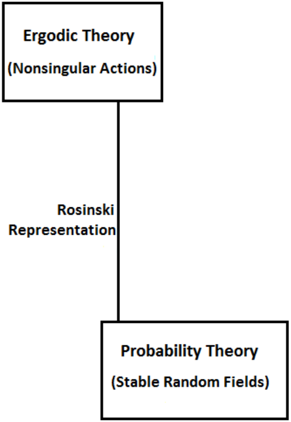
Group measure space construction, ergodicity and stable random fields

Parthanil Roy, Indian Statistical Institute

Ongoing work

June 22, 2018

What is this work about?



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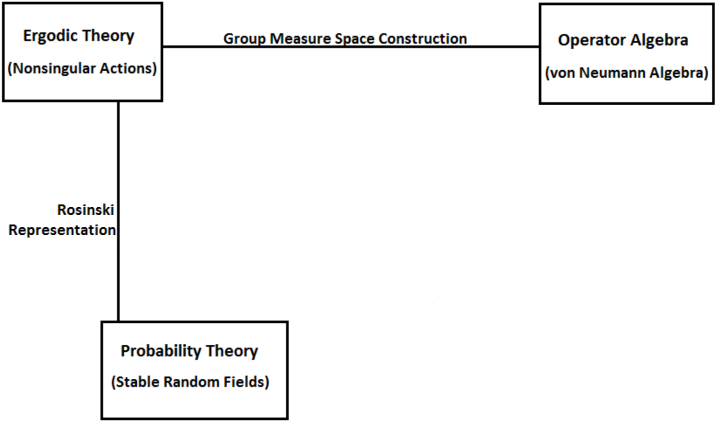

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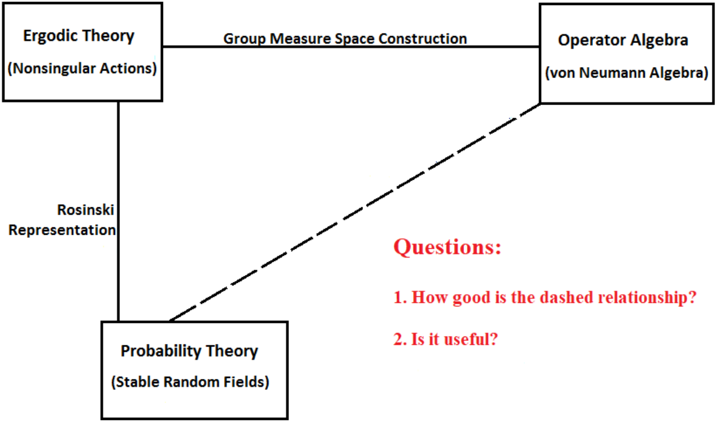

Questions:

1. How good is the dashed relationship?
2. Is it useful?

A Crash Course on Stable Random Fields

Symmetric α-stable distribution

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X follows $S\alpha S$ distribution $(0 < \alpha \leq 2)$ with scale parameter $\sigma > 0$ $\big($denoted by $X \sim S\alpha S(\sigma)$ $\big)$ if

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- **Assume:** $0 < \alpha < 2 \;\Rightarrow\; P(|X| > x) \sim c\, x^{-\alpha}$ as $x \to \infty$.
- In particular, $E(|X|^p) < \infty$ if and only if $p < \alpha$.

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Three most important cases: $G = \mathbb{Z}$, $G = \mathbb{Z}^d$ $(d > 1)$, $G = F_d$.

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- $\mu \circ \phi_t \sim \mu$ for all $t \in G$.

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(1) is a fancy way of saying that each $\sum_{i=1}^k c_i X_{t_i} \sim S\alpha S\big(\| \sum_{i=1}^k c_i f_{t_i} \|_\alpha\big)$.

A Crash Course on von Neumann Algebras

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(Here "<" means strictly weaker topology.)

Bicommutant theorem of von Neumann

Theorem (von Neumann)

Suppose M is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1, the identity operator. Then the following are equivalent:

1. *M is closed in weak operator topology.*
2. *M is closed in strong operator topology.*
3. $M = (M')' =: M''$.

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Definition

A unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ satisfying one (and hence all) of the above equivalent conditions is called a von Neumann algebra.

The central decomposition

Note that if M is a von Neumann algebra, then so is M'. We now define a very important class (building blocks) of von Neumann algebras.

Definition

A von Neumann algebra M is called a factor if $Z(M) := M \cap M' := \{T \in M : TA = AT \text{ for all } A \in M\} = \mathbb{C}1$ (i.e., the centre is trivial).

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Any von Neumann algebra can be decomposed as a direct sum (or more generally, "direct integral") of factors: there exists a measure space $(Y, \mathcal{Y}, \rho)$ such that

$$M = \int_Y M_y \, \rho(dy) \quad \text{(direct integral; see Knudby (2011))},$$

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Enough (for a von Neumann algebraist) to study and classify factors.

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- This connection is still a cutting edge research area (because of eminent mathematicians like Ioana, Popa, Vaes, etc. + their students and post-docs).
- Our work simply encashes this interplay and produces results for stationary SαS random fields.

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If Y is countable with ρ being the counting measure, then the direct integral becomes a direct sum ($M = \oplus_{y \in Y} M_y$) of factors. In this special case, the above definition is equivalent to saying no M_y is a type II_1 factor.

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In particular, $\mathcal{B}(\mathbb{C}^n) = \mathcal{M}_n(\mathbb{C})$ admits no II_1 factor in its central decomposition.

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"Group measure space construction"

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Following/extending the work of Murray and von Neumann (1936) (in the measure-preserving case), one can construct a von Neumann algebra (as a subalgebra of $\mathcal{B}(\ell^2_{\mathbb{C}}(G) \otimes \mathcal{L}^2_{\mathbb{C}}(S, \mu))$) that "encodes the ergodic theoretic features" of $\{\phi_t\}_{t \in G}$ by internalizing a crossed product relation that normalizes $\mathcal{L}^\infty_{\mathbb{C}}(S, \mu)$ inside $\mathcal{B}(\mathcal{L}^2_{\mathbb{C}}(S, \mu))$ through the Koopman representation. This von Neumann algebra is called group measure space construction.

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Notation: $\mathcal{L}^\infty_{\mathbb{C}}(S, \mu) \rtimes_{\{\phi_t\}} G$ or simply $\mathcal{L}^\infty_{\mathbb{C}}(S, \mu) \rtimes G$.

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It can be shown (nontrivial): $\mathcal{L}^{\infty}_{\mathbb{C}}(S, \mu) \rtimes G$ is a type II_1 factor.

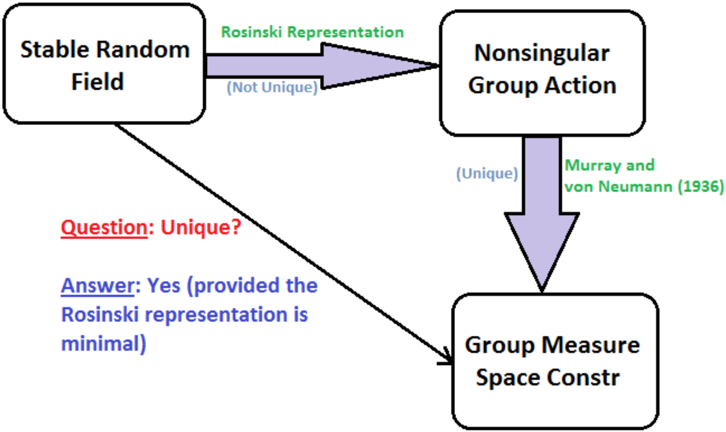
Theorem

Suppose $\{\phi_t\}_{t \in G}$ is a nonsingular action of a countable group G on a σ-finite standard measure space $(S, \mathcal{S}, \mu)$. Then the following hold:

1. *If the action $\{\phi_t\}_{t \in G}$ is free and ergodic, then $\mathcal{L}^{\infty}_{\mathbb{C}}(S, \mu) \rtimes G$ is a factor.*
2. *Conversely, if $\mathcal{L}^{\infty}_{\mathbb{C}}(S, \mu) \rtimes G$ is a factor, then the $\{\phi_t\}_{t \in G}$ is ergodic but not necessarily free.*

Main Results

How good is the connection?



The minimal group measure space construction

Theorem (R. (2018?))

Suppose $\{X_t\}_{t \in G}$ *is a (left) stationary* $S\alpha S$ *random field indexed by a countable group* G. *Let* $\{\phi_t^{(1)}\}_{t \in G}$ *and* $\{\phi_t^{(2)}\}_{t \in G}$ *be two nonsingular* G*-actions (on* $(S^{(1)}, \mu^{(1)})$ *and* $(S^{(2)}, \mu^{(2)})$*, respectively) obtained from two minimal (and hence Rosinski) representations. Then*

$$\mathcal{L}^{\infty}_{\mathbb{C}}(S^{(1)}, \mu^{(1)}) \rtimes G \cong \mathcal{L}^{\infty}_{\mathbb{C}}(S^{(2)}, \mu^{(2)}) \rtimes G$$

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$\Rightarrow$ they are "orbit equivalent" $\Rightarrow$ $\mathcal{L}^{\infty}_{\mathbb{C}}(S^{(1)}, \mu^{(1)}) \rtimes G \cong \mathcal{L}^{\infty}_{\mathbb{C}}(S^{(2)}, \mu^{(2)}) \rtimes G$. □

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- From now on $G = \mathbb{Z}^d$ (unless mentioned otherwise).

Ergodicity of $\mathbb{Z}^d$-indexed stable fields

Recall that any left-stationary SαS random field $\mathbf{X} = \{X_t\}_{t \in \mathbb{Z}^d}$ induces a measure-preserving left-shift action (of $\mathbb{Z}^d$) on $\left(\mathbb{R}^{\mathbb{Z}^d}, \mathbb{P}_{\mathbf{X}}\right)$, where

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- This work: Characterization using group measure space construction.

von Neumann algebraic characterization of ergodicty

Theorem (R. (2018?))

Suppose $\{X_t\}_{t\in\mathbb{Z}^d}$ is a stationary $S\alpha S$ random field generated by a free nonsingular action $\{\phi_t\}_{t\in\mathbb{Z}^d}$. Then $\{X_t\}_{t\in\mathbb{Z}^d}$ is ergodic if and only if the corresponding group measure space construction admits no II_1 factor in its central decomposition.

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Ergodicity of a stationary $S\alpha S$ random fields is preserved under "orbit equivalence" of the underlying free nonsingular $\mathbb{Z}^d$-actions.

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  - "free" can be replaced by "ergodically free" everywhere;
  - if the action is positive (talk of Olivier Durieu), then (almost) all the factors will be of type II_1;
  - same characterization of ergodicity holds for max-stable fields.

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- We have also calibrated the increments of SSSI SαS processes introduced by Cohen and Samorodnitsky (2006) (known to be ergodic) wrt our results - all the factors in the central decomposition is of type III. What about the ones obtained as limit by Dombry and Guillotin-Plantard (2009) and Owada and Samorodnitsky (2015)?

Thank You Very Much