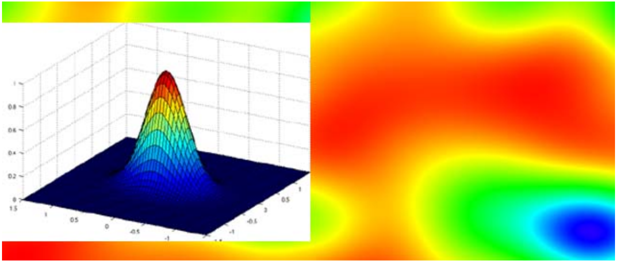

Long range dependence for random functions with infinite variance

Joint work with R. Kulik (U Ottawa)

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Overview

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Introduction: Random functions with long memory

Random function = Set of random variables indexed by $t \in T$.

Let $X = \{X_t, t \in T\}$ be a wide sense stationary random function defined on an abstract probability space $(\Omega, \mathcal{F}, P)$, e.g., $T \subseteq \mathbb{R}^d$, $d \geq 1$. The property of long range dependence (LRD) can be defined as

$$\int_T |C(t)|\, dt = +\infty$$

where $C(t) = \mathrm{cov}(X_0, X_t)$, $t \in T$ (McLeod, Hipel (1978); Parzen (1981)). Sometimes one requires that $C \in RV(-a)$, i.e., $\exists a \in (0, d)$ such that

$$C(t) = \frac{L(t)}{|t|^a}, \quad |t| \to +\infty,$$

where $L(\cdot)$ is a slowly varying function.

Various approaches to define LRD

- Unbounded spectral density at zero.
- Growth order of sums' variance going to infinity.
- Phase transition in certain parameters of the field (stability index, Hurst index, heaviness of the tails, etc.) regarding the different limiting behaviour of some statistics such as
  - Partial sums
  - Partial maxima.

These approaches are not equivalent, often statistically not tractable and tailored for a particular class of random functions (e.g., time series, square integrable, stable, etc.)

Various approaches to define LRD

LRD for heavy tailed random fields:

- Phase transitions in the limiting behaviour of partial sums and maxima of inf. divisible random processes and their ergodic properties (Samorodnitsky 2004, Samorodnitsky & Roy 2008, Roy 2010).
- α-spectral covariance approach for linear random fields with innovations lying in the domain of attraction of α–stable law (Paulauskas (2016), Damarackas, Paulauskas (2017))

LRD: Infinite variance case

For a stationary random function X with $\mathrm{E}\, X_t^2 = +\infty$ introduce

$$\mathrm{cov}_X(t,u,v) = \mathrm{cov}\left(\mathbb{1}(X_0 > u), \mathbb{1}(X_t > v)\right), \quad t \in T,\ x, v \in \mathbb{R}.$$

It is always defined as the indicators involved are bounded functions.
A random function X is called LRD (SRD, resp.) if

$$\int_T \int_{\mathbb{R}^2} |\mathrm{cov}_X(t,u,v)|\, du\, dv\, dt = +\infty \qquad (< +\infty).$$

For discrete parameter random fields (say, if $T \subseteq \mathbb{Z}^d$), the $\int_T dt$ in the above line should be replaced by a $\sum_{t \in T: t \neq 0}$.

Motivation

Assume that X is wide sense stationary with covariance function $C(t) = \text{cov}(X_0, X_t)$, $t \in T$, and moreover,

$$\text{cov}_X(t, u, v) \geq 0 \text{ or } \leq 0 \text{ for all } t \in T,\ u, v \in \mathbb{R}.$$

Examples of X with this property are all **PA** or **NA**- random functions. W. Hoeffding (1940) proved that

$$C(t) = \int_{\mathbb{R}^2} \text{cov}_X(t, u, v)\, du\, dv. \tag{1}$$

Then, X is long range dependent if

$$\int_T |C(t)|\, dt = \int_T \int_{\mathbb{R}^2} |\text{cov}_X(t, u, v)|\, du\, dv\, dt = +\infty.$$

Motivation: Checking LRD

Level (excursion) sets and their volumes:

Let $a_n(u) = \nu_d\left(A_u\left(X, W_n\right)\right)$ be the volume of the excursion set

$$A_u\left(X, W_n\right) = \{t \in T \cap W_n : X_t > u\}$$

of a random field X at level u in an observation window $W_n = n \cdot W$ where $W \subset \mathbb{R}^d$ is a convex body.

Motivation: Checking LRD

Multivariate CLT for level sets' volumes (Bulinski, S., Timmermann, Karcher, 2012):
For a stationary centered weakly dependent random field X satisfying some additional conditions (square integrable, α- or max-stable, inf. divisible) we have for any levels $u, v \in \mathbb{R}$ that

$$\frac{(a_n(u), a_n(v))^\top - (\mathrm{P}(X_0 \geq u), \mathrm{P}(X_0 \geq v))^\top \cdot \nu_d(W_n)}{\sqrt{\nu_d(W_n)}} \xrightarrow{\mathrm{d}} \mathcal{N}(\mathbf{o}, \Sigma)$$

as $n \to \infty$. Here $\Sigma = (\sigma_{ij})_{i,j=1}^2$ with $\sigma_{12} = \int_{\mathbb{R}^d} \mathrm{cov}_X(t, u, v)\, dt$.

So, $a_n(u) = \nu_d(A_u(X, W_n))$ is the right statistic to study!

Motivation: Checking LRD

The new definition is statistically feasible and easy to check. Notice that

$$\int_T \int_{\mathbb{R}^2} |\mathrm{cov}_X(t,u,v)| \, du \, dv \, dt =$$

$$\int_T \int_{\mathbb{R}^2} |F_{X_0,X_t}(u,v) - F_X(u)F_X(v)| \, du \, dv \, dt.$$

where the bivariate d.f. $F_{X_0,X_t}(u,v) = \mathrm{P}(X_0 \leq u, X_t \leq v)$ and marginal d.f. $F_X(u) = \mathrm{P}(X_0 \leq u)$ can be estimated from the data by their empirical counterparts.

Motivation: Checking LRD

For a stationary centered Gaussian random field X with Var $X_0 = 1$ and correlation function $\rho(t)$ we have (Bulinski, S., Timmermann, 2012)

$$\mathrm{cov}_X(t,u,v) = \frac{1}{2\pi}\int_0^{\rho(t)} \frac{1}{\sqrt{1-r^2}} \exp\left\{-\frac{u^2 - 2ruv + v^2}{2\,(1-r^2)}\right\} dr.$$

Ψ–Mixing

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathcal{U}, \mathcal{V})$ be two sub-$\sigma-$algebras of $\mathcal{A}$. Ψ–mixing coefficient:

$$\Psi(\mathcal{U}, \mathcal{V}) = \sup\left\{ \left| 1 - \frac{P(U \cap V)}{P(U)P(V)} \right| ; \; U \in \mathcal{U}, \; P(U) \neq 0, \; V \in \mathcal{V}, \; P(V) \neq 0 \right\}.$$

Let $X = \{X_t, t \in T\}$ be a random function, and T be a normed space with distance d. Let $X_C = \{X_t, t \in C\}$, $C \subset T$, and $\mathcal{X}_\mathcal{C}$ be the $\sigma-$algebra generated by X_C. If $|C|$ is the cardinality of C for C finite set

$$\Psi_X(k, u, v) = \sup\{\Psi(\mathcal{X}_\mathcal{A}, \mathcal{X}_\mathcal{B}) : \quad d(A, B) \geq k, \; |A| \leq u, |B| \leq v\},$$

where $u, v \in \mathbb{N}$ and $d(A, B)$ is the distance between subsets A and B.

Subordinated non–Gaussian random functions: SRD and mixing

Theorem (Kulik, S. 2017)

Let process $Y = \{Y_t, t \in T\}$ *be a stationary process with* $\Psi-$*mixing rate satisfying* $\int_T \Psi_Y(\|t\|, 1, 1)\, dt < +\infty$. *Let* $X_t = g(|Y_t|), t \in T$, *where* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\mathrm{E}\, X_0 < +\infty$. *Then* *X is SRD with*

$$\int_T \int_{\mathbb{R}^2} |cov_X(t, u, v)|\, du\, dv\, dt \leq \int_T \Psi_Y(\|t\|, 1, 1)\, dt \cdot (\mathrm{E}\, X_0)^2 < +\infty.$$

Random volatility functions

Let the random field $X = \{X_t, t \in T\}$ be given by

$$X_t = F(Y_t)Z_t$$

where $Y = \{Y_t, t \in T\}$ and $Z = \{Z_t, t \in T\}$ are independent stationary random fields, Z has property

$$\text{cov}_Z(t, u, v) \geq 0 \text{ or } \leq 0 \text{ for all } t \in T, \ u, v \in \mathbb{R},$$

$F : \mathbb{R} \to \mathbb{R}_\pm$ and $P\big(F(Y_t) = 0\big) = 0$ for all $t \in T$.
$F(Y_t)$ is called a random volatility (being a deterministic function of a random (often LRD) field $Y = \{Y_t, t \in T\}$) scaling a heavy tailed random field $Z = \{Z_t, t \in T\}$.

Random volatility functions

Theorem (Kulik, S. 2017)

A random volatility model $X = \{X_t, t \in T\}$ is LRD if one of the following holds:

(i) *Y is a white noise, $\int_{\mathbb{R}^2} |\mathrm{cov}_Z(t,u,v)|\, du\, dv > 0$ for a set of $t \in T$ with positive Lebesgue measure and either $\mathrm{E}|F(Y_0)| = +\infty$ or $\mathrm{E}|F(Y_0)| \in (0, +\infty)$, Z LRD.*

(ii) $\int_T \int_{\mathbb{R}^2} \mathrm{cov}\left(\bar{F}_Z(u/F(Y_0)), \bar{F}_Z(v/F(Y_t))\right) du\, dv\, dt = \pm\infty.$

(iii) *$\mathrm{E}\left(F(Y_0)F(Y_t)\right) = +\infty$, $\int_{\mathbb{R}^2} |\mathrm{cov}_Z(t,u,v)|\, du\, dv > 0$ for a set of $t \in T$ with positive Lebesgue measure and*

Random volatility functions

Theorem (Continuation)

(iii)

$$\int_T \int_{\mathbb{R}^2} \operatorname{cov}\left(\bar{F}_Z(u/F(Y_0)), \bar{F}_Z(v/F(Y_t))\right) du\, dv\, dt > -\infty (< +\infty).$$

(iv) $\mathrm{E}\,(F(Y_0)F(Y_t)) < +\infty$ *for all* $t \in T$ *and the above holds together with*

$$\int_T \mathrm{E}\,(F(Y_0)F(Y_t)) \int_{\mathbb{R}^2} \operatorname{cov}_Z(t,u,v)\, du\, dv\, dt = \pm\infty.$$

In all above equations, $\pm$ *is taken with the same sign as* $\operatorname{cov}_Z(t,u,v)$.

Random volatility functions

Corollary

For the random field X with $X_t = AZ_t$, $t \in T$ assume that $A > 0$ a.s., A and Z are independent and Z is stationary. Then X is LRD if one of the following holds:

1. *Z is a white noise and* $\int_{\mathbb{R}^2} \operatorname{cov}\left(\bar{F}_Z(u/A), \bar{F}_Z(v/A)\right) \, du \, dv \neq 0$.
2. *$Z \in$ **PA**(**NA**) is not a white noise, Z_0 is symmetric, and* $\mathrm{E}A^2 = +\infty$.

Random volatility functions

Examples

- In Case 1) of the above corollary, it holds

$$\int_{\mathbb{R}^2} \operatorname{cov}\left(\bar{F}_Z(u/A), \bar{F}_Z(v/A)\right) du\, dv = +\infty$$

  if e.g. $Z_0 \sim Exp(\lambda)$, $A \sim$ Frechet(1) for any $\lambda > 0$.
- Case 2) of the above corollary clearly applies to a subgaussian random function X where $A = \sqrt{B}$, $B \sim S_{\alpha/2}\left(\left(\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}, 1, 0\right)$, $\alpha \in (0, 2)$, and Z is a centered stationary Gaussian random field with covariance function $C(t) \geq 0\, (\leq 0)$ for all $t \in T$. Here Z does not need to be LRD but there should exist $t \neq 0$ such that $C(t) \neq 0$.

LT for the volume of excursion sets

Let X be a measurable real-valued random field on $\mathbb{R}^d$, $d \geq 1$ and let $W \subset \mathbb{R}^d$ be a measurable subset. Let

$$A_u(X, W) := \{t \in W : X(t) \geq u\}$$

be the excursion set of X in W over the level $u \in \mathbb{R}$.

Asymptotic (non)Gaussian behavior of $\nu_d(A_u(X, W))$ as W expands to $\mathbb{R}^d$?

Prove a more general limit theorem for integrals $\int_W g(X_t)\, dt$ of functionals g of X!

LT for the volume of excursion sets

Let X be a random volatility field of the form

$$X_t = G(Y_t)Z_t, \quad t \in T = \mathbb{R}^d,$$

where

- $\{G(Y_t), t \in T\}$ is a subordinated Gaussian random field,
- $\{Z_t, t \in T\}$ is a white noise,
- the random fields Y and Z are independent.

Let $W_n = n \cdot W$, $W \in \mathcal{K}^d$, $\nu_d(W) > 0$, $o \in W$, and g be a real valued function such that $\mathrm{E}[g(X_0)] = 0, \quad \mathrm{E}[g^2(X_0)] > 0$.
Introduce the function

$$\xi(y) = \mathrm{E}[g(G(y)Z_0)]\,.$$

It follows that $\xi(y) < \infty$ for ν_1–a. e. $y \in \mathbb{R}$, $\mathrm{E}[\xi(Y_0)] = 0$.

LT for the volume of excursion sets

Furthermore, set

$$m(y, Z_t) = g(G(y)Z_t) - \xi(y)\,, \quad \chi(y) = \mathsf{E}[m^2(y, Z_0)]\,.$$

Assume that

- $\mathrm{rank}\ (\xi) = q$, $\mathsf{E}[|g(X_0)|^2] < \infty$, $\mathsf{E}[\chi^3(Y_0)] < \infty$.
- Y is a homogeneous isotropic centered Gaussian random field with the covariance function $\rho(t) = \mathsf{E}[Y_0 Y_t] = |t|^{-\eta} L(|t|)$, $\eta \in (0, d/q)$ and L is slowly varying at infinity,
- Y has a spectral density $f(\lambda)$ which is continuous for all $\lambda \neq 0$ and decreasing in a neighborhood of 0.

LT for the volume of excursion sets

Theorem (Kulik, S. 2017)

1. *If* $\xi(y) \equiv 0$ *then*

$$n^{-d/2} \int_{W_n} g(X_t)\, dt \xrightarrow{d} \mathcal{N}(0, \sigma^2)\ , \quad n \to +\infty, \qquad (2)$$

where $\sigma^2 = \mathrm{E}[g^2(X_0)]\nu_d(W) > 0$.

2. *If* $\xi(y) \not\equiv 0$ *then*

$$n^{q\eta/2-d} L^{-q/2}(n) \int_{W_n} g(X_t)\, dt \xrightarrow{d} R\ , \quad n \to +\infty, \qquad (3)$$

where the random variable R *is a q-Rosenblatt-type random variable.*

LT for the volume of excursion sets

q–Rosenblatt-type random variable:

$$R = (\gamma(d,\eta))^{q/2} \int'_{\mathbb{R}^{dq}} \int_W e^{i\langle \lambda_1 + \ldots + \lambda_q, u\rangle} du \frac{B(d\lambda_1)\ldots B(d\lambda_q)}{(|\lambda_1| \cdot \ldots \cdot |\lambda_q|)^{(d-\eta)/2}},$$

$$\gamma(d,\eta) = \frac{\Gamma\left((d-\eta)/2\right)}{2^{\eta}\pi^{d/2}\Gamma(\eta/2)},$$

and $\int'_{\mathbb{R}^{dq}}$ is the multiple Wiener–Ito integral with respect to a complex Gaussian white noise measure B (with structural measure being the spectral measure of Y).

LT for the volume of excursion sets

Example

Assume that

$$g(y) = \mathbb{1}\{y > u\} - P\left(G(Y_0)Z_0 > u\right)$$

where G is nonnegative or nonpositive ν_1–a.e. Then

$$\xi(y) = \mathrm{E}[\mathbb{1}\{G(y)Z_0 > u\}] - P\left(G(Y_0)Z_0 > u\right).$$

- If $u = 0$ then $\xi(y) \equiv 0$, so the Gaussian case applies.
- If $u \neq 0$ then $\xi(y) \not\equiv 0$, so the non-Gaussian case applies. Let $uG(y) \geq 0$ for all y.
  $q = 1$: $G : \mathbb{R} \to \mathbb{R}_{\pm}$ is monotone right-continuous non–constant fct. with $\nu_1\left(\{x \in \mathbb{R} : G(x) = 0\}\right) = 0$.
  $q = 2$: $G(y) = G_1(|y|)$ with G_1 as above.

LT for the volume of excursion sets

Example

Let the random volatility field $X_t = G(|Y_t|)Z_t$, $t \in \mathbb{R}^d$ be s.t.

- Y is a centered Gaussian random field with unit variance and corr. function $\rho(t) \geq 0$ as above, $\rho(t) \sim |t|^{-\eta}$ as $|t| \to +\infty$
- $G(x) \geq 0$ is continuous as above with $\mathrm{E}\,|G(Y_0)|^{1+\theta} < \infty$ for some $\theta \in (0, 1)$.
- $\{Z_t\}$ is a heavy–tailed white noise, F_Z is continuous, $\mathrm{E}|Z_0| < +\infty$, $\mathrm{E}Z_0 \neq 0$, $\mathrm{E}Z_0^2 = +\infty$.

LT for the volume of excursion sets

For $\widetilde{G}(y) = G(|y|)$, It holds

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^2} \mathrm{cov}_X(t,u,v)\,du\,dv\,dt = (\mathrm{E}Z_0)^2 \sum_{k=1}^{\infty} \frac{\langle \widetilde{G}, H_k\rangle_{\varphi}^2}{k!} \int_{\mathbb{R}^d} \rho^k(t)\,dt.$$

- Since $\mathrm{rank}\,(\widetilde{G}) = 2$, X is l.r.d. if $\int_{\mathbb{R}^d} \rho^2(t)\,dt = +\infty$, that is, if $\eta \in (0, d/2)$.
- For niveau $u \neq 0$, the asymptotic behavior of the volume of the level sets $A_u(X, W_n)$ is of 2-Rosenblatt-type ($\mathrm{rank}\,(\xi) = q = 2$) if $\eta \in (0, d/2)$.

LT for the volume of excursion sets

Summary:

The correct statistics associated with the new definition of l.r.d. is the volume of excursion sets!!!!

Open problems

- Checking the new LRD definition for other classes of processes and fields with infinite variance
- Connection of LRD with LT for the volume of excursions of other stationary random fields

Literature: Historical and general expositions

[1] J. Beran, Y. Feng, S. Ghosh, R. Kulik, *Long Memory Processes: Probabilistic Properties and Statistical Methods*. Springer, 2012.

[2] H. E. Hurst, *Long term storage capacity of reservoirs*. Trans. Amer. Soc. Civil Engrs., 116, pp. 770-779, 1951.

[3] B.B. Mandelbrot, *Une classe de processus stochastiques homothétiques à soi; application à la loi climatologique de H.E. Hurst*. Comptes Rendus Acad. Sci. Paris, 260, pp. 3274-3277, 1965.

[4] D.B. Percival, *The statistics of long memory processes*. PhD Thesis, University of Washington, U.S.A, 1983.

[5] C.C. Heyde , Y. Yang, *On defining long range dependence*. Journal of Applied Probability, 34 (4), pp. 939 - 944, 1997.

[6] W. Hoeffding, *Scale-invariant correlation theory*. In N. I. Fisher and P. K. Sen (Eds.), The Collected Works of Wassily Hoeffding, New York: Springer-Verlag, pp. 57-107, 1940.

[7] F. Lavancier, *Long memory random fields*. In: Dependence in Probability and Statistics, Lecture Notes in Statist. 187, Springer, pp. 195–220, 2006.

[8] G. Samorodnitsky, *Stochastic processes and long range dependence*. Springer, Cham, 2016.

Literature: Limit theorems

[1] R. L. Dobrushin and P. Major, *Non-central limit theorems for nonlinear functionals of Gaussian fields*. Z. Wahrsch. Verw. Gebiete 50 (1979), no. 1, 27-52.

[2] J. Damarackas and V. Paulauskas, *Spectral covariance and limit theorems for random fields with infinite variance*. J. Multivariate Anal. 153 (2017), 156-175.

[3] L. Giraitis, H. L. Koul, and D. Surgailis, *Large sample inference for long memory processes*. Imperial College Press, London, 2012.

[4] L. Giraitis and D. Surgailis, *CLT and other limit theorems for functionals of Gaussian processes*. Z. Wahrsch. Verw. Gebiete 70 (1985), no. 2, 191-212.

[5] S. N. Lahiri and Peter M. Robinson, *Central limit theorems for long range dependent spatial linear processes*. Bernoulli 22 (2016), no. 1, 345-375.

[6] N. Leonenko and A. Olenko, *Sojourn measures of Student and Fisher- Snedecor random fields*. Bernoulli 20 (2014), no. 3, 1454-1483.

[7] V. Paulauskas, *Some remarks on definitions of memory for stationary random processes and fields*. Lith. Math. J. 56 (2016), no. 2, 229-250.

[8] V. Pilipauskaite and D. Surgailis, *Scaling transition for nonlinear random fields with long-range dependence*. Stochastic Process. Appl. 127 (2017), no. 8, 2751-2779.

[9] R. Kulik, E. Spodarev, *Long range dependence of heavily tailed random functions*. Preprint, arXiv:1706.00742v1, 2017.