Asymptotics of bivariate local Whittle estimators with some applications

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Bivariate long/short memory

A bivariate stationary time series $\{X_n\}_{n\in\mathbb{Z}} = \{(X_{1,n}, X_{2,n})'\}_{n\in\mathbb{Z}}$ is **long memory** if its spectral density matrix satisfies:

$$f(\lambda) = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix} \sim \begin{pmatrix} \omega_{11}\lambda^{-2d_1} & \omega_{12}e^{-i\phi}\lambda^{-(d_1+d_2)} \\ \omega_{12}e^{i\phi}\lambda^{-(d_1+d_2)} & \omega_{22}\lambda^{-2d_2} \end{pmatrix}, \ \lambda \to 0^+,$$

where $d_1, d_2 \in (0, 1/2)$, $\omega_{11}, \omega_{22} > 0$, $\omega_{12} \in \mathbb{R}$ and $\phi \in (-\pi/2, \pi/2)$, or in matrix notation,

$$f(\lambda) \sim \Phi_{D,\phi}(\lambda)^{-1}\Omega\overline{\Phi}_{D,\phi}(\lambda)^{-1}, \ \lambda \to 0^+,$$

where $\Phi_{D,\phi}(\lambda) = \mathrm{diag}(\lambda^{d_1}, \lambda^{d_2}e^{-i\phi})$, $D = \mathrm{diag}(d_1, d_2)$ and $\Omega = (\omega_{jk})$ is a real-valued, symmetric, positive semi-definite matrix. It is **short memory** when $d_1 = d_2 = 0$, in which case $\phi = 0$.

Bivariate long/short memory: special cases

Two special cases in the definition of bivariate long memory

$$f(\lambda) \sim \begin{pmatrix} \omega_{11}\lambda^{-2d_1} & \omega_{12}e^{-i\phi}\lambda^{-(d_1+d_2)} \\ \omega_{12}e^{i\phi}\lambda^{-(d_1+d_2)} & \omega_{22}\lambda^{-2d_2} \end{pmatrix} = \Phi_{D,\phi}(\lambda)^{-1}\Omega\overline{\Phi}_{D,\phi}(\lambda)^{-1}.$$

Fractal non-connectivity: $\omega_{12} = 0$. (Connectivity: $\omega_{12} \neq 0$.)
Fractional cointegration: $|\Omega| = \omega_{11}\omega_{22} - \omega_{12}^2 = 0$ and $d_1 = d_2$, $\phi = 0$. (Non-cointegration: $|\Omega| \neq 0$). Fractional (non-)cointegration is tested within the framework

$$Bf(\lambda)B' \sim \Phi_{D,\phi}(\lambda)^{-1}\Omega\overline{\Phi}_{D,\phi}(\lambda)^{-1},\ \lambda \to 0^+, \quad (d_1 < d_2)$$

with

$$B = \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix},$$

the case $\beta = 0$ corresponding to non-cointegration and the case $\beta \neq 0$ associated with cointegration. (Note that $f(\lambda) \sim \lambda^{-2d_2}[\beta^2\ \beta; \beta\ 1]$.)

Local Whittle estimation

In the **non-cointegrated case**, the local Whittle estimators of D, ϕ and Ω are defined as

$$(\widehat{D}, \widehat{\phi}, \widehat{\Omega}) = \underset{(D,\phi,\Omega)}{\operatorname{argmin}} \; Q(D, \phi, \Omega)$$

with

$$Q(D,\phi,\Omega) = \frac{1}{m}\sum_{j=1}^{m} \log \Big| \underbrace{\Phi_{D,\phi}(\lambda_j)^{-1}\Omega\overline{\Phi}_{D,\phi}(\lambda_j)^{-1}}_{\approx f(\lambda_s)} \Big| + \mathrm{tr}\Big(I(\lambda_j) \underbrace{\overline{\Phi}_{D,\phi}(\lambda_j)\Omega^{-1}\Phi_{D,\phi}(\lambda_j)}_{\approx f(\lambda_s)^{-1}} \Big),$$

where $\lambda_j = (2\pi j)/N$ are the Fourier frequencies for a sample size N, $I(\lambda) = \frac{1}{N}(\sum_{n=1}^{N} X_n e^{-in\lambda})(\sum_{n=1}^{N} X_n e^{in\lambda})'$ is the periodogram and m is the number of frequencies used in estimation. The optimization problem has been reduced explicitly to that over D, ϕ only.

In the **cointegrated case**, as above, but $I(\lambda_j)$ is replaced by $BI(\lambda_j)B'$ and β is added as another parameter.

Local Whittle estimation: previous work

Asymptotic normality: The asymptotic normality result for $\widehat{D}, \widehat{\phi}$ is provided in Robinson (2008) under suitable assumptions, in particular, on $m = m(N) \to \infty$. This is carried out in both fractionally non-cointegrated and cointegrated cases. Related work includes M.O. Nielsen (2007), M.O. Nielsen and Shimotsu (2007), Shimotsu (2007, 2012), F.S. Nielsen (2011).

Fractal connectivity: Wavelet-based and other testing procedures for fractal connectivity were considered in Achard, Bassett, Meyer-Lindenberg and Bullmore (2008), Wendt, Scherrer, Abry and Achard (2009), Kristoufek (2013), Wendt, Didier, Combrexelle and Abry (2017). Though the approach is slightly different.

Data applications: (log) spot exchange rates, realized volatilities of stocks in Finance, MEG data in Neuroscience, packet and byte counts in Internet Traffic studies.

Local Whittle estimation: our contributions

- Asymptotic normality for all model parameters $\omega_{11}, \omega_{22}, \omega_{12}, \phi, d_1, d_2$ (, β) in Parametrization P, and $\omega_{11}, \omega_{22}, r_1, r_2, d_1, d_2$ (, β) in Parametrization C, where $r_1 + ir_2 = \omega_{12} e^{-i\phi}$. The asymptotic covariance matrices in explicit form!
- Reduced optimization to that over D only.
- Resulting tests for fractal non-connectivity.
- Local Whittle plots for fractal (non-)connectivity, phase parameter.
- Local Whittle plots to consider for real data with illustrations.
- Corrected the asymptotic covariance matrix of Robinson (2008).
- Corrected the asymptotic normalization in the univariate case going back to Robinson (1995).

Glance at our contributions: asymptotic normality

E.g. Suppose that the assumptions ... hold. Then, as $N \to \infty$,

$$\sqrt{m}\begin{pmatrix} \frac{1}{\log(N/m)}(\widehat{\omega}_{11}-\omega_{11}) \\ \frac{1}{\log(N/m)}(\widehat{\omega}_{22}-\omega_{22}) \\ \frac{1}{\log(N/m)}(\widehat{\omega}_{12}-\omega_{12}) \\ \widehat{\phi}-\phi \\ \widehat{d}_1-d_1 \\ \widehat{d}_2-d_2 \end{pmatrix} \xrightarrow{d} \mathcal{N}(0,\Gamma_p),$$

where

$$\Gamma_p = \begin{pmatrix} \frac{\omega_{11}(\omega_{11}\omega_{22}+|\Omega|)}{2\omega_{22}} & \frac{\omega_{12}^2}{2} & \frac{\omega_{11}\omega_{12}}{2} & 0 & -\frac{\omega_{11}\omega_{22}+|\Omega|}{4\omega_{22}} & -\frac{\omega_{12}^2}{4\omega_{22}} \\ & \frac{\omega_{22}(\omega_{11}\omega_{22}+|\Omega|)}{2\omega_{11}} & \frac{\omega_{12}\omega_{22}}{2} & 0 & -\frac{\omega_{12}^2}{4\omega_{11}} & -\frac{\omega_{11}\omega_{22}+|\Omega|}{4\omega_{11}} \\ & & \frac{\omega_{12}^2}{2} & 0 & -\frac{\omega_{12}}{4} & -\frac{\omega_{12}}{4} \\ & & & \frac{|\Omega|}{2\omega_{12}^2} & 0 & 0 \\ & & & & \frac{\omega_{11}\omega_{22}+|\Omega|}{8\omega_{11}\omega_{22}} & \frac{\omega_{12}^2}{8\omega_{11}\omega_{22}} \\ & & & & & \frac{\omega_{11}\omega_{22}+|\Omega|}{8\omega_{11}\omega_{22}} \end{pmatrix},$$

the entries below the main diagonal are omitted but make Γ_p symmetric, and $|\Omega| = \omega_{11}\omega_{22} - \omega_{12}^2$.

Fractal (non-)connectivity tests

In connection to fractal (non-)connectivity (and fractional cointegration), consider

$$\rho^2 = \frac{\omega_{12}^2}{\omega_{11}\omega_{22}} = \frac{r_1^2 + r_2^2}{\omega_{11}\omega_{22}}, \quad \widehat{\rho}^2 = \frac{\widehat{\omega}_{12}^2}{\widehat{\omega}_{11}\widehat{\omega}_{22}} = \frac{\widehat{r}_1^2 + \widehat{r}_2^2}{\widehat{\omega}_{11}\widehat{\omega}_{22}},$$

both taking values in $[0, 1]$. Under H_0: $r_1 = r_2 = 0$ (that is, fractal non-connectivity), the asymptotic normality results yield

$$m\widehat{\rho}^2 \xrightarrow{d} \frac{\chi^2(2)}{2}$$

and under the alternative (that is, fractal connectivity),

$$\sqrt{m}\left(\widehat{\rho}^2 - \rho^2\right) \xrightarrow{d} \mathcal{N}(0, \sigma_\rho^2),$$

where $\sigma_\rho^2 = \frac{2\omega_{12}^2|\Omega|^2}{\omega_{11}^3\omega_{22}^3}$. Similar statistics are constructed in the case of fractional cointegration.

Local Whittle plots

We sugget to examine the followings 9 local Whittle plots. The first 4 plots concern the fractionally non-cointegrated case and are the local Whittle plots of:

- $\widehat{d}_1$ and $\widehat{d}_2$;
- $\widehat{\phi}$ (modified);
- $\widehat{r}_1$ and $\widehat{r}_2$;
- $\widehat{\rho}^2$.

The other 5 plots concern the fractionally cointegrated case and are the local Whittle plots of:

- $\widehat{\beta}$;
- $\widehat{d}_1$ and $\widehat{d}_2$;
- $\widehat{\phi}$ (modified);
- $\widehat{r}_1$ and $\widehat{r}_2$;
- $\widehat{\rho}^2_{fc}$.

Illustration 1: SP500 and FTSE realized volatilities

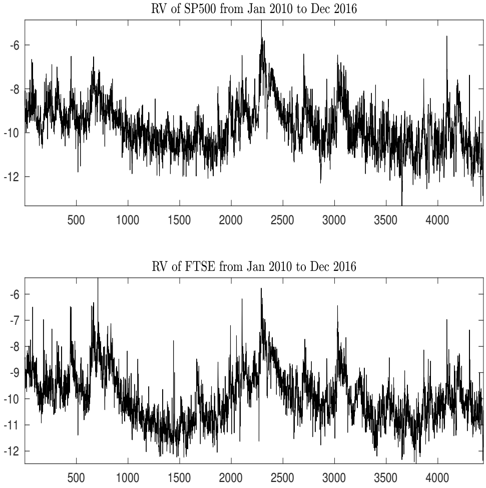
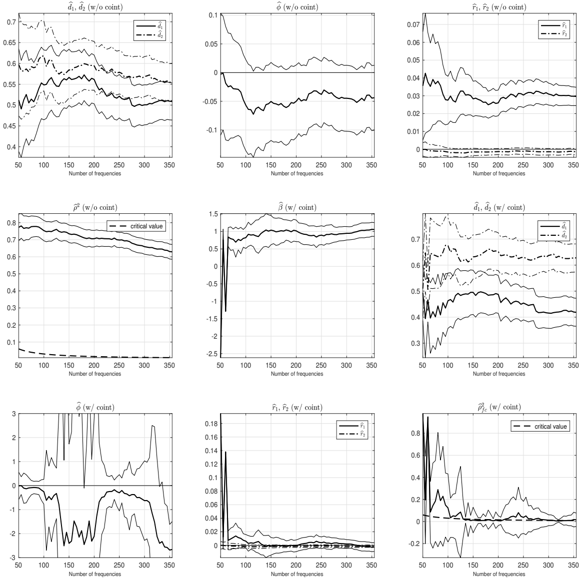
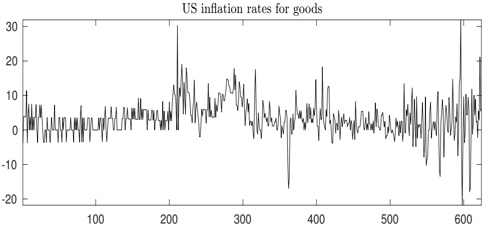


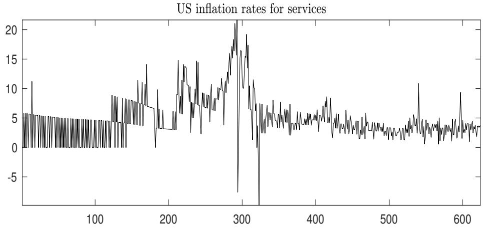
Illustration 1: SP500 and FTSE realized volatilities



Conclusion: Cointegrated but non-connected model.

Illustration 2: US inflation rates for goods and services





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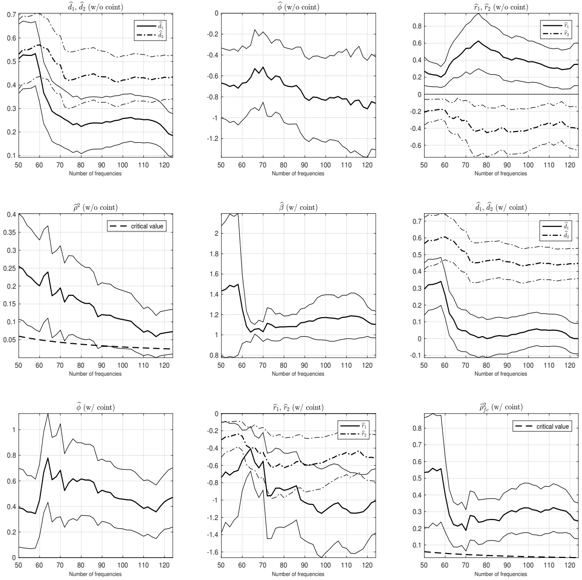
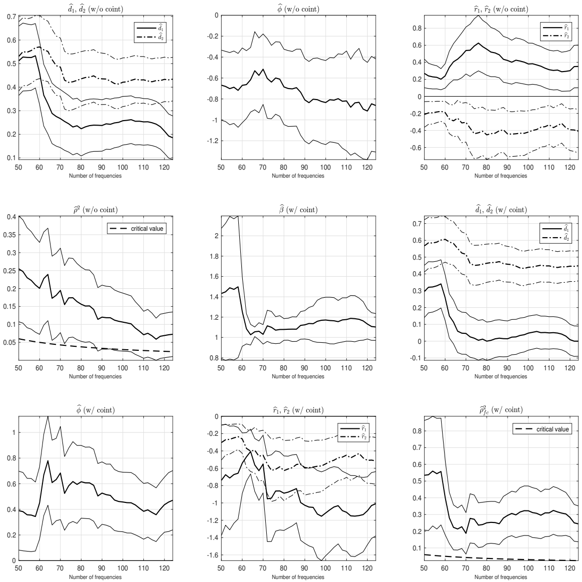


Conclusion:

Illustration 2: US inflation rates for goods and services



Conclusion: Connected either non-cointegrated or cointegrated model.

Final thoughts

- "Annoying" separate treatment of the cointegrated and non-cointegrated cases.
- Going to higher dimension (possibly with penalization) than 2.
- Extending to non-stationary case allowing for $d_1, d_2 \geq 1/2$.
- Based on "Asymptotics of bivariate local Whittle estimators with applications to fractal connectivity", C. Baek, S. Kechagias and V. Pipiras, Preprint, 2018. Available online.
- Questions?

Some other references

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