colon \mathbb{R}_{\geq 0} \to [0,1),\ x \mapsto \frac{x}{x+1}$
- use weights $\omega := (\omega^{\text{pscost}}, \omega^{\text{infer}}, \omega^{\text{prune}}, \omega^{\text{conf}}, \omega^{\text{clen}})$

Hybrid score

$$s_j := \omega * \left(f\left(\frac{s_j^{\text{pscost}}}{s_\varnothing^{\text{pscost}}}\right), f\left(\frac{s_j^{\text{infer}}}{s_\varnothing^{\text{infer}}}\right), f\left(\frac{s_j^{\text{prune}}}{s_\varnothing^{\text{prune}}}\right), f\left(\frac{s_j^{\text{conf}}}{s_\varnothing^{\text{conf}}}\right), f\left(\frac{s_j^{\text{clen}}}{s_\varnothing^{\text{clen}}}\right) \right)^T$$

SCIP implementation

$\omega^{\text{pscost}} = 1.0$, other weights $\in [10^{-4}, 10^{-2}]$

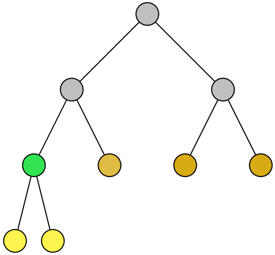
Other solvers may combine scores in a hierarchical or more complicated fashion.

Outline

Node Selection

Basic rules

- depth first search (DFS)
  $\rightarrow$ keep simplex cost small
- best bound search (BBS)
  $\rightarrow$ improve dual bound
- best estimate search (BES)
  $\rightarrow$ improve primal bound



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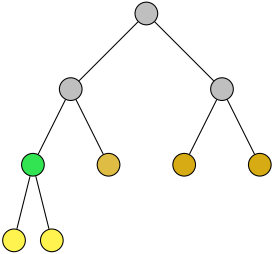


Best estimate [BGG+71]

Use learned pseudo costs to estimate objective value

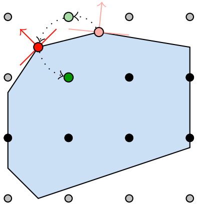
$$\hat{c} := c^T x_j^{\mathrm{LP}} + \sum_{j \in \mathcal{F}} \min\{f_j^- \Psi_j^-, f_j^+ \Psi_j^+\}$$

of the best solution in the subtree rooted at a node with LP solution x^{LP}.

Usually best bound/estimate interleaved with DFS plunges for simplex hot starting.

Outline

Primal Heuristics



Task

- improve primal bound
- effective on average
- guide remaining search

Techniques

- rounding
  - lock, randomized
  - octahedral neighborhood search
- diving
  - least infeasible
  - guided
  - solution density
- objective diving
  - objective feasibility pump
- large neighborhood search
  - RINS
  - RENS
  - local branching
- combinatorial
  - shift-and-propagate

RENS: Relaxation-Enforced Neighborhood Search [Ber14]

Input: MIP P

begin

$x^{\mathrm{LP}} \leftarrow$ relaxation optimum;

Fix all integral variables

$x_i := x_i^{\mathrm{LP}} \quad$ for all $i \in I : x_i^{\mathrm{LP}} \in \mathbb{Z}$;

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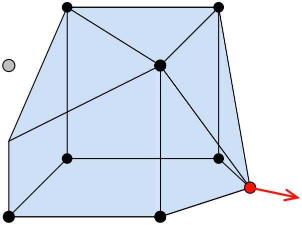

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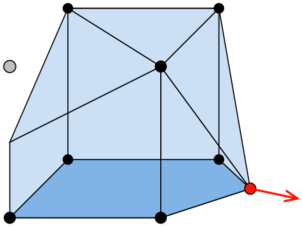

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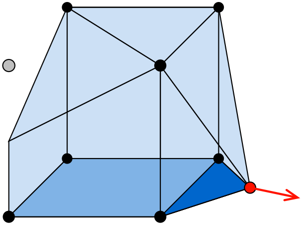

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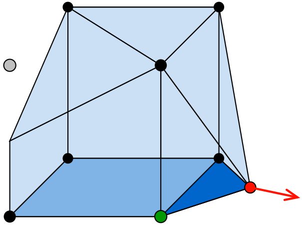

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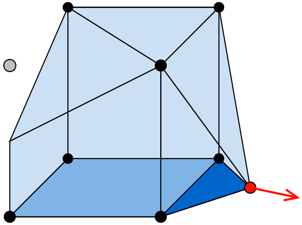

RENS is only one example of many “fix-and-MIP” LNS heuristics.

Outline

Symmetry handling

Detection

- formulation symmetry either via graph automorphism [PR15] (in SCIP via bliss)
- or via dedicated symmetry detection on the constraint matrix (in many commercial solvers)

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Alternatives:

- symmetry breaking constraints

$$\bar{c}x \geq \bar{c}\gamma(x),$$

  where $\bar{c} = (2^{n-1}, 2^{n-2}, \ldots, 2, 1) \in \mathbb{R}^n, x \in \{0,1\}^n$, or better their
- implicit enforcement via minimum cover inequalities [HP17], or recently
- derived from the Schreier-Sims table [Sal18].

Outline

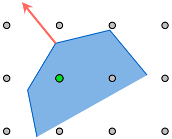
Numerics: Floating-point linear programming

Linear Program in standard form

$$\text{minimize} \;\; c^T x \;\; \text{subject to} \;\; Ax = b, x \geq \ell$$

Optimal solutions satisfy

- primal feasibility: $Ax - b = 0$ and $\hat{\ell}_i = \ell_i - x_i \leq 0$
- dual feasibility: $\hat{c}_i = c_i - y^T A_{\cdot i} \geq 0$
- complementary slackness: $\hat{c}_i \hat{\ell}_i = 0$



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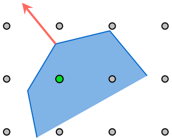

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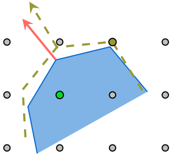

approximate solution

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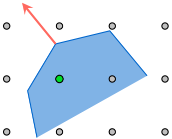
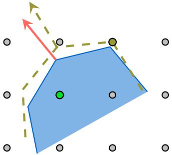
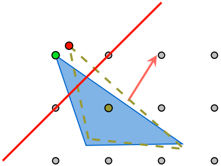
- dual feasibility: $\hat{c}_i = c_i - y^T A_{\cdot i} > -\epsilon$
- complementary slackness: $|\hat{c}_i \hat{\ell}_i| < \epsilon$



exact solution



approximate solution



invalid model strengthening

Exact Mixed-Integer Programming

Exact LP for

- primal solutions for fixed integer assignment
- dual bounding: expensive fallback
- hybrid solvers: QSopt_ex [ACDE07], SoPlex [GSW16]

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Fast and safe dual bounds

- hybrid arithmetic: floating-point– interval – rational
- correction of approximate dual solutions [NS04, SW13]

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Exact SCIP [CKSW13]

- exact LP-based branch-and-bound
- hierarchy of exact dual bounding techniques
  $\approx$ 2 – 4 times slower on numerically easy – difficult instances
- still gap to state-of-the-art MIP:
  no presolving, domain propagation, cutting planes, conflict analysis, heuristics

Outline

Motivation

Status quo: most MIP solvers do not output optimality certificates

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- practical: complex and diverse techniques in MIP solvers
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- certificate form unclear: tree, cuts, superadditive function

Goals for a practical certificate format

- expressivity: encode all (or most) MIP techniques
- simplicity: checkable by small verification code
- generality: both more general and simpler than [ABC+09] for TSP

LP certificates

LP optimality can be certified by a dual solution

- e.g.

$$\begin{array}{rl} \min & 2x + y \\ \text{s.t.} & \\ C0: & 5x - y \geq 2 \\ C1: & 3x - 2y \leq 1 \end{array}$$

Given		
$C0:$	$5x - y \geq 2$	
$C1:$	$3x - 2y \leq 1$	
Derived		**Reason**
`obj` :	$2x + y \geq 1$	$\{1 \times C0 + (-1) \times C1\}$

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- plain text syntax:

```
VAR 2
  x y
OBJ min
  2  0 2  1 1
```

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- plain text syntax:

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VAR 2
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OBJ min
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CON 2 0
  C0  G 2  2  0 5  1 -1
  C1  L 2  2  0 3  1 -2
```

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  C1  L 2  2  0 3  1 -2
RTP range 1 inf
DER 1
  C2  G 1  2  0 2  1 1  { lin 2  0 1  1 -1 }
```

LP certificates

LP optimality can be certified by a primal-dual solution

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$$\begin{array}{rl} \min & 2x + y \\ \text{s.t.} & \\ C0: & 5x - y \geq 2 \\ C1: & 3x - 2y \leq 1 \end{array}$$

Given		
$C0:$	$5x - y \geq 2$	
$C1:$	$3x - 2y \leq 1$	
Derived		**Reason**
`obj` :	$2x + y \geq 1$	$\{1 \times C0 + (-1) \times C1\}$

- plain text syntax:

```
VAR 2 x y
OBJ min 2  0 2  1 1
CON 2 0
  C0  G 2  2  0 5  1 -1
  C1  L 2  2  0 3  1 -2
RTP range 1 1
SOL 1
  2  0 3/7  1 1/7
DER 1
  C2  G 1  2  0 2  1 1  { lin 2  0 1  1 -1 }
```

Encoding Chvátal-Gomory cuts

Integer cutting planes often involve a rounding argument

- e.g.

$$\begin{array}{rl} \min & x+y \\ \text{s.t.} & \\ C0: & 4x+y \geq 1 \\ C1: & 4x-y \leq 2 \\ & x,y \in \mathbb{Z} \end{array}$$

Given		
	$x, y \in \mathbb{Z}$	
$C0:$	$4x + y \geq 1$	
$C1:$	$4x - y \leq 2$	
Derived		**Reason**
$C2:$	$y \geq -\frac{1}{2}$	$\{\frac{1}{2} \times C0 + (-\frac{1}{2}) \times C1\}$
$C3:$	$y \geq 0$	$\{\text{round up } C2\}$
$C4:$	$x + y \geq \frac{1}{4}$	$\{\frac{1}{4} \times C0 + \frac{3}{4} \times C3\}$
$C5:$	$x + y \geq 1$	$\{\text{round up } C4\}$

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- plain text syntax:

```
...
DER 4
  C2  G -1/2  1  1 1      { lin 2  0 1/2  1 -1/2 }
  C3  G 0   1  1 1        { rnd 2 }
  C4  G 1/4   2  0 1  1 1 { lin 2  0 1/4  3 3/4 }
  C5  G 1     2  0 1  1 1 { rnd 4 }
```

Encoding disjunctions

A tree-less branch-and-bound certificate

Given

$$x, y \in \mathbb{Z}$$

$$\begin{aligned} C0: &\quad 2x_1 + 3x_2 \geq 1 \\ C1: &\quad 3x_1 - 4x_2 \leq 2 \\ C2: &\quad -x_1 + 6x_2 \leq 3 \end{aligned}$$

Derived		**Reason**	**Assumptions**
$A0:$	$x_1 \leq 0$	$\{\text{assume}\}$	
$A1:$	$x_1 \geq 1$	$\{\text{assume}\}$	

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$A1:$	$x_1 \geq 1$	$\{\text{assume}\}$	
$A2:$	$x_2 \leq 0$	$\{\text{assume}\}$	
$C3:$	$0 \geq 1$	$\{C0 + (-2) \times A0 + (-3) \times A2\}$	$A0, A2$

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$C5:$	$0 \geq 1$	$\{\text{unsplit } C3, C4 \text{ on } A2, A3\}$	$A0$

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$C4:$	$0 \geq 1$	$\{(-\frac{1}{3}) \times C2 + (-\frac{1}{3}) \times A0 + 2 \times A3\}$	$A0, A3$
$C5:$	$0 \geq 1$	$\{\text{unsplit } C3, C4 \text{ on } A2, A3\}$	$A0$
$C6:$	$x_2 \geq \frac{1}{4}$	$\{(-\frac{1}{4}) \times C1 + (\frac{3}{4}) \times A1\}$	$A1$
$C7:$	$x_2 \geq 1$	$\{\text{round up } C6\}$	$A1$
$C8:$	$0 \geq 1$	$\{(-\frac{1}{3}) \times C1 + (-1) \times C2 + \frac{14}{3} \times C7\}$	$A1$

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$C8:$	$0 \geq 1$	$\{(-\frac{1}{3}) \times C1 + (-1) \times C2 + \frac{14}{3} \times C7\}$	$A1$
$C9:$	$0 \geq 1$	$\{\text{unsplit } C5, C8 \text{ on } A0, A1\}$	$\emptyset$

VIPR: a straightforward branch-and-cut certificate [CGS17]

Simplicity

- only 4 reasoning types
- no explicit tree structure
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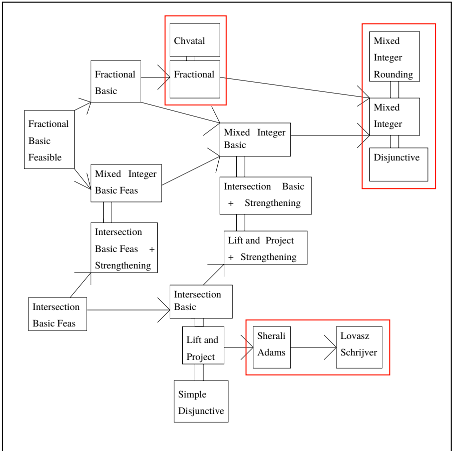

Figure 1: Comparison of Elementary Closures

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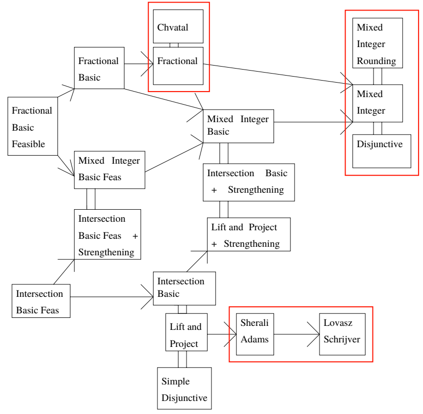

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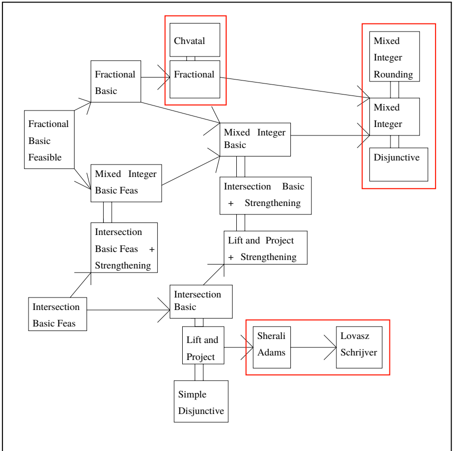

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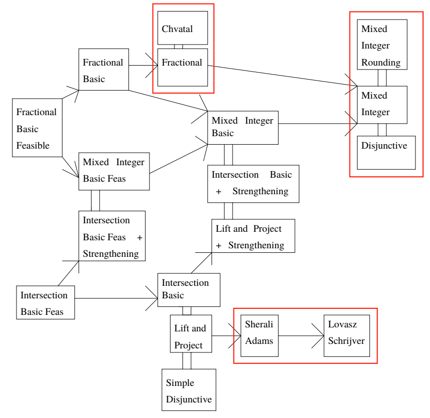
- translate MIP reasoning on-the-fly



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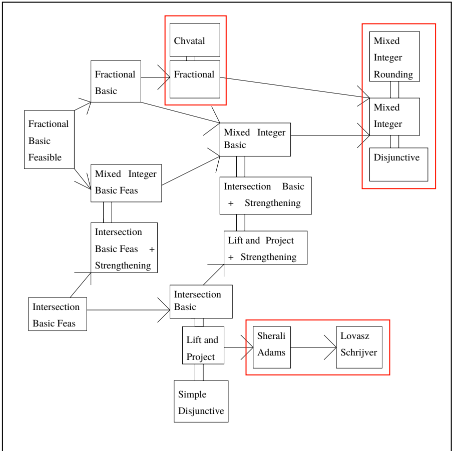
Results

- Eifler, G, Pulaj 2018: Chvátal's Conjecture Holds for Ground Sets of Seven Elements, ZIB-Report 18-49 [EGP18]



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Wrap-up

Many things not covered:

- cut generation, selection, and management
- degeneracy and performance variability
- parallelization
- column generation and decomposition methods
- restarts
- benchmarking
- ...

Wrap-up

Many things not covered:

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Many questions…? *Thank you for your attention!*

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