Representation of homogeneous measures and tail measures of regularly varying time series

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Variation on a joint work with E.Hashorva and P.Soulier (arXiv :1710.08358).

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Motivations and related works

- Homogeneous measures are interesting for themselves and play a natural role in the theory of stable laws.

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- The extremal properties of a regularly varying time series are encoded in the spectral tail process or the tail measure :

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- The tail measure is a more canonical object and provides often better insight and simpler proof for properties of the spectral tail process :

  Planinić, H. and Soulier, P. (2018+). The tail process revisited. Extremes.

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- Stochastic representation of homogeneous measures :
  - existence and uniqueness,
  - non-singular flow property for G-invariant homogeneous measures.

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- Tail measure of regularly varying time series :
  - tilt shift formula for shift-invariant homogeneous measures,
  - relationship with the spectral tail process and time change formula,
  - construction of a max-stable like time series with an arbitrary given spectral tail process satisfying the TCF (relatd to Anja's talk)
  - some properties of the constructed process in the dissipative case.

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  - tilt shift formula for shift-invariant homogeneous measures,
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  - some properties of the constructed process in the dissipative case.
- For the purpose of the proof, we consider a natural and interesting regular variation property for Poisson point processes.

Structure of the talk

1. Homogeneous measures and their representations

2. Homogeneous measures on $E^{\mathbb{Z}}$ and the time change formula

3. Connection with stationary regularly varying time series

Homogeneous measures

- **Homogeneous measure on an abstract cone :**
  - measurable cone $(\mathbb{X}, \mathcal{X})$: measurable multiplication

$$1 \cdot x = x \quad , \quad u \cdot (v \cdot x) = (uv) \cdot x \quad \text{for all } u, v > 0,\ x \in \mathbb{X}.$$

  - α-homogeneous measure ν on $(\mathbb{X}, \mathcal{X})$:

$$\nu(uB) = u^{-\alpha}\nu(B) \quad \text{for all } u > 0,\ B \in \mathcal{X}.$$

- **Example :** $\mathbb{X} = \mathbb{R}^d$, $(\mathbb{R}^d)^{\mathbb{Z}}$, $C(T, \mathbb{R})$, $D(\mathbb{R}, \mathbb{R}^d)$, $\mathcal{N}_0(\mathbb{R}^d)$, $\ldots$

- **Motivations** :
  - Regular variations : $n\mathbb{P}(X/a_n \in \cdot) \to \nu$.
  - Lévy measure of α-stable random vectors or processes.
  - Exponent measure of max-stable random vectors or processes.
  - More generally, Lévy measure of stable distributions on an abstract convex cone (Evans& Molchanov 2017).

Fundamental construction

Proposition

Let $\mathbf{X}$ be a random variable with values in a measurable cone $(\mathbb{X}, \mathcal{X})$.
For all $\alpha > 0$, the measure ν defined by

$$\nu(B) = \int_0^\infty \mathbb{P}(r\mathbf{X} \in B)\alpha r^{-\alpha-1}\mathrm{d}r \,, \quad B \in \mathcal{B} \,, \tag{1}$$

is α-homogeneous.

- We call (1) a stochastic representation for ν and $\mathbf{X}$ a generator of ν.
- For all α-homogeneous measurable function $H_\alpha : \mathbb{X} \to [0, \infty]$,

$$\nu(H_\alpha(x) > 1) = \mathbb{E}[H_\alpha(\mathbf{X})].$$

- **Question** : can any α-homogeneous measure be obtained in this way ?

Representation theorem (existence)

Let ν be α-homogeneous on $(\mathbb{X}, \mathcal{X})$.

Definition

We say that $\tau : \mathbb{X} \to [0, \infty]$ is a radial function for ν if τ is a measurable 1-homogeneous function satisfying :

$$\nu(\tau(x) = 0) = 0 \quad \text{and} \quad \nu(\tau(x) > 1) = 1.$$

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Theorem (existence of a representation)

The following statements are equivalent :

i) ν admits a radial function.

ii) ν admits a stochastic representation (1).

Remark : *i*) $\Rightarrow$ *ii*) is standard and uses the polar decomposition $x \mapsto (\tau(x, x/\tau(x))$.
ii) $\Rightarrow$ *i*) more surprising with Radon-Nykodym Theorem !

A simple criterion for existence

Corollary

Assume that :

- $\mathbb{X}$ has a zero element $0_{\mathbb{X}}$ such that $0 \cdot x = 0_{\mathbb{X}}$ for all $x \in \mathbb{X}$.
- $\mathbb{X}$ is a metric space with continuous multiplication $[0, \infty) \times \mathbb{X} \to \mathbb{X}$
- The topology of $\mathbb{X}$ is generated by a countable family of "semi-norms".

Then, any α-homogeneous measure ν such that

$$\nu(\mathbb{X} \setminus O) < \infty \quad \text{for all open neighborhood } O \ni 0_{\mathbb{X}}$$

admits a stochastic representation (1).

Remark : the condition $\nu(\mathbb{X} \setminus O) < \infty$ is natural in the framework of regular variations on a metric space (Hult and Linskog 2007).

Question : Example of homogeneous measures without a stochastic representation ?

Representation theorem (uniqueness)

Let ν be α-homogeneous on $(\mathbb{X}, \mathcal{X})$ admitting a stochastic representation (1).

Theorem (uniqueness)

For any radial function τ for ν, there is a unique (in law) generator $\mathbf{X}_\tau$ of ν such that $\tau(\mathbf{X}_\tau) \equiv 1$. Its distribution is given by

$$\mathbb{P}(\mathbf{X}_\tau \in B) = \nu(\tau(\mathbf{x}) > 1,\ \mathbf{x}/\tau(\mathbf{x}) \in B), \quad B \in \mathcal{X}.$$

Furthermore, for any $\mathbb{X}$-valued random element $\mathbf{X}$, the following statements are equivalent :

i) $\mathbf{X}$ generates ν.

ii) $\mathbb{E}[H_\alpha(\mathbf{X})] = \nu(H_\alpha(\mathbf{x}) > 1)$ for all α-homogeneous measurable H_α.

Group invariant homogeneous measure

- Let G be a group acting on the measurable cone $(\mathbb{X}, \mathcal{X})$.
- Let ν be α-homogeneous on $(\mathbb{X}, \mathcal{X})$ and G invariant.
- Let τ be a radial function for ν and $\mathbf{X}_\tau$ the associated generator.

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Theorem (non-singular flow property)

Equip $\mathbb{X}$ with the σ-algebra $\mathcal{C}$ generated by cones.
On $(\mathbb{X}, \mathcal{C}, \mathrm{P}_{\mathbf{X}_\tau})$, the group action satisfies the non-singular flow property

$$\frac{\mathrm{d}(\mathrm{P}_{\mathbf{X}_\tau} \circ g^{-1})}{\mathrm{d}\mathrm{P}_{\mathbf{X}_\tau}}(x) = \frac{\tau^\alpha(g^{-1}x)}{\tau^\alpha(x)}, \quad g \in G.$$

Remark : strong connection with non-singular flow representation of stationary α-stable processes (Rosinski 1995).

Group invariant homogeneous measure

Proof : Recall that $\mathbf{X}$ generates ν if and only if

$$\mathbb{E}[H_\alpha(\mathbf{X})] = \nu(H_\alpha(x) > 1), \quad H_\alpha \in \mathcal{H}_\alpha.$$

Since $\mathbf{X}_\tau$ generates ν,

$$\mathbb{E}\left[\tau^\alpha(g^{-1}\mathbf{X}_\tau)1_{\{g^{-1}\mathbf{X}_\tau \in C\}}\right] = \nu(\tau^\alpha(g^{-1}\mathbf{x}) > 1,\ g^{-1}\mathbf{x} \in C).$$

In particular, for $g = 1$, using $\tau(\mathbf{X}_\tau) \equiv 1$,

$$\mathbb{P}(\mathbf{X}_\tau \in C) = \nu(\tau^\alpha(\mathbf{x}) > 1,\ \mathbf{x} \in C).$$

By G-invariance of ν, the two are equal and replacing C by $g^{-1}C$ yields

$$\mathbb{P}(\mathbf{X}_\tau \in g^{-1}C) = \mathbb{E}\left[\tau^\alpha(g^{-1}\mathbf{X}_\tau)1_{\{\mathbf{X}_\tau \in C\}}\right] = \mathbb{E}\left[\frac{\tau^\alpha(g^{-1}\mathbf{X}_\tau)}{\tau^\alpha(\mathbf{X}_\tau)}1_{\{\mathbf{X}_\tau \in C\}}\right].$$

Structure of the talk

Time series framework

- Let $(E, \mathcal{E})$ be a measurable cone with origin 0_E and "norm" $\|\cdot\|$.
- $\mathbb{X} = E^{\mathbb{Z}}$ denotes the set of E-valued time series, $\mathcal{X} = \mathcal{E}^{\otimes \mathbb{Z}}$.

Definition

We call tail measure an α-homogeneous measure ν such that :

- $\nu(\{0_{E^{\mathbb{Z}}}\}) = 0$,
- $\nu(\|\mathbf{x}_h\| > 1) < \infty, h \in \mathbb{Z}$,
- $\nu(\|\mathbf{x}_0\| > 1) = 1$.

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- Owada & Samorodnitsky (2012) prove that if $\mathbf{Z}$ is a regularly varying time series (random field), then there exists a unique tail measure ν such that

$$n\,\mathbb{P}(\mathbf{Z}/a_n \in \cdot) \to \nu \quad \text{as } n \to \infty.$$

Convergence is meant in the sense of fidi vague convergence on $\overline{\mathbb{R}}^k \setminus \{0\}$ and a_n is such that $\mathbb{P}(\|\mathbf{Z}_0\| > a_n) \sim n^{-1}$.

Shift-invariant tail mesures and the tilt-shift formula

- Any tail measure ν admits a stochastic representation and we denote by $\mathbf{X} = (\mathbf{X}_h)_{h\in\mathbb{Z}}$ any generator.

Proposition (tilt shift formula)

The following statements are equivalent :

i) ν is shift-invariant ;

B is the backshift operator $B(\ldots, x_0, x_1, \ldots) = (\ldots, x_1, x_2, \ldots)$.

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iii) $\mathbb{E}[\|\mathbf{X}_0\|^\alpha H_0(B^h\mathbf{X})] = \mathbb{E}[\|\mathbf{X}_h\|^\alpha H_0(\mathbf{X})]$ for all $H_0 \in \mathcal{H}_0$. $\quad$ (TSF)

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Spectral tail process

Definition

The spectral tail process at h associated with ν is the process $\Theta^{(h)}$ with distribution

$$\mathbb{P}(\Theta^{(h)} \in \cdot\,) = \frac{\nu(\|x_h\| > 1,\ x/\|x_h\| \in \cdot\,)}{\nu(\|x_h\| > 1)}.$$

For $h = 0$, we note shortly $\Theta = \Theta^{(0)}$.

Note that $\mathbb{P}(\|\Theta_h^{(h)}\| = 1) = 1$ and $\mathbb{P}(\|\Theta_0\| = 1) = 1$

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Remark : in the framework of stationary regularly varying times series (Basrak and Segers 2009), the spectral tail process appears as the limiting distribution

$$\mathbb{P}(\Theta^{(h)} \in \cdot\,) = \lim_{u\to\infty} \mathbb{P}(\mathbf{Z}/\|\mathbf{Z}_h\| \in \cdot \mid \|\mathbf{Z}_h\| > u).$$

Time change formula for the spectral tail process

Proposition (time change formula)

Assume $\nu(\|x_h\| > 1) \equiv 1$. The following statements are equivalent :

i) ν is shift-invariant ;

Remark : time change formula *iii*) appears in Basrak & Segers (2009) in the framework of regularly time series.

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ii) $\Theta^{(h)} = B^h\Theta$, $h \in \mathbb{Z}$;

iii) $\mathbb{E}[H_0(B^h\Theta)] = \mathbb{E}[\|\Theta_h\|^\alpha H_0(\Theta)]$ for all $H_0 \in \mathcal{H}_0$ vanishing on $\{\|x_0\| = 0\}$. (TCF)

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Correspondence tail measure/spectral tail process

Proposition

The mapping $\nu \mapsto \Theta$ is a one-to-one correspondence between stationary tail measures and spectral tail processes satisfying the TCF together with $\|\Theta_0\| \equiv 1$.

Proof : To recover ν from Θ, consider a probability mass distribution on $\mathbb{Z}$

$$q = (q_h)_{h\in\mathbb{Z}},\ q_h > 0,\ \sum_{h\in\mathbb{Z}} q_h = 1$$

and the generator

$$\mathbf{X} = \frac{B^K(\Theta)}{\|\Theta\|_{q,\alpha}}, \quad K \sim q \text{ independent of } \Theta,$$

with $\|\mathbf{x}\|_{q,\alpha} = \left(\sum_{h\in\mathbb{Z}} q_h \|\mathbf{x}_h\|^\alpha\right)^{1/\alpha}$.

Structure of the talk

Stationary regularly varying time series

- Natural question (cf Anja's talk) :
  Given a shift-invariant tail measure ν, can we construct a stationary time series with tail measure ν ?

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Proposition ("max-stable like construction")

Let ν be a stationary tail measure with generator $\mathbf{X}$.
Consider the shift-invariant point process

$$\Pi = \{\Gamma_i^{-1/\alpha}\mathbf{X}^i,\ i \geq 1\} \sim \mathrm{PRM}(E^{\mathbb{Z}}, \nu).$$

Define, for $h \in \mathbb{Z}$,

$$\mathbf{Z}_h = \text{element with the largest norm within } \{\Gamma_i^{-1/\alpha}\mathbf{X}_h^i,\ i \geq 1\}.$$

Then, $\mathbf{Z} = (\mathbf{Z}_h)_{h\in\mathbb{Z}}$ is stationary and $\mathbf{Z} \in \mathrm{RV}_\alpha\left(E^{\mathbb{Z}}, (n^{1/\alpha}), \nu\right)$.

Remark : in the case $E = [0, \infty)$, $\mathbf{Z}$ is a max-stable process.

Proof : regular variations of Π together with continuous mapping theorem.

Regular variations on a metric space

Following Hult and Lindskog (2006)

- F a cone with an origin 0_F and a metric d such that

$$d(0_F, ux) \leq d(0_F, vx), \quad u \leq v, x \in F.$$

- $M_0(F)$: the space of Borel measures μ that are finite on $B(0_F, r)^c$, $r > 0$.
- M_0-convergence $\mu_n \xrightarrow{M_0} \mu$ if and only if

$$\int f d\mu_n \to \int f d\mu \quad \text{for all continuous } f \geq 0 \text{ with support separated from } 0_F.$$

- With ρ_r the Prohorov distance on the set of finite measures on $B(0_F, r)^c$,

$$\rho(\mu, \mu') = \int_0^\infty e^{-r}(\rho_r(\mu, \mu') \wedge 1) dr,$$

  metrizes the M_0-convergence
- If (F, d) is complete separable, then so is $(M_0(F), \rho)$.
- We say that $X \in \mathrm{RV}(F, \{a_n\}, \mu)$ if $n\mathbb{P}(a_n^{-1} X \in \cdot) \xrightarrow{M_0} \mu$.

Regular variations of point processes

- $\mathcal{N}_0(F)$: subspace of point measures (points may accumulate to 0_F).
- CSMP with the induced metric ρ
- Scaling : for $u > 0$ and $\pi = \sum_{i\geq 0} \delta_{x_i}$, $u\pi = \sum_{i\geq 0} \delta_{ux_i}$.
- Good control of the distance to the origin (= null measure) :

$$\frac{1}{2}(\|\pi\| \wedge 1) \leq \rho(0,\pi) \leq \|\pi\| \quad \text{with } \|\pi\| = \max_{x\in\pi} d(0_F, x).$$

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Proposition (Laplace criterion)

Let $\mu, \mu_1, \mu_2 \ldots \in \mathcal{M}_0(\mathcal{N}_0(F))$. The following are equivalent :

i) $\mu_n \to \mu$ in $\mathcal{M}_0(\mathcal{N}_0(F))$.

ii) $\int_{\mathcal{N}_0(F)} \left(1 - e^{-\pi(f)}\right) \mu_n(\mathrm{d}\pi) \to \int_{\mathcal{N}_0(F)} \left(1 - e^{-\pi(f)}\right) \mu(\mathrm{d}\pi)$ for all bounded continuous f with support bounded away from 0_F.

This extends Zhao (2016) where weak convergence of probability distribution on $\mathcal{N}_0(F)$ is considered.

Regular variations of Poisson point process

Proposition

Let $\mu \in \mathcal{M}_0(F)$ such that $n\mu(a_n^{-1}\cdot) \xrightarrow{M_0} \nu$.
Consider $\Pi \sim \mathrm{PRM}(F,\mu)$ as a random element of $\mathcal{N}_0(F)$. Then,

$$\Pi \in \mathrm{RV}_\alpha\left(\mathcal{N}_0(F), \{a_n\}, \nu^*\right) \quad \text{with} \quad \nu^*(\cdot) = \int \mathbf{1}_{\{\delta_x \in \cdot\}}\, \nu(\mathrm{d}x).$$

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Proof : Laplace functional of PRM is explicit and

$$\begin{aligned} n\mathbb{E}\left[1 - e^{-\int_F f(x/a_n)\Pi(dx)}\right] &= n\left(1 - \exp\left[\int_F (e^{-f(x/a_n)} - 1)\mu(dx)\right]\right) \\ &= n\left(1 - \exp\left[n^{-1}\int_F (e^{-f(x)} - 1)n\mu(a_n dx)\right]\right) \\ &\longrightarrow \int_F (1 - e^{-f(x)})\nu(dx). \end{aligned}$$

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Comment : single large point heuristic similar to the single big jump heuristic for RV Lévy processes.

Moving shift representation

Definition

We say that a shift invariant tail measure ν has a moving shift representation if there exists a stochastic process $\widetilde{\mathbf{X}}$ such that

$$\nu(A) = \sum_{h\in\mathbb{Z}} \int_0^\infty \mathbb{P}(rB^h\widetilde{\mathbf{X}} \in A)\alpha r^{-\alpha-1} dr, \quad A \in \mathcal{X}.$$

The condition $\nu\{0_{E^{\mathbb{Z}}}\} = 0$ and $\nu(\|\mathbf{x}_0\| > 1) = 1$ imply

$$\mathbb{P}(\widetilde{\mathbf{X}} = 0_{E^{\mathbb{Z}}}) = 0 \quad \text{and} \quad \sum_{h\in\mathbb{Z}} \mathbb{E}[\|\widetilde{\mathbf{X}}_h\|] = 1.$$

Conversely, if $\widetilde{\mathbf{X}}$ satisfies these conditions, ν is a well defined shift invariant tail measure.

Existence of a moving shift representation

Theorem

Let ν be a shift invariant tail measure. The following are equivalent :

i) ν has a moving shift representation ;

ii) ν is supported by $\{\sum_{h\in\mathbb{Z}} \|\mathbf{x}_h\|^\alpha < \infty\}$;

iii) ν is supported by $\{\lim_{|h|\to\infty} \|\mathbf{x}_h\| = 0\}$;

iv) ν is supported by $\{\inf \operatorname{argmax}(x) \in \mathbb{Z}\}$;

v) the dynamical system $(E^{\mathbb{Z}}, \mathcal{C}, \nu, B)$ is dissipative, with $\mathcal{C}$ the σ-algebra of cones.

Then, the moving shift representation holds with

$$\widetilde{\mathbf{X}} = \frac{\Theta}{\|\Theta\|_\alpha} \quad \text{with } \|\Theta\|_\alpha = \Big(\sum_{h\in\mathbb{Z}} \|\Theta_h\|^\alpha\Big)^{1/\alpha}.$$

Properties of $\mathbf{Z}$

We go back to the "max-stable like" process

$$\mathbf{Z}_h = \text{element with the largest norm within } \{\Gamma_i^{-1/\alpha}\mathbf{X}_h^i,\ i \geq 1\}, \quad h \in \mathbb{Z},$$

where $\Pi = \{\Gamma_i^{-1/\alpha}\mathbf{X}^i,\ i \geq 1\} \sim \mathrm{PRM}(E^{\mathbb{Z}}, \nu)$.

Proposition

$\mathbf{Z}$ satisfies the anti-clustering condition if and only if ν is dissipative.

Anti-Clustering condition : for some intermediate sequence $1 \ll r_n \ll n$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\max_{m \leq |h| \leq r_n} \|\mathbf{X}_h\| > a_n u \,\Big|\, \|\mathbf{X}_0\| > a_n u \right) = 0, \quad u > 0. \tag{AC}$$

Mixing and existence of extremal indices

Theorem

Assume ν is dissipative. Then,

- $\mathbf{Z}$ is mixing ;
- $\mathbf{Z}$ admits an m-dependent tail equivalent approximation :
- For all non-negative Lipschitz $H \in \mathcal{H}_1$ with $\nu(H(\mathbf{x}) > 1) = 1$, the sequence $(H(\mathbf{Z}_h))_{h\in\mathbb{Z}}$ has a positive extremal index given by

$$\theta(H) = \mathbb{E}\left[\max_{i\in\mathbb{Z}} H^\alpha(\widetilde{\mathbf{X}}_i)\right] = \mathbb{E}\left[\frac{\max_{i\in\mathbb{Z}} H^\alpha(\Theta_i)}{\sum_{i\in\mathbb{Z}} \|\Theta_i\|^\alpha}\right] \in (0,1],$$

Here, the extremal index appears in the Fréchet limit

$$\mathbb{P}\left(n^{-1/\alpha} \max_{0\le h\le n-1} H(\mathbf{Z}_h) \le x\right) \longrightarrow \exp(-\theta(H)x^{-\alpha}), \quad x > 0.$$