Stream function formulation of surface Stokes equations

Arnold Reusken

Chair for Numerical Mathematics
RWTH Aachen

Oaxaca, July 30th 2018

Joint work with Philip Brandner, Thomas Jankuhn (RWTH), Maxim Olshanskii (Houston)

Outline

- Modeling of fluidic surfaces: surface (Navier-)Stokes equations
- Well-posedness of surface Stokes equations
- **Stream function formulation of surface Stokes equations**
- Finite element discretization based on stream function formulation.

Material surfaces (e.g., biomembranes)

Initialization: smooth closed surface $\Gamma(0)$. Mass density $\rho(x,0)$.
Velocity $\mathbf{u}(x,0)$.
Small material subdomain: $\gamma(t) \subset \Gamma(t)$.

Modeling principles

- Inextensibility: $\frac{d}{dt}\int_{\gamma(t)} 1\,ds = 0$.
- Mass conservation: $\frac{d}{dt}\int_{\gamma(t)} \rho(x,t)\,ds = 0$.
- Momentum conservation:

$$\frac{d}{dt}\int_{\gamma(t)} \rho\mathbf{u}\,ds = \int_{\partial\gamma(t)} f_\nu\,ds + \int_{\gamma(t)} \mathbf{b}\,ds$$

with line contact force f_ν (ν: conormal), area force $\mathbf{b}$.

ρ (area) density: constant; $\mathbf{u}$: velocity (tangential and normal to $\Gamma(t)$).

Modeling of Newtonian surface fluid

Continuum mechanics on 2D surface [Gurtin, Murdoch].

$$\mathbf{P} = \mathbf{P}(x) := \mathbf{I} - \mathbf{n}\mathbf{n}^T \quad \text{(projection on tangential plane at } x \in \Gamma)$$

$$E_s(\mathbf{u}) := \frac{1}{2}\mathbf{P}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\mathbf{P} \quad \text{(surface strain tensor)}$$

$$f_\nu = \boldsymbol{\sigma}_\Gamma \nu \quad \text{(contact force)}$$

$$\boldsymbol{\sigma}_\Gamma = -\pi\mathbf{P} + 2\mu E_s(\mathbf{u}) \quad \text{(Newtonian surface stress)}$$

Modeling of Newtonian surface fluid

Continuum mechanics on 2D surface [Gurtin, Murdoch].

$$\mathbf{P} = \mathbf{P}(x) := \mathbf{I} - \mathbf{n}\mathbf{n}^T \quad \text{(projection on tangential plane at } x \in \Gamma\text{)}$$

$$E_s(\mathbf{u}) := \frac{1}{2}\mathbf{P}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)\mathbf{P} \quad \text{(surface strain tensor)}$$

$$f_\nu = \boldsymbol{\sigma}_\Gamma \nu \quad \text{(contact force)}$$

$$\boldsymbol{\sigma}_\Gamma = -\pi \mathbf{P} + 2\mu E_s(\mathbf{u}) \quad \text{(Newtonian surface stress)}$$

Combining this results in

Surface incompressible Navier-Stokes equations

$$\begin{aligned} \rho \dot{\mathbf{u}} &= -\nabla_\Gamma \pi + 2\mu \,\mathrm{div}_\Gamma (E_s(\mathbf{u})) + \mathbf{b} + \pi \kappa \mathbf{n} \\ \mathrm{div}_\Gamma \mathbf{u} &= 0 \end{aligned}$$

$\nabla_\Gamma \pi = \mathbf{P}\nabla \pi^e$, $\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt} + \mathbf{u} \cdot \nabla_\Gamma \mathbf{u}$ (material derivative), κ : mean curvature

Modeling of fluidic surfaces

More info: Jankuhn, Olshanskii, AR: *Incompressible Fluid Problems on Embedded Surfaces: Modeling and Variational Formulations*, IFB 2018

Modeling of fluidic surfaces

More info: Jankuhn, Olshanskii, AR: *Incompressible Fluid Problems on Embedded Surfaces: Modeling and Variational Formulations*, IFB 2018

Other modeling approaches:

- differential geometry [Scriven], [Arroyo,DeSimone]
- energetic approaches [Koba,Liu et al.], [Barrett,Garcke et al.]
- other contributions [Reuther,Voigt, et al.], [Bothe, Prüss], [Gurtin,Murdoch], ...

Note: Resulting models agree for stationary Γ.

There are differences for evolving Γ.

Surface Stokes equations

Deleting the $\mathbf{u} \cdot \nabla_\Gamma \mathbf{u}$ term we get for the stationary case:

Surface Stokes equations (Γ stationary)

$$-2\mu\mathbf{P}\,\mathrm{div}_\Gamma(E_s(\mathbf{u})) + \nabla_\Gamma \pi = \mathbf{b} \quad (\mathbf{Pb} = \mathbf{b})$$
$$\mathrm{div}_\Gamma \mathbf{u} = 0$$

$E_s(\mathbf{u}) = \frac{1}{2}\mathbf{P}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\mathbf{P} =: \frac{1}{2}(\nabla_\Gamma\mathbf{u} + (\nabla_\Gamma\mathbf{u})^T)$.

Only very few results on surface Stokes are available.

Well-posedness

$$V := H^1(\Gamma)^n, \quad \mathbf{H}_t^1 := \{\mathbf{v} \in V : \ \mathbf{v} \cdot \mathbf{n} = 0\},$$

$$E := \{\mathbf{v} \in \mathbf{H}_t^1 : \ E_s(\mathbf{v}) = 0\} \quad (\text{“killling fields”}; n = 3\text{: } \dim E \le 3).$$

$$V_t^0 := \mathbf{H}_t^1 / E.$$

Weak formulation

Find $(\mathbf{u}, p) \in V_t^0 \times L_0^2(\Gamma)$ s.t.

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = f(\mathbf{v}) \qquad \text{for all } \mathbf{v} \in V_t^0,$$

$$b(\mathbf{u}, q) = 0 \qquad \text{for all } q \in L_0^2(\Gamma).$$

$$a(\mathbf{u}, \mathbf{v}) := 2\mu \int_\Gamma \mathrm{tr}(E_s(\mathbf{u}) E_s(\mathbf{v}))\, ds, \quad b(\mathbf{u}, p) := -\int_\Gamma p\, \mathrm{div}_\Gamma \mathbf{u}\, ds$$

Well-posedness

$$V := H^1(\Gamma)^n, \quad \mathbf{H}_t^1 := \{\mathbf{v} \in V : \ \mathbf{v}\cdot\mathbf{n} = 0\},$$
$$E := \{\mathbf{v} \in \mathbf{H}_t^1 : \ E_s(\mathbf{v}) = 0\} \quad (\text{“killing fields”}; n=3\text{: } \dim E \leq 3).$$
$$V_t^0 := \mathbf{H}_t^1 / E.$$

Weak formulation

Find $\quad (\mathbf{u}, p) \in V_t^0 \times L_0^2(\Gamma) \quad$ s.t.

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v}, p) = f(\mathbf{v}) \qquad \text{for all } \mathbf{v} \in V_t^0,$$
$$b(\mathbf{u}, q) = 0 \qquad \text{for all } q \in L_0^2(\Gamma).$$

$$a(\mathbf{u},\mathbf{v}) := 2\mu \int_\Gamma \operatorname{tr}(E_s(\mathbf{u})E_s(\mathbf{v}))\, ds, \quad b(\mathbf{u}, p) := -\int_\Gamma p \,\mathrm{div}_\Gamma \mathbf{u}\, ds$$

Well-posedness

$$V := H^1(\Gamma)^n, \quad \mathbf{H}_t^1 := \{\mathbf{v} \in V : \ \mathbf{v}\cdot\mathbf{n} = 0\},$$
$$E := \{\mathbf{v} \in \mathbf{H}_t^1 : \ E_s(\mathbf{v}) = 0\} \quad (\text{“killing fields”}; n=3\text{: } \dim E \leq 3).$$
$$V_t^0 := \mathbf{H}_t^1 / E.$$

Weak formulation is well-posed

Find $(\mathbf{u}, p) \in V_t^0 \times L_0^2(\Gamma)$ s.t.

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = f(\mathbf{v}) \qquad \text{for all } \mathbf{v} \in V_t^0,$$
$$b(\mathbf{u},q) = 0 \qquad \text{for all } q \in L_0^2(\Gamma).$$

$$a(\mathbf{u},\mathbf{v}) := 2\mu \int_\Gamma \mathrm{tr}(E_s(\mathbf{u})E_s(\mathbf{v}))\, ds, \quad b(\mathbf{u},p) := -\int_\Gamma p\, \mathrm{div}_\Gamma \mathbf{u}\, ds$$

Well-posedness

Analysis of well-posedness

- Surface Korn's inequality for $a(\cdot,\cdot)$.
- Inf-sup property for $b(\cdot,\cdot)$. ("easy"!)

$$\inf_{p \in L^2_0(\Gamma)} \sup_{\mathbf{v} \in V_t^0} \frac{b(\mathbf{v}, p)}{\|\mathbf{v}\|_1 \|p\|_{L^2}} \geq c > 0.$$

Stream function formulation

Surface differential operators

Assumption. Γ is a C^2 connected compact oriented hypersurface in $\mathbb{R}^3$ without boundary.

$\mathbf{P}(x) = \mathbf{I} - \mathbf{n}(x)\mathbf{n}(x)^T$, $x \in \Gamma$.
For $u \in C(\Gamma)$: $u^e :=$ constant extension along $\mathbf{n}$.

Differential operators (based on Euclidean space operators)

$$\nabla_\Gamma \phi := \mathbf{P}\nabla\phi^e, \quad \text{(scalar } \phi\text{)}$$

$$\nabla_\Gamma \mathbf{u} := \mathbf{P}\nabla\mathbf{u}^e\mathbf{P} \quad \text{(vector } \mathbf{u}\text{)}$$

$$\mathrm{div}_\Gamma \mathbf{u} := \mathrm{tr}(\nabla_\Gamma \mathbf{u}), \quad \mathrm{div}_\Gamma A := \begin{pmatrix} \mathrm{div}_\Gamma(e_1^T A) \\ \mathrm{div}_\Gamma(e_2^T A) \\ \mathrm{div}_\Gamma(e_3^T A) \end{pmatrix} \quad \text{(matrix } A\text{)}$$

$$\mathrm{curl}_\Gamma \mathbf{u} := (\nabla_\Gamma \times \mathbf{u}^e) \cdot \mathbf{n}$$

$$\mathbf{curl}_\Gamma \phi := \mathbf{n} \times \nabla_\Gamma \phi \quad \text{(tangential vector)}$$

Surface differential operators

Properties, for smooth *tangential* $\mathbf{u}$, A:

Partial integration; $\nabla_\Gamma = -\operatorname{div}_\Gamma^T$, $\operatorname{curl}_\Gamma = -\mathbf{curl}_\Gamma^T$

$$\int_\Gamma \operatorname{div}_\Gamma \mathbf{u}\, \phi \, ds = -\int_\Gamma \mathbf{u} \cdot \nabla_\Gamma \phi \, ds$$

$$\int_\Gamma (\operatorname{div}_\Gamma A) \cdot \mathbf{u} \, ds = -\int_\Gamma \operatorname{tr}(A^T \nabla_\Gamma \mathbf{u}) \, ds$$

$$\int_\Gamma \operatorname{curl}_\Gamma \mathbf{u}\, \phi \, ds = -\int_\Gamma \mathbf{u} \cdot \mathbf{curl}_\Gamma \phi \, ds$$

Surface differential operators

Properties, for smooth *tangential* **u**, A:

Partial integration; $\nabla_\Gamma = -\operatorname{div}_\Gamma^T,\ \operatorname{curl}_\Gamma = -\mathbf{curl}_\Gamma^T$

$$\int_\Gamma \operatorname{div}_\Gamma \mathbf{u}\, \phi\, ds = -\int_\Gamma \mathbf{u} \cdot \nabla_\Gamma \phi\, ds$$

$$\int_\Gamma (\operatorname{div}_\Gamma A) \cdot \mathbf{u}\, ds = -\int_\Gamma \operatorname{tr}(A^T \nabla_\Gamma \mathbf{u})\, ds$$

$$\int_\Gamma \operatorname{curl}_\Gamma \mathbf{u}\, \phi\, ds = -\int_\Gamma \mathbf{u} \cdot \mathbf{curl}_\Gamma \phi\, ds$$

Basic identities

$$\operatorname{div}_\Gamma(\mathbf{curl}_\Gamma \phi) = 0$$

$$\operatorname{curl}_\Gamma(\nabla_\Gamma \phi) = 0$$

$$\operatorname{curl}_\Gamma(\mathbf{curl}_\Gamma \phi) = \operatorname{div}_\Gamma(\nabla_\Gamma \phi) = \Delta_\Gamma \phi$$

$$\mathbf{curl}_\Gamma(\operatorname{curl}_\Gamma \mathbf{u}) = \mathbf{P} \operatorname{div}_\Gamma(\nabla_\Gamma \mathbf{u}) - \nabla_\Gamma(\operatorname{div}_\Gamma \mathbf{u}) - K\mathbf{u}$$

with Gaussian curvature K.

Surface Helmholtz decomposition

$$\mathbf{L}_t^2(\Gamma) := \{\, \mathbf{u} \in L^2(\Gamma)^3 \mid \mathbf{n} \cdot \mathbf{u} = 0 \quad \text{a.e. on } \Gamma \,\},$$
$$\mathbf{H}_t^1(\Gamma) := \{\, \mathbf{u} \in H^1(\Gamma)^3 \mid \mathbf{n} \cdot \mathbf{u} = 0 \quad \text{a.e. on } \Gamma \,\}.$$

∇_Γ, $\mathbf{curl}_\Gamma : H^1(\Gamma) \to \mathbf{L}_t^2(\Gamma)$, defined by bounded extension.

Surface Helmholtz decomposition

$$\mathbf{L}_t^2(\Gamma) := \{\, \mathbf{u} \in L^2(\Gamma)^3 \mid \mathbf{n}\cdot\mathbf{u} = 0 \quad \text{a.e. on } \Gamma \,\},$$
$$\mathbf{H}_t^1(\Gamma) := \{\, \mathbf{u} \in H^1(\Gamma)^3 \mid \mathbf{n}\cdot\mathbf{u} = 0 \quad \text{a.e. on } \Gamma \,\}.$$

∇_Γ, $\mathbf{curl}_\Gamma : H^1(\Gamma) \to \mathbf{L}_t^2(\Gamma)$, defined by bounded extension.

div_Γ, $\mathrm{curl}_\Gamma : \mathbf{L}_t^2(\Gamma) \to H^{-1}(\Gamma)$ defined via duality:

$$\langle\, \mathrm{div}_\Gamma \mathbf{u}, \phi \rangle := -\int_\Gamma \mathbf{u}\cdot\nabla_\Gamma \phi \, ds \quad \forall \;\; \phi \in H^1(\Gamma), \mathbf{u} \in \mathbf{L}_t^2(\Gamma),$$
$$\langle\, \mathrm{curl}_\Gamma \mathbf{u}, \phi \rangle := -\int_\Gamma \mathbf{u}\cdot \mathbf{curl}_\Gamma \phi \, ds \quad \forall \;\; \phi \in H^1(\Gamma), \mathbf{u} \in \mathbf{L}_t^2(\Gamma)$$

Surface Helmholtz decomposition

$$\mathbf{L}_t^2(\Gamma) := \{\, \mathbf{u} \in L^2(\Gamma)^3 \mid \mathbf{n} \cdot \mathbf{u} = 0 \quad \text{a.e. on } \Gamma \,\},$$
$$\mathbf{H}_t^1(\Gamma) := \{\, \mathbf{u} \in H^1(\Gamma)^3 \mid \mathbf{n} \cdot \mathbf{u} = 0 \quad \text{a.e. on } \Gamma \,\}.$$

∇_Γ, $\mathbf{curl}_\Gamma$: $H^1(\Gamma) \to \mathbf{L}_t^2(\Gamma)$, defined by bounded extension.

div_Γ, curl_Γ : $\mathbf{L}_t^2(\Gamma) \to H^{-1}(\Gamma)$ defined via duality:

$$\langle\, \mathrm{div}_\Gamma \mathbf{u}, \phi \rangle := -\int_\Gamma \mathbf{u} \cdot \nabla_\Gamma \phi \, ds \quad \forall \;\; \phi \in H^1(\Gamma), \mathbf{u} \in \mathbf{L}_t^2(\Gamma),$$
$$\langle\, \mathrm{curl}_\Gamma \mathbf{u}, \phi \rangle := -\int_\Gamma \mathbf{u} \cdot \mathbf{curl}_\Gamma \phi \, ds \quad \forall \;\; \phi \in H^1(\Gamma), \mathbf{u} \in \mathbf{L}_t^2(\Gamma)$$

Harmonic fields

$$\mathcal{H} = \{\, \mathbf{u} \in \mathbf{L}_t^2(\Gamma) \mid \mathrm{div}_\Gamma \mathbf{u} = 0 \quad \text{and} \quad \mathrm{curl}_\Gamma \mathbf{u} = 0 \,\},$$

Surface Helmholtz decomposition

Lemma (application of Peetre-Tartar Lemma)

$$\dim(\mathcal{H}) < \infty$$

Surface Helmholtz decomposition

Lemma (application of Peetre-Tartar Lemma)

$$\dim(\mathcal{H}) < \infty$$

Main Theorem

$\forall \quad \mathbf{u} \in \mathbf{L}^2_t(\Gamma)$: $\exists_1 \ \psi, \phi \in H^1_*(\Gamma) := \{ \phi \in H^1(\Gamma) \mid \int_\Gamma \phi \, ds = 0 \}$ and $\boldsymbol{\xi} \in \mathcal{H}$:

$$\mathbf{u} = \nabla_\Gamma \psi + \mathbf{curl}_\Gamma \phi + \boldsymbol{\xi}.$$

The range spaces $\nabla_\Gamma(H^1_*(\Gamma))$ and $\mathbf{curl}_\Gamma(H^1_*(\Gamma))$ are closed in $\mathbf{L}^2_t(\Gamma)$.

$$\mathbf{L}^2_t(\Gamma) = \nabla_\Gamma(H^1_*(\Gamma)) \oplus \mathbf{curl}_\Gamma(H^1_*(\Gamma)) \oplus \mathcal{H}$$

is L^2-orthogonal direct sum.

Surface Helmholtz decomposition

Lemma (application of Peetre-Tartar Lemma)

$$\dim(\mathcal{H}) < \infty$$

Main Theorem

$\forall \quad \mathbf{u} \in \mathbf{L}^2_t(\Gamma)$: $\exists_1 \ \psi, \phi \in H^1_*(\Gamma) := \{ \phi \in H^1(\Gamma) \mid \int_\Gamma \phi \, ds = 0 \}$ and $\boldsymbol{\xi} \in \mathcal{H}$:

$$\mathbf{u} = \nabla_\Gamma \psi + \mathbf{curl}_\Gamma \phi + \boldsymbol{\xi}.$$

The range spaces $\nabla_\Gamma(H^1_*(\Gamma))$ and $\mathbf{curl}_\Gamma(H^1_*(\Gamma))$ are closed in $\mathbf{L}^2_t(\Gamma)$.

$$\mathbf{L}^2_t(\Gamma) = \nabla_\Gamma(H^1_*(\Gamma)) \oplus \mathbf{curl}_\Gamma(H^1_*(\Gamma)) \oplus \mathcal{H}$$

is L^2-orthogonal direct sum.

Corollary: $\mathbf{u} = \mathbf{curl}_\Gamma \phi + \boldsymbol{\xi}$ if $\mathrm{div}_\Gamma \mathbf{u} = 0$ $\quad$ (ϕ: stream function)

Surface Helmholtz decomposition

Theorem

Γ simply connected $\Rightarrow \dim(\mathcal{H}) = 0$.

Proof is "elementary" (elliptic regularity theory + properties of geodesics).

Surface Helmholtz decomposition

Theorem

Γ simply connected $\Rightarrow \dim(\mathcal{H}) = 0$.

Proof is "elementary" (elliptic regularity theory + properties of geodesics).

Corollary

Let Γ be simply connected. For $\mathrm{curl}_\Gamma,\ \mathrm{div}_\Gamma : \mathbf{L}_t^2(\Gamma) \to H^{-1}(\Gamma)$ and $\mathbf{curl}_\Gamma,\ \nabla_\Gamma : H^1(\Gamma) \to \mathbf{L}_t^2(\Gamma)$:

$$\ker(\mathrm{div}_\Gamma) = \mathrm{im}(\mathbf{curl}_\Gamma),$$
$$\ker(\mathrm{curl}_\Gamma) = \mathrm{im}(\nabla_\Gamma).$$

Surface Helmholtz decomposition

Theorem

Γ simply connected $\Rightarrow \dim(\mathcal{H}) = 0$.

Proof is "elementary" (elliptic regularity theory + properties of geodesics).

Corollary

Let Γ be simply connected. For $\mathrm{curl}_\Gamma,\ \mathrm{div}_\Gamma : \mathbf{L}^2_t(\Gamma) \to H^{-1}(\Gamma)$ and $\mathbf{curl}_\Gamma,\ \nabla_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma)$:

$$\begin{aligned} \ker(\mathrm{div}_\Gamma) &= \mathrm{im}(\mathbf{curl}_\Gamma), \\ \ker(\mathrm{curl}_\Gamma) &= \mathrm{im}(\nabla_\Gamma). \end{aligned}$$

Corollary

Assume that Γ is simply connected.

$$\|\mathbf{u}\|^2_{\mathbf{H}^1} \leq c\left(\|\mathrm{div}_\Gamma \mathbf{u}\|^2_{L^2(\Gamma)} + \|\mathrm{curl}_\Gamma \mathbf{u}\|^2_{L^2(\Gamma)}\right) \quad \text{for all } \mathbf{u} \in \mathbf{H}^1_t(\Gamma).$$

Relation to Hodge decomposition (differential geometry)

Relation Hodge-Helmholtz

For $\mathbf{u} \in \mathbf{L}_t^2(\Gamma)$:

$$\mathbf{u} = \nabla_\Gamma \psi + \mathbf{curl}_\Gamma \phi + \boldsymbol{\xi}$$
$$\text{iff} \quad \omega_{\mathbf{u}} = \mathrm{d}\psi - \delta(\phi v^g) + \omega_{\boldsymbol{\xi}}$$

with $\omega_{\mathbf{u}}$ ($\omega_{\boldsymbol{\xi}}$) the 1-form associated to $\mathbf{u}$ ($\boldsymbol{\xi}$), v^g area 2-form,
d: exterior derivative, δ: codifferential.

$$\omega_{\boldsymbol{\xi}} \in H_1(\Gamma) := \{\, \text{1-form } \ \omega \,|\, \mathrm{d}\omega = 0 \ \text{ and } \ \delta\omega = 0 \,\} \quad \text{(1-harmonics)}$$

Relation to Hodge decomposition (differential geometry)

Relation Hodge-Helmholtz

For $\mathbf{u} \in \mathbf{L}_t^2(\Gamma)$:

$$\mathbf{u} = \nabla_\Gamma \psi + \mathbf{curl}_\Gamma \phi + \boldsymbol{\xi}$$
$$\text{iff} \;\; \omega_{\mathbf{u}} = \mathrm{d}\psi - \delta(\phi v^g) + \omega_{\boldsymbol{\xi}}$$

with $\omega_{\mathbf{u}}$ ($\omega_{\boldsymbol{\xi}}$) the 1-form associated to $\mathbf{u}$ ($\boldsymbol{\xi}$), v^g area 2-form, d: exterior derivative, δ: codifferential.

$$\omega_{\boldsymbol{\xi}} \in H_1(\Gamma) := \{\, \text{1-form} \;\; \omega |\; \mathrm{d}\omega = 0 \;\; \text{and} \;\; \delta\omega = 0 \,\} \quad \text{(1-harmonics)}$$

Fundamental results

$H_1(\Gamma) \cong H^1_{dR}(\Gamma)$ (first de Rham cohomology group)

First Betti number $b_1(\Gamma) := \dim(H^1_{dR}(\Gamma))$ depends only on topology of Γ

$b_1(\Gamma) = 0$ if $\Gamma =$ sphere, $b_1(\Gamma) = 2n$ if $\Gamma = n$-torus

Classification thm: Γ homeomorphic to either a sphere or an n-torus

Stokes in stream function formulation

Assumption. Γ is **simply connected** (essential!) $\Rightarrow \mathbf{u} = \mathbf{curl}_\Gamma \phi$.

Recall: well-posed Stokes variational problem.
$E := \{\mathbf{v} \in \mathbf{H}_t^1 : \ E_s(\mathbf{v}) = 0\}, \quad V_t^0 = \mathbf{H}_t^1 / E$.

Find $\quad (\mathbf{u}, p) \in V_t^0 \times L_0^2(\Gamma) \quad$ s.t.

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= f(\mathbf{v}) && \text{for all } \mathbf{v} \in V_t^0, \\ b(\mathbf{u}, q) &= 0 && \text{for all } q \in L_0^2(\Gamma). \end{aligned}$$

Solution $\mathbf{u}^*$.

Stokes in stream function formulation

Assumption. Γ is **simply connected** (essential!) $\Rightarrow \mathbf{u} = \mathbf{curl}_\Gamma \phi$.

Recall: well-posed Stokes variational problem.
$E := \{\mathbf{v} \in \mathbf{H}_t^1 : \ E_s(\mathbf{v}) = 0\}, \quad V_t^0 = \mathbf{H}_t^1 / E.$

Find $\quad (\mathbf{u}, p) \in V_t^0 \times L_0^2(\Gamma) \quad$ s.t.

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = f(\mathbf{v}) \qquad \text{for all } \mathbf{v} \in V_t^0,$$
$$b(\mathbf{u}, q) = 0 \qquad \text{for all } q \in L_0^2(\Gamma).$$

Solution $\mathbf{u}^*$.

$$H_*^2(\Gamma) := H^2(\Gamma) \cap H_*^1(\Gamma), \quad \tilde{E} := \mathbf{curl}_\Gamma^{-1}(E) \subset H_*^2(\Gamma),$$
$$\mathbf{H}_{t,\mathrm{div}}^1 := \{\, \mathbf{u} \in \mathbf{H}_t^1(\Gamma) \mid \mathrm{div}_\Gamma \mathbf{u} = 0 \,\}.$$

Lemma

$$\mathbf{curl}_\Gamma : H_*^2(\Gamma) \to \mathbf{H}_{t,\mathrm{div}}^1 \quad \text{is an homeomorphism,}$$

Stokes in stream function formulation

Define for $\phi, \psi \in H^2(\Gamma)$:

$$\tilde{a}(\phi, \psi) := a(\,\mathbf{curl}_\Gamma \phi,\, \mathbf{curl}_\Gamma \psi) = \int_\Gamma \frac{1}{2}\Delta_\Gamma \phi \Delta_\Gamma \psi - K \nabla_\Gamma \phi \cdot \nabla_\Gamma \psi \, ds$$

Theorem

Take unique stream function $\phi^* \in H^1_*(\Gamma)$ such that $\mathbf{u}^* = \mathbf{curl}_\Gamma \phi^*$.
This ϕ^* is the unique solution of: $\phi \in H^2_*(\Gamma)/\tilde{E}$ such that

$$\tilde{a}(\phi, \psi) = (\mathbf{f}, \mathbf{curl}_\Gamma \psi)_{L^2(\Gamma)} \quad \text{for all} \quad \psi \in H^2_*(\Gamma)/\tilde{E}.$$

Furthermore

$$\|\phi^*\|_{H^3(\Gamma)} \leq c \|\mathbf{f}\|_{L^2(\Gamma)}$$

Stokes in stream function formulation

Reformulation as coupled system of second order problems.

Determine $\phi \in H^1_*(\Gamma)/\tilde{E}$, $\xi \in H^1(\Gamma)$ such that

$$\int_\Gamma \frac{1}{2}\nabla_\Gamma \xi \cdot \nabla_\Gamma \psi + K \nabla_\Gamma \phi \cdot \nabla_\Gamma \psi \, ds = -(\mathbf{f}, \mathbf{curl}_\Gamma \psi)_{L^2(\Gamma)} \quad \forall\, \psi \in H^1_*(\Gamma)/\tilde{E}$$

$$\int_\Gamma \nabla_\Gamma \phi \cdot \nabla_\Gamma \eta + \xi \eta \, ds = 0 \quad \forall\, \eta \in H^1(\Gamma).$$

This problem has a unique solution given by $\phi = \phi^*, \xi = \Delta_\Gamma \phi^*$.

This formulation is suitable for a (surface) finite element discretization.

Stokes in stream function formulation

Reformulation as coupled system of second order problems.

Determine $\phi \in H^1_*(\Gamma)/\tilde{E}$, $\xi \in H^1(\Gamma)$ such that

$$\int_\Gamma \frac{1}{2}\nabla_\Gamma \xi \cdot \nabla_\Gamma \psi + K \nabla_\Gamma \phi \cdot \nabla_\Gamma \psi \, ds = -(\mathbf{f}, \mathbf{curl}_\Gamma \psi)_{L^2(\Gamma)} \quad \forall\, \psi \in H^1_*(\Gamma)/\tilde{E}$$

$$\int_\Gamma \nabla_\Gamma \phi \cdot \nabla_\Gamma \eta + \xi \eta \, ds = 0 \quad \forall\, \eta \in H^1(\Gamma).$$

This problem has a unique solution given by $\phi = \phi^*, \xi = \Delta_\Gamma \phi^*$.

This formulation is suitable for a (surface) finite element discretization.
Remarks:

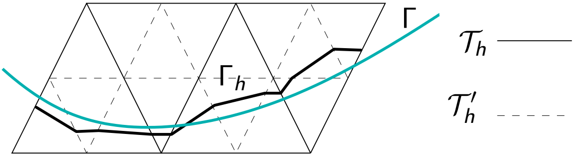
- Test space $H^1_*(\Gamma)/\tilde{E}$ can be replaced by $H^1(\Gamma)$.
- Issue related to projection onto $\tilde{E}$ (not caused by stream function).
- This difficulty vanishes for operator $-\mathbf{P}\,\mathrm{div}_\Gamma(E_s(\cdot)) + cI$ (time-dependent problem).
- Gaussian curvature K is needed.

Finite element discretization: TraceFEM (or SFEM)

$\Gamma =$ zero level of g (level set function)

$g \approx I_h(g)$ (piecewise P_1 interpolation).

$\Gamma \approx \Gamma_h :=$ zero level of $I_h(g)$ (planar segments).



Under reasonable assumptions: $\mathrm{dist}(\Gamma, \Gamma_h) \leq c\, h^2$.

Trace FE space

$$V_h: \quad \text{piecewise linears on } \mathcal{T}_h$$

$$V_h^\Gamma := \{\, (\phi_h)_{|\Gamma_h} \mid \phi_h \in V_h \,\} \subset H^1(\Gamma_h).$$

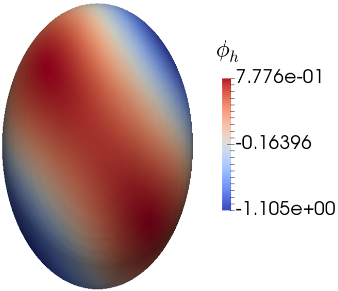
Numerical experiment

Use Galerkin technique with $\Gamma_h \approx \Gamma$ and trace space V_h^Γ for ϕ and ξ.
Some technicalities related to discrete kernel $\tilde{E}_h \approx \tilde{E}$.

Γ: ellipsoid with known Gaussian curvature.

Prescribed smooth solution ϕ.

ℓ	$\lVert\phi_h - \phi^e\rVert_{L^2(\Gamma_h)}$	EOC
1	$6.63 \cdot 10^{-1}$	
2	$2.04 \cdot 10^{-1}$	1.70
3	$5.81 \cdot 10^{-2}$	1.81
4	$1.50 \cdot 10^{-2}$	1.95
5	$3.67 \cdot 10^{-3}$	2.03

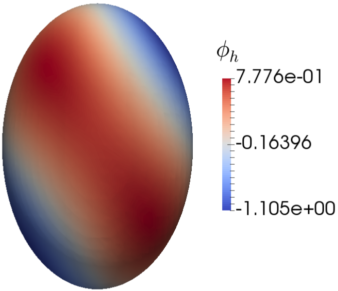


Extension to time-dependent Stokes

Straightforward approach:

- Stream function formulation applies (Γ simply connected).
- Coupled parabolic equations for $\phi(x,t)$, $\xi(x,t)$.
- Method of lines: TraceFEM in x, implicit Euler (CN) in t.

$\mathbf{u}_h = \mathbf{curl}_\Gamma \phi_h$
reconstructed from ϕ_h



Further development (error analysis): current research.

Concluding remarks

- Derivation of surface Navier-Stokes equations.
- Well-posed variational formulation of surface Stokes problem.
- Surface Helmholtz decomposition.
- Stream function formulation of surface Stokes problem.
- Trace FE discretization.

Further issues:

- Extension to (time-dependent) Navier-Stokes.
- Efficient reconstruction of $\mathbf{u}_h$ from stream function ϕ_h.
- Extension to evolving (simply connected) surface.
- Linear algebra issues (preconditioner).

Reference:

A. Reusken, *Stream Function Formulation of Surface Stokes Equations*, IGPM report 478, RWTH Aachen (2018)