Asymptotic behavior of Berry random wave model

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Based on a collaboration with G. Peccati and M. Rossi

High Dimensional Probability

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Chladni plate experiments

Let us start this talk with a video reproducing Chladni plate experiments from 1787: if you sprinkle fine sand uniformly over a drumhead and then make it vibrate, the grains of sand will collect in characteristic spots, called **Chladni patterns.**

If we speed up the vibration frequency, then the pattern formed by the grains lacks its structure and becomes diffuse again, until another mode of pure vibration is reached, so that the sprinkled salt organizes itself again into a new pattern, which is more complex than the previous one.

Nodal set

- ⋆ Terminology: the **nodal set** of an **eigenfunction** f of the Laplacian is simply the set of its zeros, that is, $f^{-1}(0)$.

- ⋆ In original **Chladni plate experiments**, we actually visualize several nodal sets, corresponding to different eigenfunctions of the Laplacian on the square. (The video corresponds rather to the bilaplacian, because the vibration is delivered at the center of the plate, and the boundary is not fixed.)

- ⋆ Nodal sets are also of interest in **quantum mechanics**. In this context, an L^2-normalized eigenfunction (that is, $\|f\|_{L^2} = 1$) can be seen as the probability density of a free particle in the energy state associated with f. The nodal set $f^{-1}(0)$ may be interpreted as the set of locations where the particle is least likely to be found.

Eigenfunctions of the Laplacian

- ⋆ Fix $E > 0$ (energy), and consider the eigenspace of the usual Laplacian $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ on $\mathbb{R}^2$ associated with the eigenvalue $-4\pi^2 E$, that is, the set of eigenfunctions $f : \mathbb{R}^2 \to \mathbb{R}$ that satisfy
$$\Delta f = -4\pi^2 E\, f.$$

- ⋆ It is a huge set, that in particular contains any (limit of) linear combination(s) of
$$x \mapsto e^{2i\pi\sqrt{E}\langle z,x\rangle}$$
with $z \in \mathbb{S}^1$ (unit circle of $\mathbb{R}^2$), for instance
$$\sum_{n=1}^{N} \left\{ a_n e^{2i\pi\sqrt{E}\langle z_n,x\rangle} + \overline{a_n} e^{-2i\pi\sqrt{E}\langle z_n,x\rangle} \right\},$$
where $(z_n) \subset \mathbb{S}^1$ and $(a_n) \subset \mathbb{C}$.

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OUR RANDOM MODEL – BERRY, 1977

- ⋆ Fix $E > 0$. The **Berry random wave model** on $\mathbb{R}^2$ with parameter E, written $B_E = \{B_E(x) : x \in \mathbb{R}^2\}$, is defined as

$$B_E(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^1} e^{2i\pi\sqrt{E}\langle z,x\rangle} G(dz),$$

  where G = Hermitian Gaussian measure on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$.
- ⋆ Equivalently, B_E is the unique (in law) centered, isotropic Gaussian field on $\mathbb{R}^2$ such that $\Delta B_E = -4\pi^2 E\, B_E$.
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$$\mathbb{E}[B_E(x)B_E(y)] = J_0(2\pi\sqrt{E}\|x-y\|)$$

  where J_0 = Bessel function of the 1st kind.

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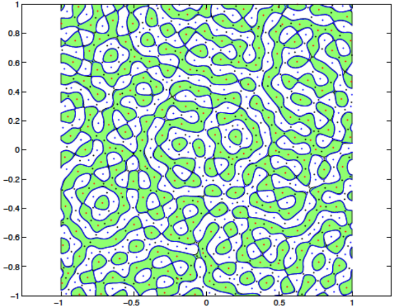
Geometry of random nodal sets

In this talk, we are interested in the **high-energy** (as $E \to \infty$) **geometry** of the random nodal set

$$B_E^{-1}(0) \cap \mathcal{D} = \{x \in \mathcal{D} : \, B_E(x) = 0\},$$

where $\mathcal{D}$ is a given compact set with smooth boundary.



Courtesy of D. Belyaev

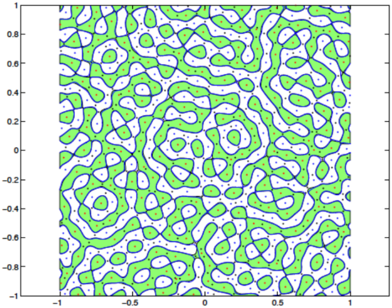
($B_E^{-1}(0)$ is the reunion of smooth curves with no intersection.)

Nodal length

- ⋆ More precisely our aim is to study the asymptotic behaviour, as $E \to \infty$, of the (random) **nodal length** $\mathcal{L}_E$, defined as

$$\mathcal{L}_E := \text{total length}\{B_E^{-1}(0) \cap \mathcal{D}\}.$$

- ⋆ Can we evaluate $\mathbb{E}[\mathcal{L}_E]$? $\mathbf{Var}(\mathcal{L}_E)$? Can we prove a CLT?



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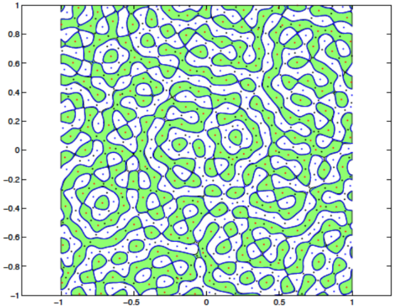

Expectation, variance and fluctuations

⋆ *Berry* (J. Phys. A, 2002): semi-rigorous computations lead first to

$$\mathbb{E}[\mathcal{L}_E] = \frac{\pi\sqrt{E}}{\sqrt{2}}.$$

⋆ For the order of magnitude of the **variance**, the natural **guess** would be $\sim \sqrt{E}$, since we might legitimately expect that

$$E^{1/4}\left\{\frac{\mathcal{L}_E}{\sqrt{E}} - \frac{\pi}{\sqrt{2}}\right\} \overset{\text{law}}{\to} \text{n.d. limit}$$

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Expectation, variance and fluctuations

- ⋆ *Berry* (J. Phys. A, 2002): semi-rigorous computations lead in fact to

$$\mathbf{Var}(\mathcal{L}_E) \sim \frac{\text{area}\mathcal{D}}{512\pi} \log E \quad \text{as } E \to \infty.$$

- ⋆ According to Berry himself, such a variance reduction "... *results from a cancellation whose meaning is still obscure*..." (Berry (2002), p. 3032)

- ⋆ Nothing (rigorous or semi-rigorous) is known for this model about fluctuations (CLT ?).

- ⋆ Several other related models have been studied so far. We cite only two such references: Rudnick and Wigman (*Ann. IHP* 2007), Krishnapur, Kurlberg and Wigman (*Ann. Math.* 2013).

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OUR RESULT

Theorem (Nourdin, Peccati, Rossi) As $E \to \infty$,

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Strategy of proof

We use the representation (*coarea formula*)

$$\mathcal{L}_E = \int_{\mathcal{D}} \delta_0(B_E(x)) \|\nabla B_E(x)\| dx \quad (\text{in } L^2(\mathbb{P}))$$

to deduce the Wiener chaos expansion of $\mathcal{L}_E$.

A small digression: two situations

- ⋆ In many instances, second order results for sequences of the form $F_k = \mathbb{E}[F_k] + \sum_{q=1}^{\infty} \text{proj}(F_k|C_q)$ can be deduced from the behaviour of chaotic projections.
- ⋆ Situation 1: F_k is **dominated by one of its projection**, and it inherits the rigid asymptotic structure of sequences inside a Wiener chaos (see the next slide).
- ⋆ Situation 2: **no single projection dominates**, and interactions have to be dealt with.

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A SMALL DIGRESSION: A RIGID STRUCTURE

Fix $q \geq 2$, and let $F_k \in C_q$, $k \geq 1$ with variance 1 (say).

- $\star$ *Nourdin and Poly (2013)*: If $F_k \overset{\text{law}}{\to} Z$, then Z has a density.
- $\star$ *Nualart and Peccati (2005)*: $F_k \overset{\text{law}}{\to} N(0,1)$ iff $\mathbb{E}F_k^4 \to 3$.
- $\star$ *Peccati and Tudor (2005)*: componentwise convergence towards Gaussian implies joint convergence.
- $\star$ *Nourdin and Peccati (2009)*: $F_k \overset{\text{law}}{\to} (N(0,1)^2 - 1)/\sqrt{2}$ iff $\mathbb{E}F_k^4 - 12\mathbb{E}F_k^3 \to -36$.
- $\star$ *Nourdin and Rosiński (2014) & Nourdin, Nualart and Peccati (2015)*: if $H_k \in C_p$ (with variance 1), then F_k, H_k are asymptotically independent iff $\mathbf{Cov}(H_k^2, F_k^2) \to 0$.
- $\star$ In many instances, one can also give a bound for strong distances, such as the total variation distance.

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Cancellation and domination phenomenon

Theorem (Nourdin, Peccati, Rossi, 2017)

1. **(Exact cancellation)** For every fixed $E > 0$,

$$\mathrm{proj}(\mathcal{L}_E \,|\, C_{2q+1}) = 0, \quad q \geq 0,$$

and $\mathrm{proj}(\mathcal{L}_E \,|\, C_2)$ reduces to a "negligible boundary term" as $E \to \infty$ ($\to$ Berry obscure cancellation)

2. **(4^{th} chaos dominates)** Set $\widetilde{\mathcal{L}_E} = \frac{\mathcal{L}_E - \mathbb{E}(\mathcal{L}_E)}{\mathbf{Var}(\mathcal{L}_E)^{1/2}}$. Then, as $E \to \infty$,

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Sketch of the proof

What is the chaotic expansion of

$$\mathcal{L}_E = \int_{\mathcal{D}} \delta_0(B_E(x)) \|\nabla B_E(x)\| dx \quad ?$$

Sketch of the proof: covariances

For any $k \in \{1,2\}$ and $x \neq y$, we have

$$\begin{aligned}
\mathbb{E}[B_E(x)B_E(y)] &= J_0(2\pi\sqrt{E}\|x-y\|)\\
\mathbb{E}[B_E(x)\partial_k B_E(y)] &= \sqrt{2}\,\frac{x_k-y_k}{\|x-y\|}\,J_1(2\pi\sqrt{E}\|x-y\|)\\
\mathbb{E}[\partial_k B_E(x)\partial_k B_E(y)] &= J_0(2\pi\sqrt{E}\|x-y\|)\\
&\quad + \left(1-2\frac{(x_k-y_k)^2}{\|x-y\|^2}\right)J_2(2\pi\sqrt{E}\|x-y\|)\\
\mathbb{E}[\partial_1 B_E(x)\partial_2 B_E(y)] &= -2\frac{(x_1-y_1)(x_2-y_2)}{\|x-y\|^2}J_2(2\pi\sqrt{E}\|x-y\|).
\end{aligned}$$

In particular, $\mathbb{E}[B_E(x)^2]=1$ and $\mathbf{Var}(\partial_k B_E(x)) = 2\pi^2 E$. Due to this latter fact and because Hermite polynomials behave well with respect to the *standard* Gaussian, we introduce the renormalized gradient as

$$\widetilde{\nabla} = (\widetilde{\partial_1}, \widetilde{\partial_2}) := \frac{\nabla}{\sqrt{2\pi^2 E}} = \left(\frac{\partial_1}{\sqrt{2\pi^2 E}}, \frac{\partial_2}{\sqrt{2\pi^2 E}}\right).$$

Sketch of the proof: updated expression

Using $\widetilde{\nabla}$, the expression of $\mathcal{L}_E$ becomes

$$\mathcal{L}_E = \sqrt{2\pi^2 E} \int_{\mathcal{D}} \delta(B_E(x)) \|\widetilde{\nabla} B_E(x)\| dx.$$

We observe that $B_E(x)$, $\widetilde{\partial_1} B_E(x)$ and $\widetilde{\partial_2} B_E(x)$ are independent standard Gaussian random variables for each <u>fixed</u> x.

Sketch of the proof: chaotic expansion

- $\star$ We have

$$\delta(B_E(x)) = \sum_{l=0}^{\infty} \beta_{2l} H_{2l}(B_E(x)),$$

where $\beta_k = \frac{1}{k!}\mathbb{E}[\delta(N)H_k(N)]$ with $N \sim N(0,1)$. In particular, $\beta_0 = \frac{1}{\sqrt{2\pi}}$, $\beta_2 = -\frac{1}{2\sqrt{2\pi}}$ and $\beta_4 = \frac{1}{8\sqrt{2\pi}}$.

Sketch of the proof: chaotic expansion

⋆ We have

$$\begin{aligned} \|\widetilde{\nabla} B_E(x)\| &= \sum_{k,l=0}^{\infty} \alpha_{2k,2l} H_{2k}(\widetilde{\partial}_1 B_E(x)) H_{2l}(\widetilde{\partial}_2 B_E(x)), \\ \text{where } \alpha_{0,0} &= \mathbb{E}\left[\|\mathbf{N}\|\right] = \frac{\sqrt{2\pi}}{2} \\ \alpha_{2,0} &= \alpha_{0,2} = \frac{1}{2}\mathbb{E}\left[\|\mathbf{N}\| H_2(N_1)\right] = \frac{\sqrt{2\pi}}{8} \\ \alpha_{4,0} &= \alpha_{0,4} = \frac{1}{4!}\mathbb{E}\left[\|\mathbf{N}\| H_4(N_1)\right] = -\frac{\sqrt{2\pi}}{128} \\ \alpha_{2,2} &= \frac{1}{4}\mathbb{E}\left[\|\mathbf{N}\| H_2(N_1) H_2(N_2)\right] = -\frac{\sqrt{2\pi}}{64}, \end{aligned}$$

with $\mathbf{N} = (N_1, N_2) \sim N_2(0, I_2)$.

Sketch of the proof: second chaos

- $\star$ The projection $\mathcal{L}_E[2]$ of $\mathcal{L}_E$ onto the second chaos is given by

$$\begin{aligned}\mathcal{L}_E[2] \;&=\; \sqrt{2\pi^2 E}\left\{\beta_2\alpha_{0,0}\int_{\mathcal{D}} H_2(B_E(x))dx + \beta_0\alpha_{0,2}\int_{\mathcal{D}} H_2(\widetilde{\partial}_1 B_E(x))dx\right.\\ &\qquad\qquad \left. + \beta_0\alpha_{2,0}\int_{\mathcal{D}} H_2(\widetilde{\partial}_2 B_E(x))dx\right\}\\ &=\; \frac{\pi}{8}\sqrt{2E}\left\{-2\int_{\mathcal{D}} B_E(x)^2 dx + \int_{\mathcal{D}} \|\widetilde{\nabla} B_E(x)\|^2 dx\right\}.\end{aligned}$$

- $\star$ The first Green identity asserts that

$$\int_{\mathcal{D}} \|\nabla B_E(x)\|^2 dx = -\int_{\mathcal{D}} B_E(x)\Delta B_E(x)dx + \int_{\partial\mathcal{D}} B_E(x)\langle \nabla B_E(x), n(x)\rangle dx$$

where $n(x)$ denotes the outward pointing unit normal at x.

Sketch of the proof: second chaos

- ⋆ As a result,

$$\begin{aligned}\int_{\mathcal{D}} \|\widetilde{\nabla} B_E(x)\|^2 dx &= \frac{1}{2\pi^2 E} \int_{\mathcal{D}} \|\nabla B_E(x)\|^2 dx \\ &= 2\int_{\mathcal{D}} B_E(x)^2 dx + \frac{1}{2\pi^2 E}\int_{\partial\mathcal{D}} B_E(x)\langle \nabla B_E(x), n(x)\rangle dx,\end{aligned}$$

  implying in turn that

$$\mathcal{L}_E[2] = \frac{1}{8\pi\sqrt{2E}} \int_{\partial\mathcal{D}} B_E(x)\langle \nabla B_E(x), n(x)\rangle dx.$$

- ⋆ We deduce

$$\begin{aligned}\mathbf{Var}(\mathcal{L}_E[2]) &\leq \frac{1}{128\pi^2 E}\int_{\partial\mathcal{D}} \mathbb{E} B_E(x)^2 dx \times \int_{\partial\mathcal{D}} \mathbb{E}\|\nabla B_E(x)\|^2 dx \\ &= \frac{1}{64}\,\text{perimeter}(\mathcal{D})^2 = O(1).\end{aligned}$$

Sketch of the proof: fourth chaos

⋆ We have that $\mathcal{L}_E[4]$ is given by

$$\frac{\sqrt{2\pi^2 E}}{128}\left\{8\int_{\mathcal{D}} H_4(B_E(x))dx - \int_{\mathcal{D}} \big(H_4(\widetilde{\partial}_1 B_E(x)) + H_4(\widetilde{\partial}_2 B_E(x))\big)dx \right.$$
$$-2\int_{\mathcal{D}} H_2(\widetilde{\partial}_1 B_E(x))H_2(\widetilde{\partial}_2 B_E(x))dx$$
$$\left. -8\int_{\mathcal{D}} H_2(B_E(x))\big(H_2(\widetilde{\partial}_1 B_E(x)) + H_2(\widetilde{\partial}_2 B_E(x))\big)dx\right\}.$$

⋆ For instance we have, setting $a(x) = J_0(2\pi\|x\|)$,

$$\mathbf{Var}\Big(\int_{\mathcal{D}} H_4(B_E(x))dx\Big) = 24\int_{(\sqrt{E}\mathcal{D})^2} J_0(2\pi\|x-y\|)^4 dxdy$$

$$= \; 24\int_{\sqrt{E}\mathcal{D}} dx \int_{-x+\sqrt{E}\mathcal{D}} J_0(2\pi\|u\|)^4 du \sim 9\,\frac{\text{area}(\mathcal{D})}{\pi^3} \times \frac{\log E}{E},$$

since $J_0(2\pi r) \sim \frac{1}{\pi\sqrt{r}}\cos(2\pi r - \frac{\pi}{4})$ as $r \to \infty$.

Sketch of the proof: CLT

- ⋆ We use the seminal results of Nualart-Peccati (aka the fourth moment theorem) and Peccati-Tudor .
- ⋆ It has strong similarities with Breuer-Major theorem.

Nodal points

- Another quantity of interest for physicians is the number $\mathcal{N}_E$ of **nodal points**, that is, the number of intersection points between B_E and an independent copy $\widehat{B}_E$ on the region $\mathcal{D} \subset \mathbb{R}^2$.
- It is given (*coarea formula*) by

$$\mathcal{N}_E = \int_{\mathcal{D}} \delta(B_E(x))\delta(\widehat{B}_E(x))\left|\mathrm{Jac}_{B_E,\widehat{B}_E}(x)\right|dx.$$

- Can we characterize the fluctuations of $\mathcal{N}_E$ as $E \to \infty$?

Expectation, variance and fluctuations

⋆ *Berry* (J. Phys. A, 2002): semi-rigorous computations lead to

$$\mathbf{Var}(\mathcal{N}_E) \sim \frac{11\mathrm{area}\mathcal{D}}{32\pi} E \log E.$$

⋆ Nothing (rigorous or semi-rigorous) is known for this model about fluctuations (CLT ?).

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Our result

Theorem (Nourdin, Peccati, Rossi, 2017) As $E \to \infty$,

- $$\mathbf{Var}(\mathcal{N}_E) \sim \frac{11\text{area}\mathcal{D}}{32\pi} E \log E.$$

- A CLT for $\mathcal{N}_E$ holds:

$$\frac{\mathcal{N}_E - \mathbb{E}[\mathcal{N}_E]}{\mathbf{Var}(\mathcal{N}_E)^{1/2}} \to N(0,1).$$

THE END!
Questions?

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