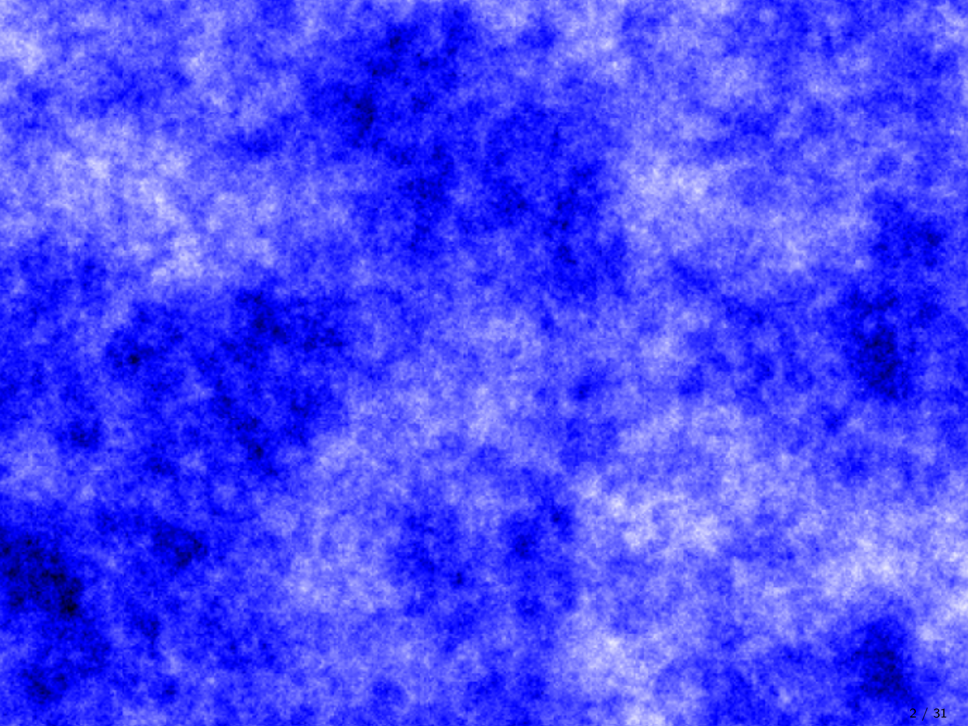
Efficient simulation of Brown-Resnick processes based on variance reduction of Gaussian processes

Kirstin Strokorb

joint work with Marco Oesting

Oaxaca – June 21, 2018

Why would you want to simulate a Brown-Resnick process?



Brown-Resnick process

"Original" definition

PPP $\sim u^{-2}\mathrm{d}u$

$\{U_i\}_{i=1}^{\infty}$

&

i.i.d. log-Gaussian

$V^{(i)}(x) = e^{W^{(i)}(x)-\sigma^2(x)/2}$

$\longrightarrow$

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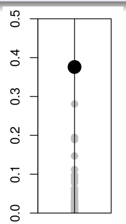
$Z(x) = \bigvee_{i=1}^{\infty} U_i V^{(i)}(x)$



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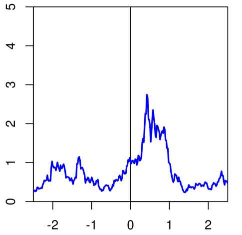

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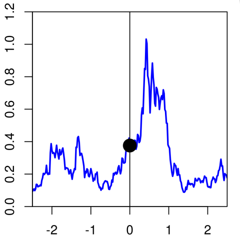

[Brown/Resnick '77, Kabluchko/Schlather/de Haan '09]

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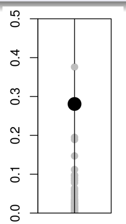

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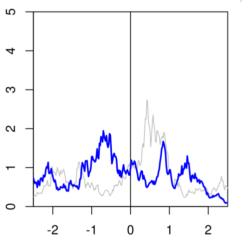

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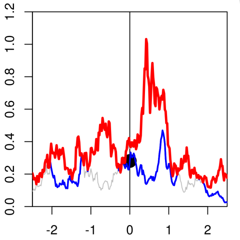

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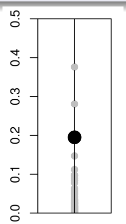

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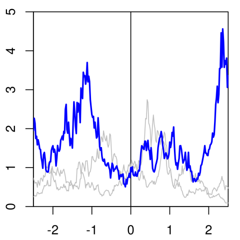

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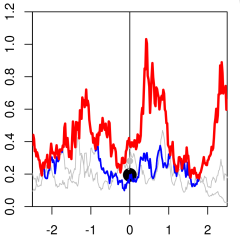

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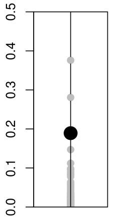

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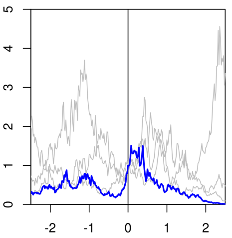

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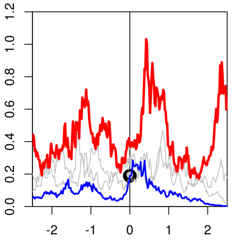

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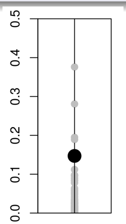

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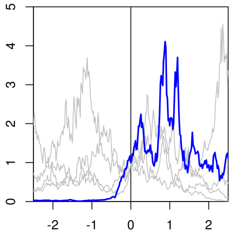
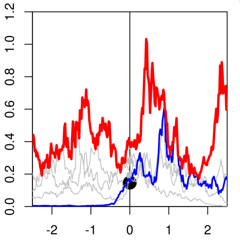


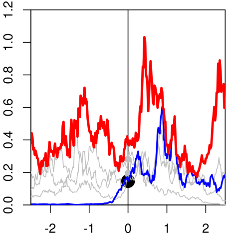


[Brown/Resnick '77, Kabluchko/Schlather/de Haan '09]

Properties

Brown-Resnick process

$$Z(x) = \bigvee_{i=1}^{\infty} U_i e^{W^{(i)}(x) - \sigma^2(x)/2}$$



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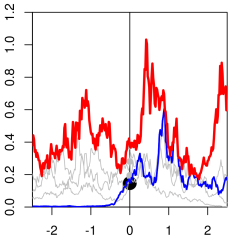

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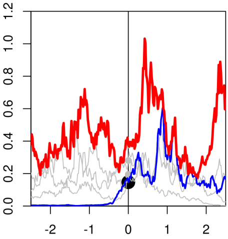

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$$\gamma(x - y) = \mathbb{E}(W(x) - W(y))^2$$

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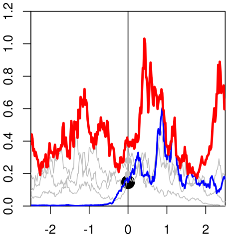

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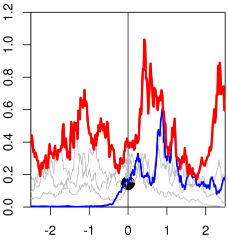

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$\Rightarrow$ **popular (benchmark) model for spatial extremes**
(consistent, parsimonious, tractable, flexible, smoothness control, ...)

Simulation approaches so far

Notation.

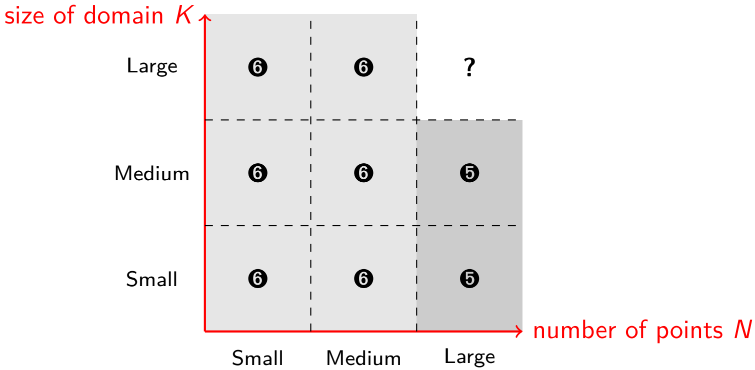
- K simulation domain
- N number of points in K
  on which Z shall be simulated

Overview

Method/Reference	Stopping rule for exact simulation	Expected number of Gaussian processes
❶ Original definition Kabluchko/Schlather/deHaan '09	no	unclear
❷ Random shift Oesting/Kabluchko/Schlather '12	no	unclear
❸ M3 representation Oesting/Kabluchko/Schlather '12	no	unclear
❹ L1-normalized spectral process Dieker/Mikosch '15	yes	$N \cdot C_K$
❺ Sup-normalized spectral process Oesting/Schlather/Zhou '18	(yes/no)	$\theta_K \cdot C_K \cdot$ # MCMC steps $= \mathcal{O}(1)$ wrt N
❻ Iterative extremal functions Dombry/Engelke/Oesting '16	yes	N
❼ Record breakers Liu/Blanchet/Dieker/Mikosch 16+	yes	$o(N^{\varepsilon})$, $\varepsilon > 0$

Which to use for exact simulation?

Heuristic[1] (on average fastest algorithm)



❻ Iterative extremal functions Dombry/Engelke/Oesting '16

❺ Sup-normalized spectral process Oesting/Schlather/Zhou '18

[1] not taking ❼ Record breakers Liu/Blan./Diek./Mik. 16+ into account, difficult to compare

What if an error is allowed?

(E.g. to speed up simulation
or make it feasible at all.)

Threshold Stopping as in [Schlather '02]

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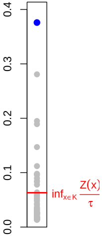

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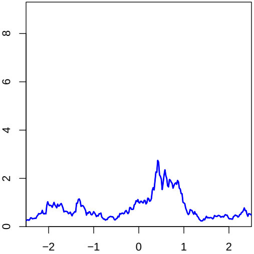

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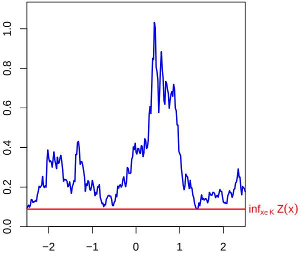

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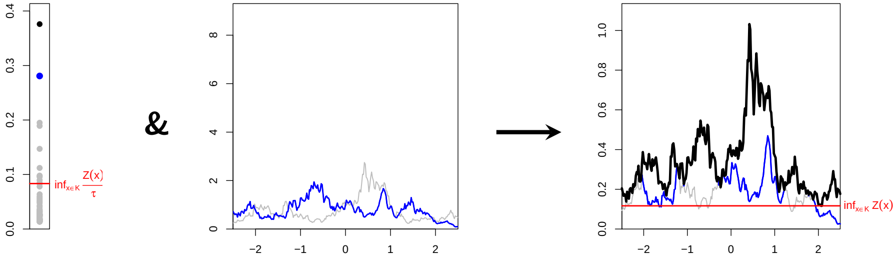

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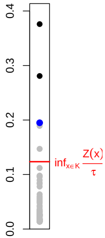

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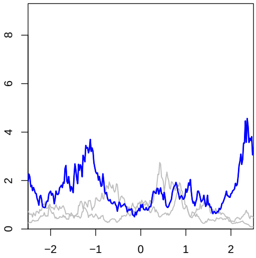

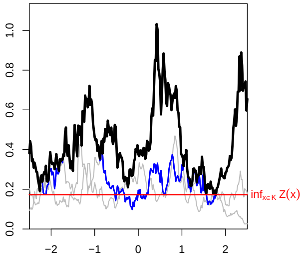

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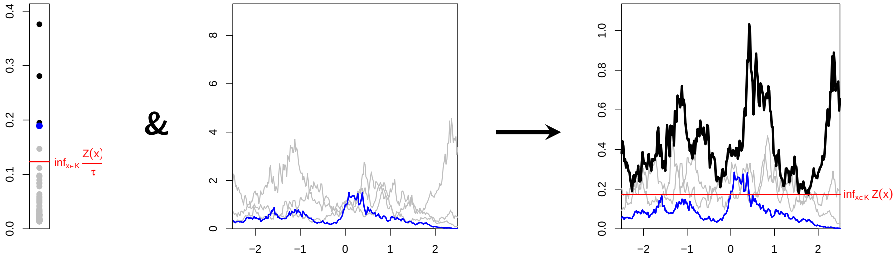

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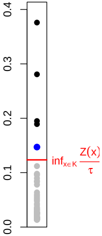

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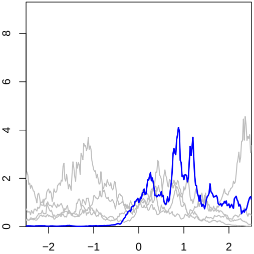

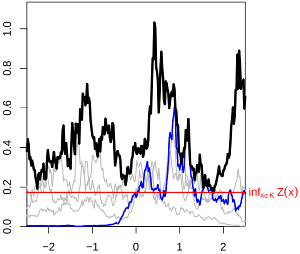

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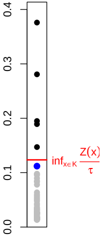

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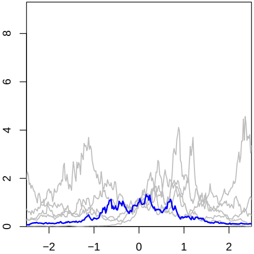

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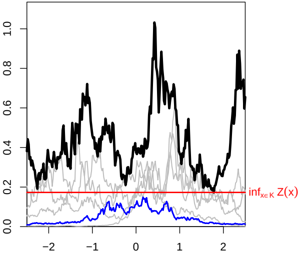

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First observations (from Slivnyak-Mecke Theorem)

- expected threshold stopping time

$$\geqslant \tau \, \mathbb{E}\left\{ 1 \Big/ \inf_{\boldsymbol{x} \in K} Z(\boldsymbol{x}) \right\}$$

- expected number of missing extremal functions

$$\leqslant \mathbb{E}\left(\sup_{\boldsymbol{x} \in K} \frac{V'(\boldsymbol{x})}{Z'(\boldsymbol{x})} - \sup_{\boldsymbol{x} \in K} \frac{\tau}{Z'(\boldsymbol{x})} \right)_+$$

for independent stochastic processes Z' and V' with the same distributions as Z and V, respectively.

Error bounds

$$\mathcal{P}^{(\text{abs})}_{K,\tau,\varepsilon} = \mathbb{P}\Big(\sup_{\boldsymbol{x}\in K}|Z(\boldsymbol{x}) - Z^{(T)}(\boldsymbol{x})| > \varepsilon\Big)$$

$$= 1 - \mathbb{E}_{Z^{(T)}}\left\{\exp\left(-\mathbb{E}_V\left(\sup_{\boldsymbol{x}\in K}\frac{V(\boldsymbol{x})}{Z^{(T)}(\boldsymbol{x})+\varepsilon} - \sup_{\boldsymbol{x}\in K}\frac{\tau}{Z^{(T)}(\boldsymbol{x})}\right)_+\right)\right\}$$

$$\mathcal{P}^{(\text{rel})}_{K,\tau,\varepsilon} = \mathbb{P}\left(\sup_{\boldsymbol{x}\in K}\frac{|Z(\boldsymbol{x}) - Z^{(T)}(\boldsymbol{x})|}{Z^{(T)}(\boldsymbol{x})} > \varepsilon\right)$$

$$= 1 - \mathbb{E}_{Z^{(T)}}\left\{\exp\left(-\mathbb{E}_V\left(\sup_{\boldsymbol{x}\in K}\frac{V(\boldsymbol{x})}{(1+\varepsilon)Z^{(T)}(\boldsymbol{x})} - \sup_{\boldsymbol{x}\in K}\frac{\tau}{Z^{(T)}(\boldsymbol{x})}\right)_+\right)\right\}$$

$$\mathcal{P}_{K,\tau} = \mathbb{P}(Z^{(T)} \neq Z^{(\infty)} \text{ on } K)$$

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$$\leqslant \mathbb{E}\left(\sup_{x\in K}\frac{V(x)}{Z(x)} - \sup_{x\in K}\frac{\tau}{Z(x)}\right)_+ \leqslant C_K \cdot \underbrace{\mathbb{E}\sup_{x\in K}(V(x)-\tau)_+}_{\int_\tau^\infty \mathbb{P}(\sup_{x\in K} V(x) > u)du}$$

Minimal Gaussian Variance

Idea. Choose spectral rep. $V(x) = exp(W(x) - \sigma^2(x)/2)$ such that

$\sup_{x \in K} W(x) - \sigma^2(x)/2$ becomes as "light tailed" as possible.

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Proposition (MO/KS: Application of [Debicki/Kosinski/Mandjes/Rolski '10])

Let $\{W_i(x), x \in K\}$, $i = 1, 2$ be centered Gaussian processes with a.s. bounded sample paths and variance functions $\sigma_i^2(x) = \mathrm{Var}(W_i(x))$ and

$$\sup_{x \in K} \sigma_1^2(x) < \sup_{x \in K} \sigma_2^2(x) < \infty$$

Then

$\sup_{x \in K} W_1(x) - \sigma_1^2(x)/2$ has lighter tail than $\sup_{x \in K} W_2(x) - \sigma_2^2(x)/2$.

Minimal Gaussian Variance

Problem. Find centred sample-continuous Gaussian process W

- minimizing $\sup_{x \in K} Var(W(x))$ **(I)**
- subject to $\gamma(x-y) = \mathbb{E}(W(x)-W(y))^2$, $x, y \in K$ **(II)**

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[Matheron '74]
Let W_0 be any (reference) process satisfying **(II)**.
Then the solution can be represented as

$$W^{\lambda}(x) = W_0(x) - \int_K W_0(x')\lambda(dx'), \quad x \in K$$

for some probability measure λ on K.

$\Rightarrow$ *"Parametrization by probability measures λ on K."*

Example

- $W_0 = B =$ std. Brownian motion on $K = [-R, R]$ (variogram $\gamma(x) = |x|$)

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- Modified Brownian motion with $\lambda = \frac{1}{2}\delta_{-R} + \frac{1}{2}\delta_R$

$$W^\lambda(x) = W_0(x) - \int_K W_0(x')\lambda(dx') = B(x) - \left(\frac{1}{2}B(-R) + \frac{1}{2}B(R)\right)$$

has the same variogram.

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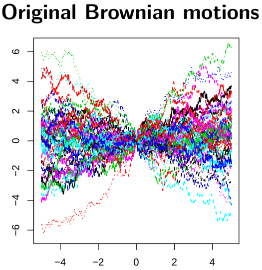

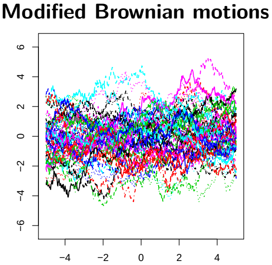

Example

- $W_0 = B =$ std. Brownian motion on $K = [-R, R]$ (variogram $\gamma(x) = |x|$)
- Modified Brownian motion with $\lambda = \frac{1}{2}\delta_{-R} + \frac{1}{2}\delta_R$

$$W^\lambda(x) = W_0(x) - \int_K W_0(x')\lambda(dx') = B(x) - \left(\frac{1}{2}B(-R) + \frac{1}{2}B(R)\right)$$

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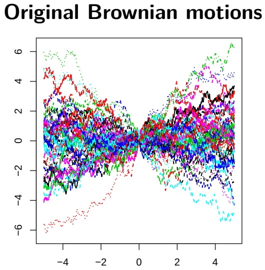
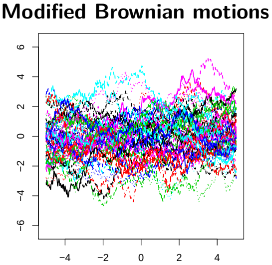




This choice minimizes $\lambda \mapsto \sup_{x\in[-R,R]} Var(W^\lambda(x))$.
It is even locally stationary.

More generally . . .

Proposition (MO/KS: Application of [Matheron '74])

Let

- $\gamma(x) = \psi(\|x\|^2)$ be a convex variogram on $\mathbb{R}^d$ and W_0 a reference process with variogram γ,
- $K \subset \mathbb{R}^d$ compact, such that $S(\mathrm{Ex}(K))$ acts transitively on $\mathrm{Ex}(K)$

Then the modified process

$$W^\lambda(x) = W_0(x) - \int_K W_0(x')\lambda(dx'), \quad x \in K$$

with $\lambda =$ uniform distribution on $\mathrm{Ex}(K)$ minimizes $\lambda \mapsto \sup_{x \in K} Var(W^\lambda(x))$.

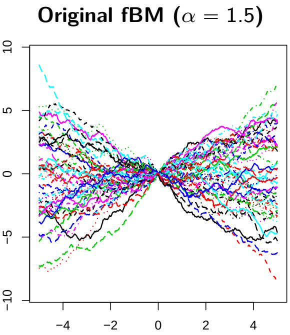
Example. $\gamma(x) = \|x\|^\alpha$, $\alpha \in [1, 2)$ (fractional Brownian sheet) on a hyperrectangle $K = \prod_{i=1}^d [-R_i, R_i]$ (d-dim'l simulation window)

Example

For $\alpha \geqslant 1$ the modified fractional Brownian motion

$$\widetilde{B}_\alpha(x) = B_\alpha(x) - \left(\frac{1}{2}B_\alpha(-R) + \frac{1}{2}B_\alpha(R)\right)$$

minimizes $W \mapsto \sup_{x\in[-R,R]} Var(W(x))$
(among Gaussian processes with variogram $\gamma(x) = \|x\|^\alpha$).



Problem. Find centred sample-continuous Gaussian process W

- minimizing $\sup_{x \in K} Var(W(x))$ **(I)**
- subject to $\gamma(x-y) = \mathbb{E}(W(x) - W(y))^2$, $x, y \in K$ **(II)**

What if the variogram is not convex?

Still subtracting vertices reduces the variance

Proposition (MO/KS)

Let

- $\gamma(x) = \psi(\|x\|^2)$ for a Bernstein function ψ and W_0 the reference process on $\mathbb{R}^d$ with $W_0(o) = 0$,
- $K = \prod_{i=1}^{d}[-R_i, R_i]$ be a hyperrectangle.

Then the process

$$W^\lambda(x) = W_0(x) - \int_K W_0(x')\lambda(dx'), \quad x \in K$$

with $\lambda =$ uniform distribution on the vertices of K reduces $W \mapsto \sup_{x\in K} Var(W(x))$, i.e.,

$$\sup_{x\in K} Var(W(x)) \leqslant \sup_{x\in K} Var(W_0(x)).$$

Remark. Can replace K with any subset containing the vertices of the hyperrectangle.

Still subtracting vertices reduces the variance

Proof. Need to show $\mathrm{Var}(W(x)) \leqslant \sup_{x \in K} Var(W_0(x))$ for all $x \in K$.

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$$\Leftrightarrow \qquad \frac{1}{2^d} \sum_{A \subset \{1,\dots,d\}} \gamma(x - v_A) - \frac{1}{2}\gamma(v_\emptyset - v_A) \quad \leqslant \quad \gamma(v_\emptyset)$$

(label the vertices $(\pm R_1, \pm R_2, \dots, \pm R_d)$ of K

by $A \subset \{1, \dots, d\}$ according to $\pm$)

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$$\Leftarrow \quad \frac{1}{2^d} \sum_{A \subset \{1,\dots,d\}} \psi(3 \sum_{i \in A} R_i^2 + \sum_{j \in A^c} R_j^2) - \frac{1}{2}\psi(4 \sum_{i \in A} R_i^2) \quad \leqslant \quad \psi(\sum_{i=1}^{d} R_i^2)$$

(using 2-alternation of ψ iteratively d times)

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$$\Leftarrow \qquad \psi(3a+b) - \frac{1}{2}\psi(4a) \quad \leqslant \quad \psi(a+b)$$

which is true for Bernstein functions
(uses combination of 2-alternation and 3-alternation). $\square$

More about fractional Brownian sheets ...

Proposition (Combining [Matheron '74] and [Gneiting '00 (Addendum)])

For $\alpha \in (0,2)$ the function

$$C(x-y) = a - \|x-y\|^{\alpha}, \qquad x, y \in B_R(o)$$

is a covariance function if and only if

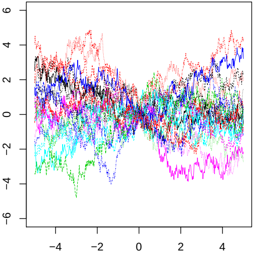
$$a \geqslant \frac{\Gamma\left(\frac{2-\alpha}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} R^{\alpha} =: A_{\alpha,d}(R).$$

(locally stationary rep. on $B_R(o)$ for the variogram $\gamma(x-y) = \|x-y\|^{\alpha}$).

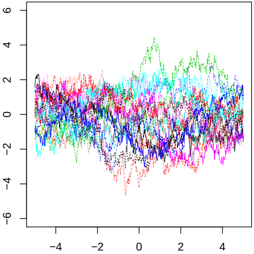
Choosing $a = A_{\alpha,d}(R)$ minimizes $W \mapsto \sup_{x \in B_R(o)} Var(W(x))$ among Gaussian representations for γ if and only if $d = 1$ and $\alpha \leqslant 1$.

Example

Original fBM
($\alpha = 0.7$)

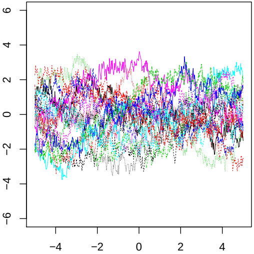


Modified fBM
($\alpha = 0.7$)



Reduced variance.

Locally stationary fBM
($\alpha = 0.7$)



Minimal variance.

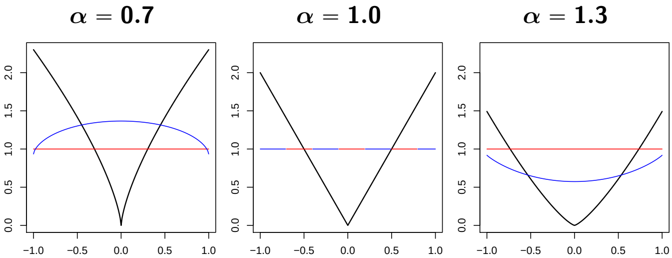
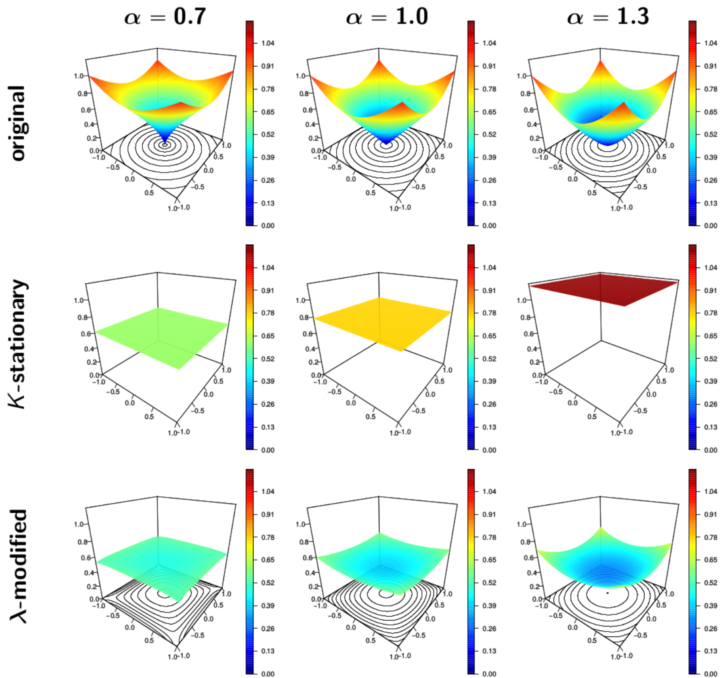


Figure : (above) Variances $\sigma^2(t)$ of the Gaussian representations of the variogram $\gamma(h) = |h/s|^\alpha$ on the domain $K = [-1, 1]$. The plots show the variance for the original representation with $W_0(0) = 0$ (black), the minimal K-stationary representation (red) and the λ-modified representation with $\lambda = \text{Unif}(\text{Ex}(K))$ (blue). For $\alpha = 1$ the last two coincide. The scale $s > 0$ is chosen such that the variance of the minimal K-stationary representation (red) is normalized to 1.

Figure : (next page) Variances $\sigma^2(t)$ of the Gaussian representations of the variogram $\gamma(\boldsymbol{h}) = \|\boldsymbol{h}/\sqrt{2}\|^\alpha$ on the domain $K = [-1, 1]^2$ for $\alpha \in \{0.7, 1.0, 1.3\}$ (left to right). The plots show the variance for the original representation with $W_0(\mathbf{0}) = 0$, the minimal K-stationary representation and the λ-modified representation with $\lambda = \text{Unif}(\text{Ex}(K))$ (top to bottom). Minimality of the K-stationary representation refers to the minimal ball $B_{\sqrt{2}}(\mathbf{0})$ containing K.


α = 0.7
α = 1.0
α = 1.3
original
K-stationary
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Quick wrap up.

- several situations in which we understand
  how to reduce the maximal variance a of Gaussian processes
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How can this be useful for BR-simulation?

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- which reduces either the error or simulation time
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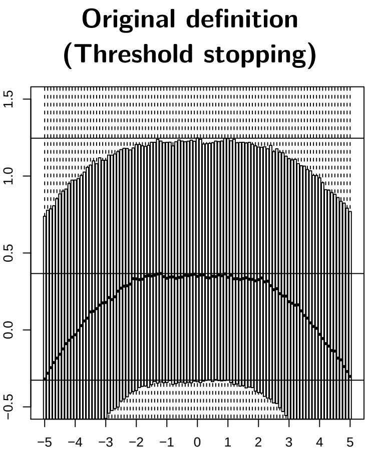
To what extent?

Comparison with existing methods?

What can go wrong? (Typical phenomena)



Pointwise boxplots of 10000 simulations, Gumbel scale, each stopped “too early”.

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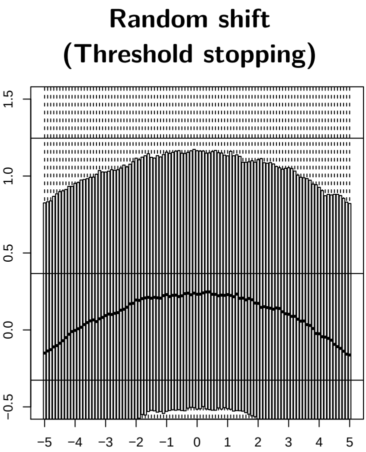

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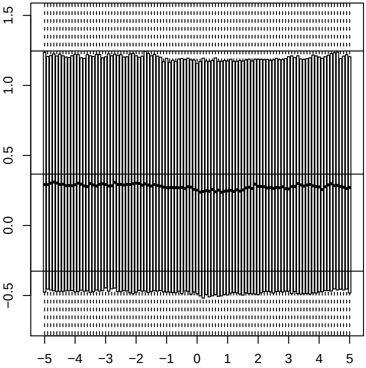
Reduced/Minimal variance (Threshold stopping)



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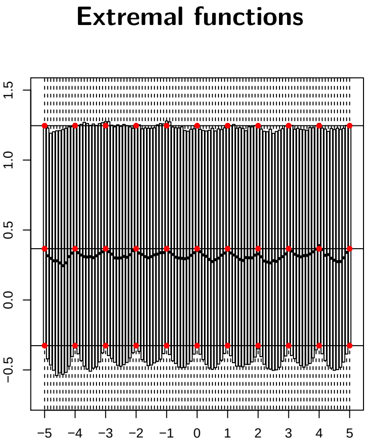

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Numerical experiments

Fair comparison?

Tradeoff

Efficiency

- Time = Expected number of Gaussian processes to be simulated

Accuracy

- Error = Expected number of missing extremal functions

Fix time.

Observe error.

Numerical results (dimension 1)

Table : Benchmark error terms $\widehat{P}_{K,\tau}$ for the simulation of BR processes on the interval $K = [-1, 1]$ (step size 0.004) for the variogram $\gamma(h) = |h/s|^{\alpha}$.

Scenario		Original definition	K-stationary	$\lambda = \mathrm{Unif}(\mathrm{Ex}(K))$ modification	Extremal functions
Scale 1	$\alpha = 0.7$	0.33	**0.07**	0.17	0.77
	$\alpha = 1.0$	0.21	**0.08**	**0.09**	0.55
	$\alpha = 1.3$	0.09	0.32	**0.03**	0.32
Scale 2	$\alpha = 0.7$	0.76	**0.33**	0.55	0.85
	$\alpha = 1.0$	0.51	**0.31**	**0.29**	0.64
	$\alpha = 1.3$	0.26	0.31	**0.13**	0.37
Scale 3	$\alpha = 0.7$	0.97	0.84	0.96	**0.81**
	$\alpha = 1.0$	0.90	0.79	0.81	**0.70**
	$\alpha = 1.3$	0.76	0.72	0.46	**0.42**

Remark. "True" minimizing measure for $\alpha < 1$ of discrete problem available. Even better.

Ongoing: comparison with Dieker-Mikosch and others. DM often extremely good.

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  - B) Usually possible to solve

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  Useful for $d = 1$ and $\alpha \in (0,1)$ or α close to 0.
  C) Remaining cases.

$$\min_{\boldsymbol{\lambda}} \quad \max_{i=1}^{N} \frac{1}{2}(\boldsymbol{\lambda} - \boldsymbol{e}^i)^\mathsf{T}(-\boldsymbol{\Gamma})(\boldsymbol{\lambda} - \boldsymbol{e}^i) \qquad \text{subject to} \qquad \boldsymbol{e}^\mathsf{T}\boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda} \geqslant \boldsymbol{0}.$$

  (Reformulations, augmented problem, dual problem, ...)

Open problem

Let $\Gamma_{ij} = \|x_i - x_j\|^\alpha$, $i, j \in \{1, \ldots, N\}$, $\{x_1, \ldots, x_N\} \subset \mathbb{R}^d$
Consider

$$\lambda = \Gamma^{-1}(1, 1, \ldots, 1)^T.$$

Open problem

Let $\Gamma_{ij} = \|x_i - x_j\|^\alpha$, $i, j \in \{1, \ldots, N\}$, $\{x_1, \ldots, x_N\} \subset \mathbb{R}^d$
Consider

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Conjecture 1.
For $d = 1$ and $\alpha \in (0, 1]$ we have $\lambda \geqslant 0$.

Conjecture 2.
For $d > 2$ there exists $\alpha_{\text{critical}} = \alpha_{\text{critical}}(x_1, \ldots, x_N) \in (0, 1)$ such that $\lambda \geqslant 0$ for $\alpha \leqslant \alpha_{\text{critical}}$.

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- Often worthwile doing: Solve **discrete optimization problem** first.
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- **Ongoing:** Comparison with other normalizations (can perform very well).

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Thank you!

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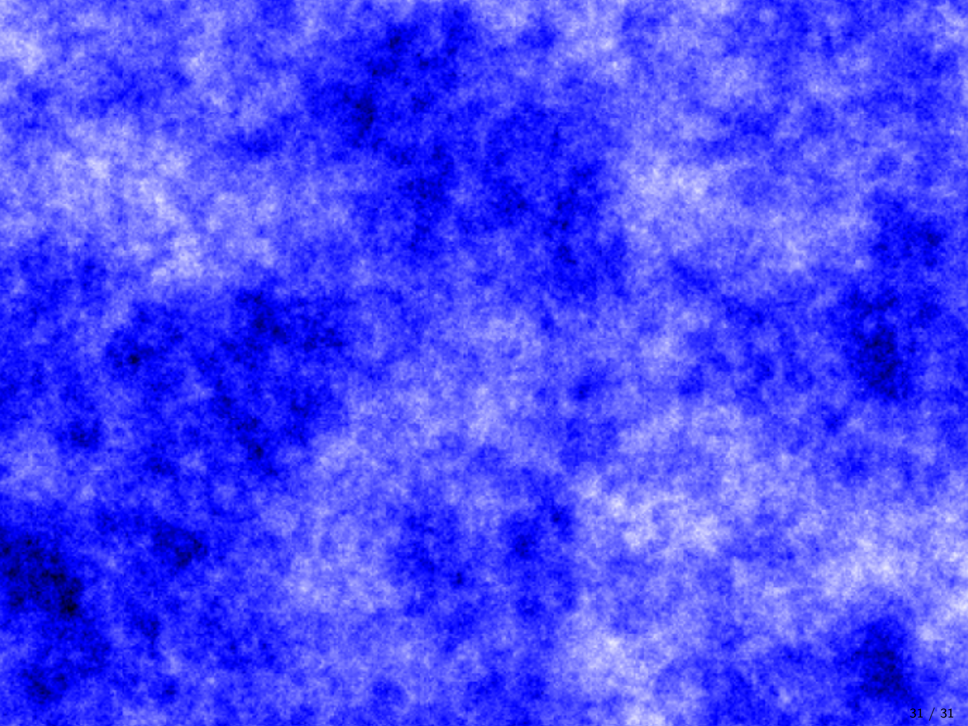
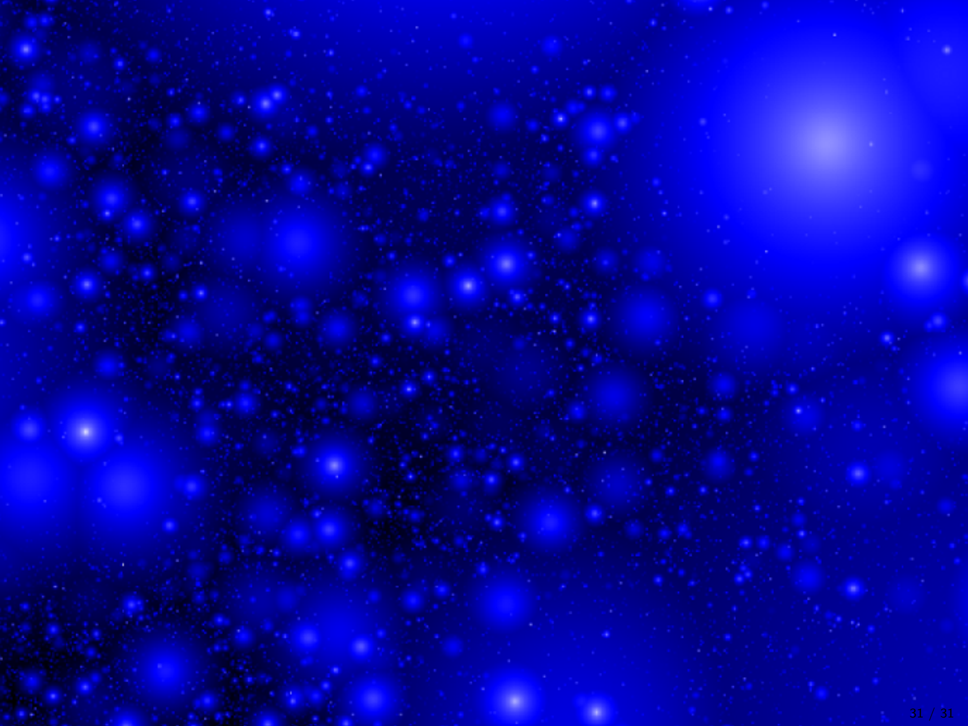




Numerical experiments

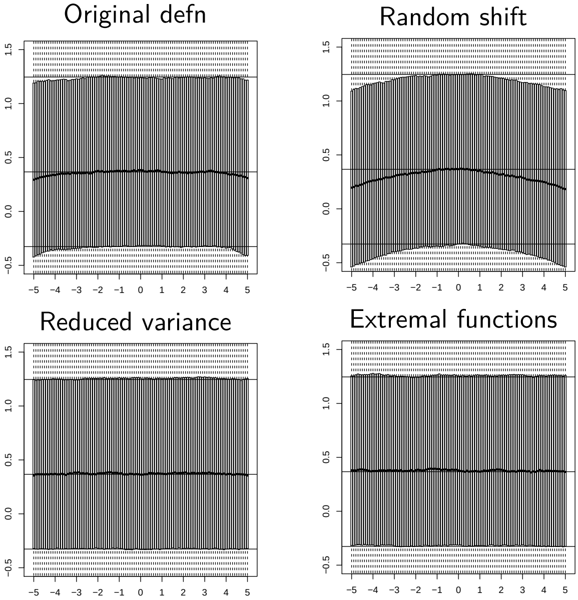
Setting.

- Variogram $\gamma(x) = |x|^\alpha$
- Simulation domain $K = [-5, 5]$
- Step size 0.02
- Threshold for "Reduced variance": $\tau = \exp(2\sqrt{5^\alpha/2})$

Error = Expected # of missing extremal functions
(based on 25 000 simulations)

	Threshold stopping			Extremal
	Original defn.	Random shift	Reduced variance	functions
$\alpha = 0.7$	1.11	1.35	**0.70**	3.03
$\alpha = 1.0$	0.97	1.22	**0.61**	1.24
$\alpha = 1.3$	0.95	1.08	0.63	**0.27**

Boxplots $\alpha = 1.3$



Numerical results (dimension 2)

Table : Benchmark error terms $\widehat{P}_{K,\tau}$ for the simulation of BR processes on the square $K = [-1, 1]^2$ for the variogram $\gamma(\boldsymbol{h}) = (2\sqrt{2}/\pi)\|\boldsymbol{h}\|$.

Scenario	**Original definition**	**$\boldsymbol{K}$-stationary**	**$\boldsymbol{\lambda} = \mathrm{Unif}(\mathrm{Ex}(K))$ modification**	**Extremal functions**
$\sigma^2_{\mathrm{LS}} = 1,\ \alpha = 1.0$	0.07	0.09	**0.01**	0.73