Talagrand's inequalities of higher order and KKL's Theorem

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Outline of the talk

Gaussian case

- Talagrand's inequality and representation formula of the variance with semigroup.
- Another representation formula of the variance and Talagrand's inequality at order 2.

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Gaussian case

- Talagrand's inequality and representation formula of the variance with semigroup.
- Another representation formula of the variance and Talagrand's inequality at order 2.

Discrete case

- Talagrand's inequality on the discrete cube.
- Influence in Boolean analysis and KKL's Theorem.
- Talagrand's inequality at order 2 : from the Gaussian case to the discrete case.
- KKL's Theorem of order 2.

Talagrand's inequality

γ_n standard Gaussian measure on $\mathbb{R}^n$.

[Talagrand]

$f : \mathbb{R}^n \to \mathbb{R}$ smooth enough

$$\mathrm{Var}_{\gamma_n}(f) \leq C \sum_{i=1}^{n} \frac{\|\partial_i f\|_2^2}{1 + \log \frac{\|\partial_i f\|_2}{\|\partial_i f\|_1}}$$

Improves upon Poincaré's inequality.
proof ?

Semigroup tools

Ornstein-Uhlenbeck

$$P_t(f) = \int_{\mathbb{R}^n} f(xe^{-t} + \sqrt{1 - e^{-2t}}y)d\gamma_n(y) \quad t \geq 0, x \in \mathbb{R}^n$$

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Few properties

- Integration by parts $\int_{\mathbb{R}^n} f(-Lf)d\gamma_n = \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n$ with $L = \Delta - x \cdot \nabla$ and $|\cdot|$ Euclidean norm.
- Ergodicity $P_t(f) \to \mathbb{E}_{\gamma_n}[f] \quad t \to \infty$.
- Commutation $\nabla P_t = e^{-t} P_t \nabla \quad t \geq 0$.
- Hypercontractivity,

$$\|P_t f\|_q \leq \|f\|_{p(t)}, \quad p(t) = (q-1)e^{-2t} + 1, t > 0$$

Note : $p(t) < q$ (improves upon Jensen's inequality).

Representation formula

Interpolation by semigroup

$$\mathrm{Var}_{\gamma_n}(f) = 2\int_0^{\infty} e^{-2t} \int_{\mathbb{R}^n} |P_t \nabla f|^2 d\gamma_n dt$$

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Hypercontractivity

For $i = 1, \ldots, n$

$$\|P_t(\partial_i f)\|_2 \le \|\partial_i f\|_{p(t)} \quad p(t) = 1 + e^{-2t},\ t > 0.$$

Yields Talagrand's inequality (after some Hölder interpolation arguments)

Application

$X_1, \ldots, X_n$ i.i.d. $\mathcal{N}(0,1)$, $M_n = \max_{i=1,\ldots,n} X_i$

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Superconcentration

Talagrand's inequality (and some variants) useful tool in superconcentration theory to get subdiffusive variance bounds (cf. Chatterjee's book).

Examples

- First passage percolation.
- Gaussian polymers.
- maximum of stationary Gaussian sequences.
- ...

(Roughly superconcentration = classical concentration tools gives sub-optimal bounds)

Question :

Alternative variance representation formula

$$\Downarrow$$

Talagrand's inequality of order 2 ?

Representation formula, order one

$f : \mathbb{R}^n \to \mathbb{R}$ smooth enough, $|\cdot|$ Euclidean norm.

Theorem [Tanguy 2017]

$$\begin{aligned} \mathrm{Var}_{\gamma_n}(f) &= \left| \int_{\mathbb{R}^n} \nabla f d\gamma_n \right|^2 \\ &+ 2\int_0^\infty e^{-2u}(1-e^{-2u}) \int_{\mathbb{R}^n} \left| P_u(\nabla^2 f)\right|^2 d\gamma_n du \end{aligned}$$

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- L^2 decomposition (Hermite polynomials) + remainder with Ornstein-Uhlenbeck semi-group.
- Notice : inverse Poincaré's inequality immediate.

Proof

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$$K(s) - K(t) = \int_t^s K'(u)du$$

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$$K(s) - K(t) = \int_t^s K'(u)du$$

$s \to \infty$ by ergodicity $K(\infty) = \left| \int_{\mathbb{R}^n} \nabla f d\gamma_n \right|^2.$

By integration by parts ($\int_{\mathbb{R}^n} f(-Lf) d\gamma_n = \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n$)

and commutation property ($\nabla P_t = e^{-t} P_t \nabla$)

$$K'(u) = \frac{d}{du} \int_{\mathbb{R}^n} |P_u \nabla f|^2 d\gamma_n =$$

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Finally

$$K(t) = \left| \int_{\mathbb{R}^n} \nabla f d\gamma_n \right|^2 + 2\int_t^{\infty} e^{-2u} \int_{\mathbb{R}^n} e^{-2u} |P_u \nabla^2 f|^2 d\gamma_n du$$

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Then

$$\begin{aligned}\mathrm{Var}_{\gamma_n}(f) &= 2\int_0^\infty e^{-2t}\left|\int_{\mathbb{R}^n} \nabla f d\gamma_n\right|^2 dt \\ &+ 4\int_0^\infty e^{-2t}\int_t^\infty e^{-2u}\int_{\mathbb{R}^n} |P_u \nabla^2 f|^2 d\gamma_n du dt\end{aligned}$$

Conclude with Fubini's Theorem

First iteration

$$\begin{aligned}\mathrm{Var}_{\gamma_n}(f) &= \left|\int_{\mathbb{R}^n} \nabla f d\gamma_n\right|^2 \\ &+ 2\int_0^\infty e^{-2u}(1-e^{-2u})\int_{\mathbb{R}^n} \left|P_u(\nabla^2 f)\right|^2 d\gamma_n du\end{aligned}$$

First iteration

$$
\begin{aligned}
\mathrm{Var}_{\gamma_n}(f) &= \Big|\int_{\mathbb{R}^n} \nabla f d\gamma_n\Big|^2 \\
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\end{aligned}
$$

Note : iterate the procedure (set $K_2(t) = \int_{\mathbb{R}^n} \big|P_u(\nabla^2 f)\big|^2 d\gamma_n \ldots$)

First iteration

$$\mathrm{Var}_{\gamma_n}(f) = \left| \int_{\mathbb{R}^n} \nabla f d\gamma_n \right|^2 + 2\int_0^\infty e^{-2u}(1-e^{-2u}) \int_{\mathbb{R}^n} \left| P_u(\nabla^2 f) \right|^2 d\gamma_n du$$

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Théorème [T. 2017]

$p \geq 1$

$$\mathrm{Var}_{\gamma_n}(f) = \sum_{k=1}^{p} \frac{1}{k!} \left| \int_{\mathbb{R}^n} \nabla^k f d\gamma_n \right|^2 + \frac{2}{p!} \int_0^\infty e^{-2t}(1-e^{-2t})^p \int_{\mathbb{R}^n} \left| P_t(\nabla^{p+1} f) \right|^2 d\gamma_n dt$$

Talagrand's inequality at order 2

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Use hypercontractivity to bound the remainder term

$$
\begin{aligned}
R &= 2\sum_{i,j=1}^{n}\int_0^{\infty} e^{-2u}(1-e^{-2u})\int_{\mathbb{R}^n}\Big[P_u\big(\partial_{ij}f\big)\Big]^2 d\gamma_n du \\
&= 2\sum_{i,j=1}^{n}\int_0^{\infty} e^{-2u}(1-e^{-2u})\|P_u\big(\partial_{ij}f\big)\|_2^2 du
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Same proof as Talagrand's inequality with an improvement thanks to the additional factor $1-e^{-2u}$

Talagrand order 2

Theorem [T. 2017]

$$\mathrm{Var}_{\gamma_n}(f) \leq \left| \int_{\mathbb{R}^n} \nabla f d\gamma_n \right|^2 + C \sum_{i,j=1}^{n} \frac{\|\partial_{ij} f\|_2^2}{\left[1 + \log \frac{\|\partial_{ij} f\|_2}{\|\partial_{ij} f\|_1}\right]^2}$$

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Open questions

- Comparison between Talagrand's inequalities of order 1 and 2 ?
- Application in superconcentration theory ?

Boolean analysis

Historically Talagrand's inequality on $C_n = \{-1, 1\}^n$ with $\mu^n = (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)^{\otimes n}$.

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(Can also be proven by semi group argument)

with $D_i f(x) = \frac{f(x) - f\left(\tau_i(x)\right)}{2}$ $\quad \tau_i(x) = (x_1, \ldots, -x_i, \ldots, x_n)$, $x \in C_n$.

Influence and KKL's Theorem

$f : C_n \to \{0,1\}, \quad \mu^n = (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)^{\otimes n}$

Influence

$$I_i(f) = \mathbb{P}\big(f(X) \neq f\left(\tau_i(X)\right)\big), \quad \mathcal{L}(X) = \mu^n$$

Probability that coordinate i is pivotal for input X

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Theorem [Kalai-Kahn-Linial]

$$\forall f : C_n \to \{0,1\}, \exists i \in \{1,\dots,n\} \quad I_i(f) \geq c\frac{\log n}{n}$$

(optimal on Tribes functions)

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KKL's theorem can be proved by Talagrand's inequality

Link with Talagrand's inequality

$f \ : \ C_n \to \{0,1\}$

$$I_i(f) = \|D_i f\|_1 = \|D_i f\|_2^2, \quad i = 1, \dots, n$$

(up to numerical constants)

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Talagrand inequality in terms of influence

$$\mathrm{Var}_{\mu^n}(f) \le C \sum_{i=1}^{n} \frac{I_i(f)}{1 + \log \frac{1}{1/\sqrt{I_i(f)}}}.$$

application : KKL's Theorem

KKL's Theorem

If $\exists i \in \{1, \dots, n\}$ s.t. $I_i(f) \geq \frac{C}{\sqrt{n}}$ then $I_i(f) \geq C\frac{\log n}{n}$.

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$$\text{Assume that } \forall i \in \{1, \ldots, n\} \quad I_i(f) \leq \frac{C}{\sqrt{n}} \qquad (1)$$

Talagrand's inequalities implies

$$\exists i \in \{1, \ldots, n\} \quad \text{s.t.} \quad \frac{C}{n} \leq \frac{I_i(f)}{1+\log \frac{1}{1/\sqrt{I_i(f)}}} \qquad (2)$$

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Use assumption (1) to deduce $\frac{C}{n} \leq \frac{I_i(f)}{\log n}$ from (2).

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$f\ :\ \{-1,1\}^n \to \{0,1\}$ define

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$(i,j) \in \{1,\ldots,n\}^2$.

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Beware $I_{(i,i)}(f) = I_i(f)$! Similarly (up to numerical constants)

$$I_{(i,j)}(f) = \|D_{ij}f\|_2^2 = \|D_{ij}f\|_1, \qquad (\text{with } D_{ij} = D_i \circ D_j)$$

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Talagrand of order 2 on the cube ?

Semigroup proof ?

Similarities with Gaussian setting

Bonami-Beckner semigroup

$$Q_t f(x) = \int_{C_n} f(y) \prod_{i=1}^{n} (1 + e^{-t} x_i y_i) d\mu^n(y)$$

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Few properties

- Integration by parts $\int_{C_n} f(-Lf) d\mu^n = \int_{C_n} |Df|^2 d\mu^n$ with $L = \frac{1}{2} \sum_{i=1}^{n} D_i$.
- Ergodicity $Q_t(f) \underset{t \to \infty}{\longrightarrow} \int_{C_n} f d\mu^n$.
- $(Q_t)_{t \geq 0}$ hypercontractive [Bonami-Beckner].

Representation formula

Variance representation formula [Bobkov-Götze-Houdré]

$$\mathrm{Var}_{\mu^n}(f) = 2\int_0^{\infty}\sum_{i=1}^{n}\int_{C_n}\left[Q_s(D_i f)\right]^2 d\mu^n ds$$

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Difference with Gaussian setting

- $D_{ii} = D_i \circ D_i = D_i$.
- $D_i Q_s = Q_s D_i$, (no e^{-s} with commutation).

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Same proof as Gaussian case ?

Some issues

Issue I

$D_i Q_s = Q_s D_i$, (no e^{-s} with commutation)

Solution : μ^n satisfies Poincaré inequality $\Rightarrow$ exponentiel decay for the variance along $(Q_t)_{t\geq 0}$.

$$\mathrm{Var}_{\mu^n}(Q_t f) \leq e^{-2t}\|f\|_2^2, \quad t \geq 0$$

Issue I

First step

- $D_i f$ is centered :

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  $2\int_{C_n} D_i f d\mu^n = \int_{C_n} f(x)d\mu^n(x) - \int_{C_n} f(\tau_i x)d\mu^n(x) = 0.$

Issue I

First step

- $D_i f$ is centered :
  $2\int_{C_n} D_i f d\mu^n = \int_{C_n} f(x)d\mu^n(x) - \int_{C_n} f(\tau_i x)d\mu^n(x) = 0.$
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Double the time $+$ Poincaré inéquality

$$\mathrm{Var}_{\mu^n}(f) = 4\int_0^\infty \sum_{i=1}^n \int_{C_n} \left[Q_s \circ Q_s(D_i f)\right]^2 d\mu^n ds$$

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&\leq 4\int_0^\infty \sum_{i=1}^n e^{-2s} \int_{C_n} \left[Q_s(D_i f)\right]^2 d\mu^n ds
\end{aligned}$$

Follow the Gaussian case

Set $K(s) = \int_{C_n} |Q_s(Df)|^2 d\mu^n$ with $Df = (D_1 f, \dots, D_n f)$.

Proceed as the Gaussian case, use Poincaré's trick again

$$\mathrm{Var}_{\mu^n}(f) \leq 8 \int_0^\infty e^{-2s}(1 - e^{-4s}) \sum_{i,j=1}^n \int_{C_n} Q_s^2(D_{ij} f) d\mu^n ds$$

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On the diagonal : $D_{ii} = D_i$.

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$$I^{\leq s_0} = 8 \int_0^{s_0} e^{-2s}(1 - e^{-4s}) \sum_{i=1}^n \int_{C_n} Q_s^2(D_i f) d\mu^n ds$$

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Deal with $I^{\leq s_0}$

$$I^{\leq s_0} = 8\int_0^{s_0} e^{-2s}(1-e^{-4s})\sum_{i=1}^{n}\int_{C_n} Q_s^2(D_i f)\,d\mu^n\,ds$$

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&\leq 32 s_0 \int_0^{\infty}\sum_{i=1}^{n}\int_{C_n} Q_s^2(D_i f)\,d\mu^n ds
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Choose s_0 s.t. $16 s_0 \leq \frac{1}{2}$.

Update estimates

So far

$$\mathrm{Var}_{\mu^n}(f) \leq I^{\leq s_0} + I^{\geq s_0} + C \sum_{i \neq j} \frac{\|D_{ij} f\|_2^2}{\left[1 + \log \frac{\|D_{ij} f\|_2}{\|D_{ij} f\|_1}\right]^2}.$$

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With $16 s_0 \leq \frac{1}{2}$, we obtain $I^{\leq s_0} \leq \frac{1}{2} \mathrm{Var}_{\mu^n}(f)$.

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$$\frac{1}{2}\mathrm{Var}_{\mu^n}(f) \leq I^{\geq s_0} + C \sum_{i \neq j} \frac{\|D_{ij} f\|_2^2}{\left[1 + \log \frac{\|D_{ij} f\|_2}{\|D_{ij} f\|_1}\right]^2}.$$

Deal with $I^{\geq s_0}$

For the other term

$$I^{\geq s_0} = 8 \sum_{i=1}^{n} \int_{s_0}^{\infty} e^{-2s}(1 - e^{-4s}) \int_{C_n} Q_s^2(D_i f) d\mu^n ds.$$

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Set for $i = 1, \ldots, n$

$$I_i^{\geq s_0} = 8 \int_{s_0}^{\infty} e^{-2s}(1 - e^{-4s}) \int_{C_n} Q_s^2(D_i f) d\mu^n ds.$$

That is to say : $I^{\geq s_0} = \sum_{i=1}^{n} I_i^{\geq s_0}$.

Deal with $I^{\geq s_0}$

Use hypercontractivity : $\|Q_u D_i f\|_2^2 \leq \|D_i f\|_{1+e^{-2u}}^2, \quad u > 0.$

on $I_i^{\geq s_0}, \quad \forall i = 1, \ldots, n.$

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Conclusion : $I^{\geq s_0} = \sum_{i=1}^n I_i^{\geq s_0} \leq C \sum_{i=1}^n \|D_i f\|_{1+e^{-2s_0}}^2.$

Conclusion of the proof

Finally, we have proven

Talagrand inequality of order 2 [T. 2017]

$$\mathrm{Var}_{\mu^n}(f) \leq C \sum_{i=1}^{n} \|D_i f\|_{1+e^{-s_0}}^2 + C \sum_{i \neq j} \frac{\|D_{ij} f\|_2^2}{\left[1 + \log \frac{\|D_{ij} f\|_2}{\|D_{ij} f\|_1}\right]^2}$$

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Application : KKL of order 2

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$f\ :\ C_n \to \{0,1\}$

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Either $\exists i \in \{1, \ldots, n\}$

$$I_i(f) \geq c\left(\frac{1}{n}\right)^{1/1+\eta} \quad 0 < \eta < 1$$

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Or $\exists i \neq j \in \{1,\ldots,n\}$

$$I_{(i,j)}(f) \geq c\left(\frac{\log n}{n}\right)^2$$

Same proof as original KKL's Theorem.
Tribes functions optimal for the 2nd alternative.

Open questions and research projects

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Prove superconcentration for

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- Branching random walk ?

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Research projects

Threshold phenomenon ?
(Russo/Margulis's Lemma for biased measure μ_p^n on $\{-1,1\}^n$ + Talagrand of order 2 ?).

(Talagrand and Russo/Margulis of order 2 okay for $\mu_p^n = \left(p\delta_{-1} + q\delta_{+1}\right)^{\otimes n}$ with $0 < p < 1, \quad q = 1 - p$)

Thank you for your attention