Hierarchical Convex Optimization with Proximal Splitting Operators

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Based on joint work with Masao Yamagishi

Dedicated to the memory of Jonathan M. Borwein

Splitting Algorithms, Modern Operator Theory, & Applications
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Convex Optimization - A typical form :

$(\mathcal{X}, \langle\cdot,\cdot\rangle_{\mathcal{X}}, \|\cdot\|_{\mathcal{X}}), (\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}}, \|\cdot\|_{\mathcal{K}})$: Real Hilbert Spaces

$f \in \Gamma_0(\mathcal{X}),\ g \in \Gamma_0(\mathcal{K}),\ A : \mathcal{X} \to \mathcal{K}$: Bdd linear

$$\text{(P)} \quad \underset{x \in \mathcal{X}}{\text{minimize}}\ f(x) + g(Ax)$$

has been playing a central role in Inverse Problems because

$f \in \Gamma_0(\mathcal{X}),\ g_i \in \Gamma_0(\mathcal{K}_i),\ A_i : \mathcal{X} \to \mathcal{K}_i$: Bdd linear

$$\textbf{(Q)} \quad \underset{\boldsymbol{x \in \mathcal{X}}}{\textbf{minimize}}\ \boldsymbol{f(x) + \sum_{i=1}^{M} g_i(A_i x)}$$

can be handled as an instance of (P) by

$$\mathcal{K} := \mathcal{K}_1 \times \cdots \times \mathcal{K}_M,\ g := \bigoplus_{i=1}^{M} g_i \text{ and } Ax := (A_1 x, \ldots, A_M x)$$

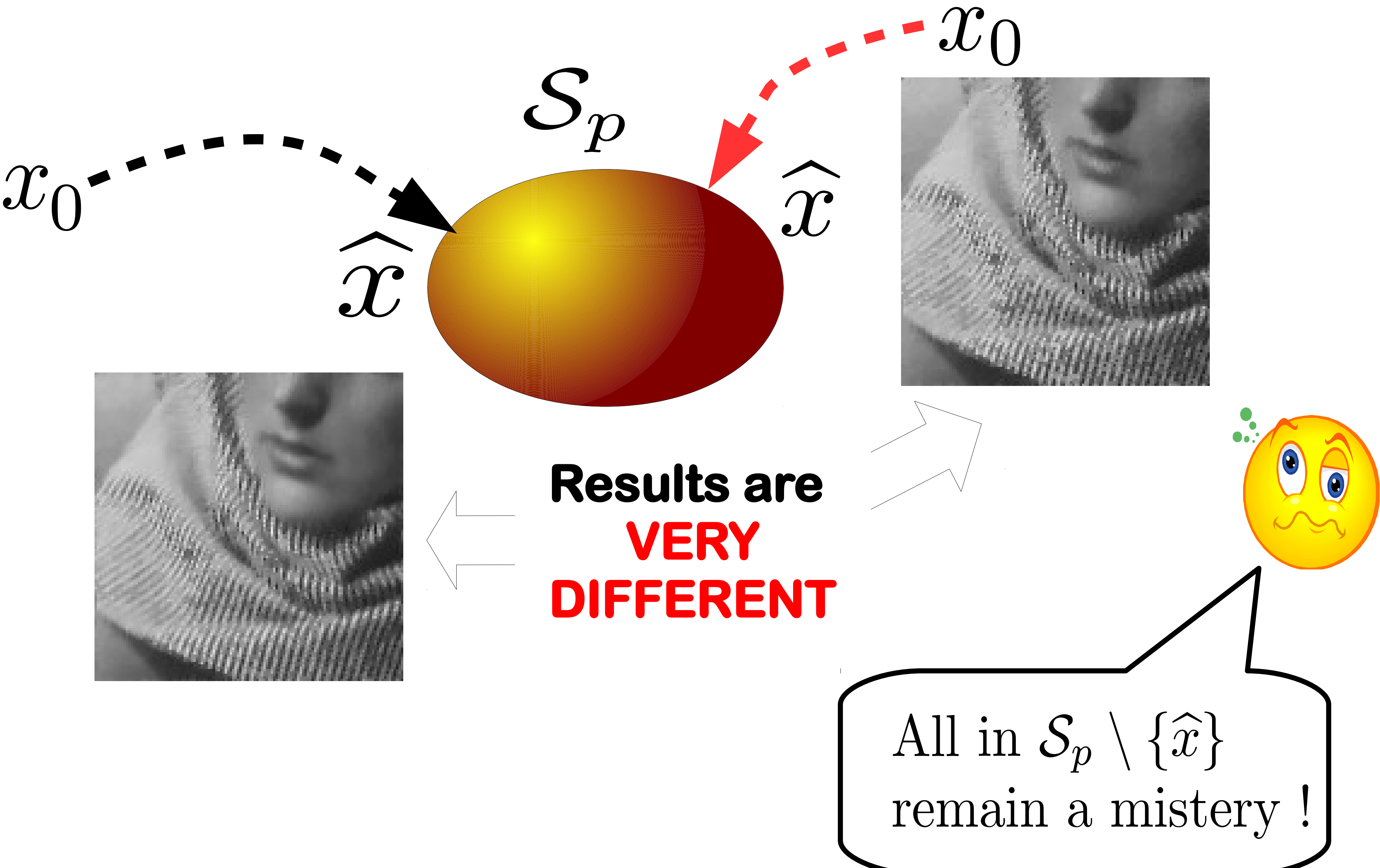
But

Almost all existing algorithms achieve convergence to only one unspecial solution :

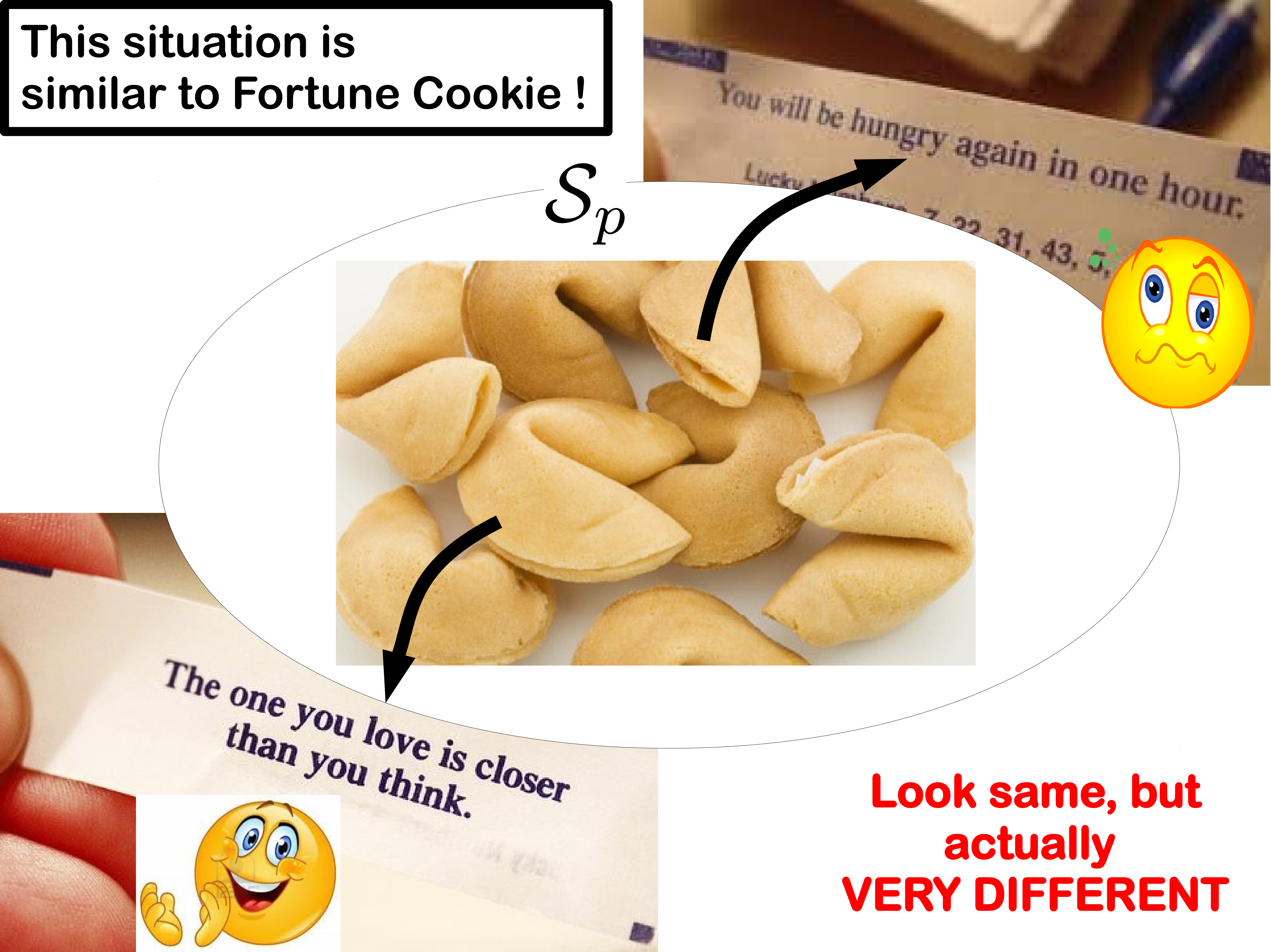
$$x^\star \in \mathcal{S}_p := \arg\min_{x \in \mathcal{X}} f(x) + g(Ax) \neq \emptyset.$$

Other solutions in $\mathcal{S}_p \setminus \{x^\star\}$ remain mistery !

Imagine, e.g., convex feasibility problems !


x_0
$\mathcal{S}_p$
x_0
$\widehat{x}$
$\widehat{x}$
Results are
VERY
DIFFERENT
All in $\mathcal{S}_p \setminus \{\widehat{x}\}$
remain a mistery !

This situation is similar to Fortune Cookie !



Look same, but actually VERY DIFFERENT

Challenges for strategic convergence are found, e.g,

Better limit

Superiorization:

[Censor-Davidi-Herman '10], [Herman-Garduno-Davidi-Censor '12]

An idea to incorporate a faborable attribute into a given iterative algorithm, without changing the inherent desired properties of the algorithm.

We are trying to find Best limit :

Hierarchical convex optimization :

[Yamada-Ogura-Shirakawa '02], [Yamada-Yukawa-Yamagishi '11], [Ono-Yamada '15], [Yamagishi-Yamada '17]

Best limit in what sense ?

Hierarchical Convex Optimization

Suppose $\mathcal{S}_p := \arg\min_{x \in \mathcal{X}} f(x) + g(Ax) \neq \varnothing$ &

$\Psi \in \Gamma_0(\mathcal{X})$ **is desired to be minimized additionally, i.e.,**

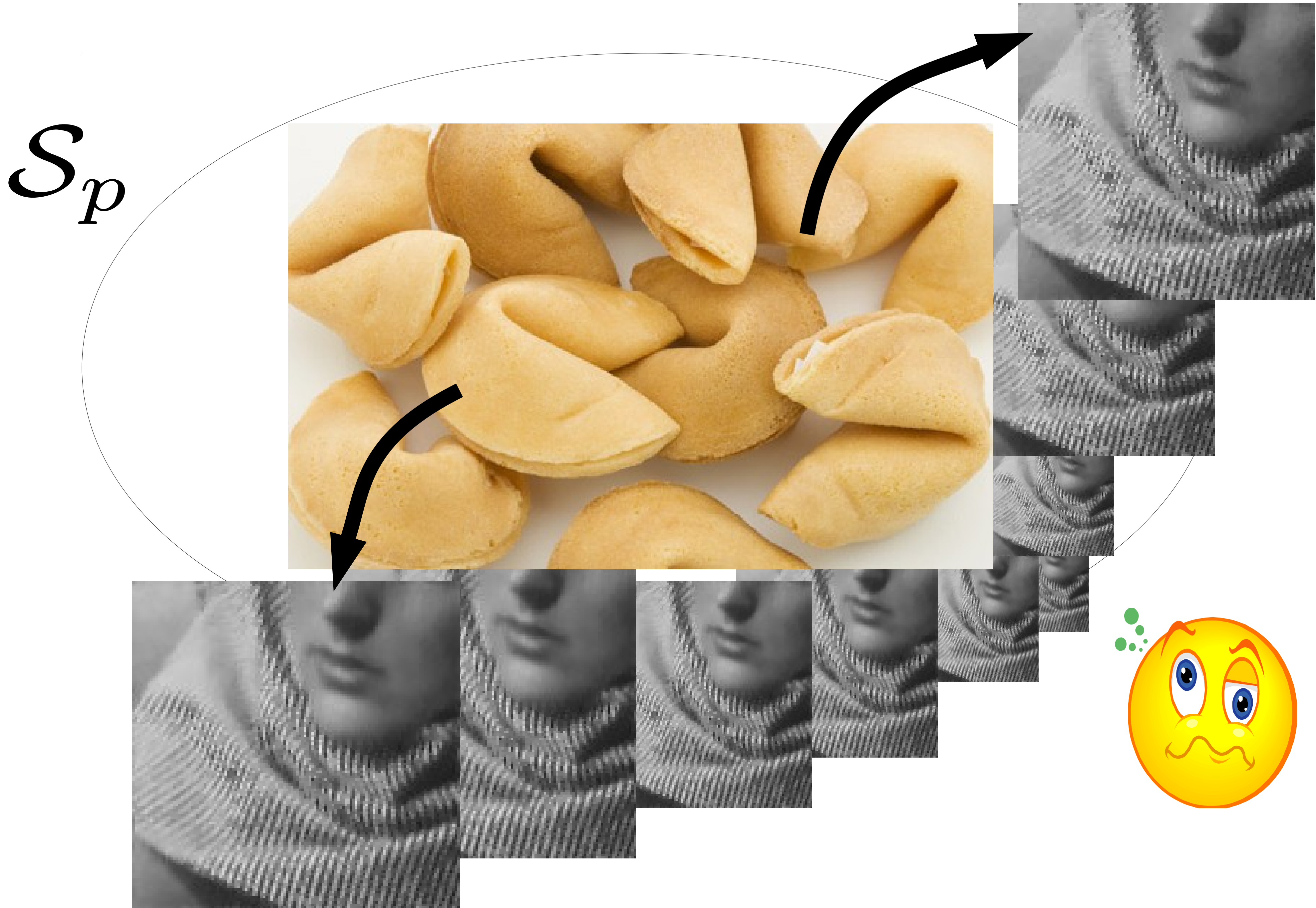
Minimize	$\Psi(x^\star)$	⇦ 2nd stage optimization
Subject to	$x^\star \in \mathcal{S}_p$	⇦ The set of all solutions of 1st stage optimization

For this challenging mission impossible, we need at least

1. Exploiting **Full information** on $\mathcal{S}_p$ (usually infinite set in $\mathcal{X}$).
2. Mathematically sound **algorithmic ideas** to minimize Ψ over $\mathcal{S}_p$.

Hierarchical convex optimization casts a question:

Can we choose a best one without crunching all cookies ?



PART I

Preliminaries :

How can we capture full information on the solution set of Convex Optimization Problem ?

PART II

Hierarchical Convex Optimization :

How can we choose a Best Fortune Cookie

without crunching all cookies ?

PART III

Application to

State-of-the-art Statistiacal Estimation Technique

A Hierarchical Enhancement of Lasso

Convex Optimization Problem

defined on a Real Hilbert Space $\mathcal{X}$

Minimize $\varphi : \mathcal{X} \to (-\infty, \infty]$

where $\varphi \in \Gamma_0(\mathcal{X})$

Proper	$\mathrm{dom}\varphi := \{x \in \mathcal{X} \mid \varphi(x) < \infty\} \neq \varnothing$
Lower Semi-continuous	$(\forall \alpha \in \mathbb{R})\ \mathrm{lev}_{\leq\alpha}(\varphi) := \{x \in \mathcal{X} \mid \varphi(x) \leq \alpha\}$ is Closed in $\mathcal{X}$
Convex	$(\forall x, y \in \mathrm{dom}\varphi, \forall \lambda \in (0,1))$ $\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$

The solution sets of convex optimization problems can often be expressed as

$$\arg\min_{x \in \mathcal{X}} \varphi(x) = \Xi\left(\mathrm{Fix}(T)\right)$$

where

$T : \mathcal{H} \to \mathcal{H}$: a **nonexpansive operator** defined on a certain Hilbert space $\mathcal{H}$

i.e. $\|T(x) - T(y)\| \leq \|x - y\| \qquad (\forall x, y \in \mathcal{H})$

$\Xi : \mathcal{H} \to 2^{\mathcal{X}}$: a certain set-valued operator

Computable Nonexpansive Operators for Convex Optimization

. . . Primal-Dual splitting Operator . . .

ADMM Operator (Dual Variant of Douglas-Rachford splitting operator)

Great !

Augmented Lagrangian Operator

Forward-Backward splitting operator

Proximity operator

$$\mathrm{prox}_f := (\mathrm{Id} + \partial f)^{-1} \quad (f \in \Gamma_0(\mathcal{H}))$$

$$: \mathbf{x} \mapsto \arg\min_{\mathbf{y} \in \mathcal{H}} \left[f(\mathbf{y}) + \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2 \right]$$

Metric projection

Nonexpansive Operator

$$(\forall \mathbf{x}, \mathbf{y} \in \mathcal{H}) \|T(\mathbf{x}) - T(\mathbf{y})\| \leq \|\mathbf{x} - \mathbf{y}\|$$

Proximity Operator (J.J.Moreau '62)

$f \in \Gamma_0(\mathcal{X})$

$$\mathrm{prox}_f : \mathcal{X} \to \mathcal{X} : x \mapsto \arg\min_{y \in \mathcal{X}} \left\{ f(y) + \frac{1}{2}\|x - y\|^2 \right\}$$

is $1/2$- averaged nonexpansive operator, i.e.,

$\mathrm{rprox}_f := 2\mathrm{prox}_f - \mathrm{Id}$ is nonexpansive.

$$\begin{aligned}
& z \in \arg\min_{x \in \mathcal{X}} f(x) \\
\Leftrightarrow \quad & 0 \in \partial f(z) := \{p \in \mathcal{X} \mid f(z) + \langle p, x - z \rangle \le f(x)\ (\forall x \in \mathcal{X})\} \in 2^{\mathcal{X}} \\
\Leftrightarrow \quad & z \in z + \partial f(z) = (\mathrm{Id} + \partial f)(z) \in 2^{\mathcal{X}} \\
\Leftrightarrow \quad & z = (\mathrm{Id} + \partial f)^{-1}(z) = \mathrm{prox}_f(z) \\
\Leftrightarrow \quad & z \in \mathrm{Fix}\left(\mathrm{prox}_f\right)
\end{aligned}$$

Subdifferential of f at z

Proximity operator of f

Resolvent of ∂f

Proximity Operator of Conjugate function

$$\forall f \in \Gamma_0(\mathcal{X}),\ f^*: \mathcal{X} \ni y \mapsto \sup_{x \in \mathcal{X}} \left(\langle y, x \rangle - f(x) \right) \in (-\infty, \infty]$$

is called **Fenchel-Rockafellar Conjugate** of f

and satisfies $f^* \in \Gamma_0(\mathcal{X})$ &

Inverse Resolvent Identity

$$\mathrm{Id} = \mathrm{prox}_f + \mathrm{prox}_{f^*}$$

If $f \in \Gamma_0(\mathcal{X})$ is **prox-friendly** (i.e., prox_f is easily computable), $f^* \in \Gamma_0(\mathcal{X})$ is also **prox-friendly**.

Most splitting algorithms more or less rely on ...

Fact (Krasnosel'skii-Mann, e.g.[Mann'53,Dotson'70,Groetsch'72])

Suppose $T : \mathcal{H} \to \mathcal{H}$ is $\begin{cases} \text{Nonexpansive} \\ \mathrm{Fix}(T) \neq \emptyset \end{cases}$

Then for any $\begin{cases} \forall x_0 \in \mathcal{H} \\ (\alpha_n)_{n=0}^{\infty} \subset (0,1) \text{ s.t. } \sum_{n=0}^{\infty} \alpha_n (1-\alpha_n) = \infty \end{cases}$

$$x_{n+1} := (1-\alpha_n) x_n + \alpha_n T(x_n) \rightharpoonup \exists \hat{x} \in \mathrm{Fix}(T)$$

In fact, after careful observations, we can interpret

Proximal Splitting Algorithms
(Forward backward splitting/Primal-dual splitting/ Douglus -Rachford splitting / ADMM etc)

as applications of **K-M Alg**

to

$$\arg\min_{x \in \mathcal{X}} f(x) + g(Ax) = \Xi(\mathrm{Fix}(T))$$

where

$T : \mathcal{H} \to \mathcal{H}$: a computable nonexpansive operator defined on a certain Hilbert space $\mathcal{H}$

$\Xi : \mathcal{H} \to 2^{\mathcal{X}}$: a certain set-valued operator

Example (ADMM e.g. [Gabay '83])

$(\mathcal{X}, \langle\cdot,\cdot\rangle, \|\cdot\|), (\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}}, \|\cdot\|_{\mathcal{K}})$: Real Hilbert Spaces

$f \in \Gamma_0(\mathcal{X}),\ g \in \Gamma_0(\mathcal{K}),\ A : \mathcal{X} \to \mathcal{K}$: Bdd linear

$$\text{(P)} \quad \underset{x \in \mathcal{X}}{\text{minimize}}\ f(x) + g(Ax)$$

$$\begin{cases} x_{k+1} \in \underset{x \in \mathcal{X}}{\arg\min} \left(f(x) + \frac{1}{2} \|Ax - y_k - \nu_k\|_{\mathcal{K}}^2 \right) \\ y_{k+1} \in \underset{y \in \mathcal{K}}{\arg\min} \left(g(y) + \frac{1}{2} \|Ax_{k+1} - y - \nu_k\|_{\mathcal{K}}^2 \right) \\ \nu_{k+1} = \nu_k - Ax_{k+1} + y_{k+1} \end{cases}$$

A Fixed Point Theoretic View of ADMM [Eckstein-Bertsekas'92]

$(\mathcal{X}, \langle\cdot,\cdot\rangle_{\mathcal{X}}, \|\cdot\|_{\mathcal{X}}), (\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}}, \|\cdot\|_{\mathcal{K}})$: Real Hilbert Spaces

$f \in \Gamma_0(\mathcal{X}),\ g \in \Gamma_0(\mathcal{K}),\ A : \mathcal{X} \to \mathcal{K}$: Bdd linear

$$\text{(P)} \quad \underset{x\in\mathcal{X}}{\text{minimize}}\ f(x) + g(Ax)$$

$$\text{(D)} \quad \underset{u\in\mathcal{K}}{\text{minimize}}\ f^*(A^*u) + g^*(-u)$$

$$\theta_1 := f^* \circ A^*,\ \theta_2 := g^* \circ (-\mathrm{Id})$$

$$\mathcal{S}_d := \arg\min_{\nu\in\mathcal{K}} f^*(A^*\nu) + g^*(-\nu) = \mathrm{prox}_{\theta_2}\left(\mathrm{Fix}\left(\mathrm{rprox}_{\theta_1}\mathrm{rprox}_{\theta_2}\right)\right)$$

$$\mathcal{S}_p := \arg\min_{x\in\mathcal{X}} f(x) + g(Ax) = \partial f^*(A^*\nu^\star) \cap A^{-1}\left(\partial g^*(-\nu^\star)\right) \quad (\forall \nu^\star \in \mathcal{S}_d)$$

$$\text{ADMM} = \text{K-M alg for a point in } \mathrm{Fix}(\mathrm{rprox}_{\theta_1}\mathrm{rprox}_{\theta_2})$$

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Application to State-of-the-art Statistiacal Estimation Technique

A Hierarchical Enhancement of Lasso

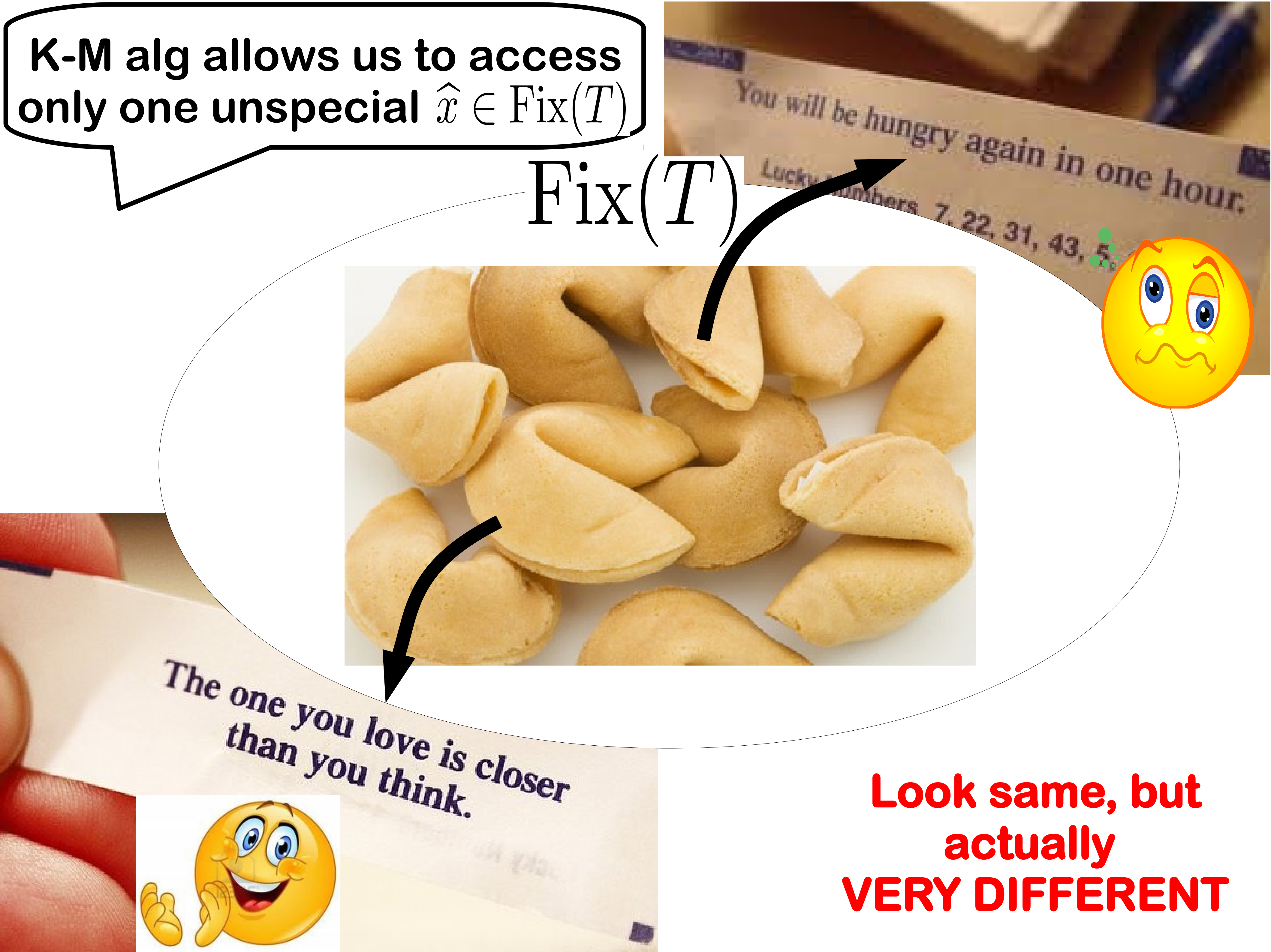
We have seen the solution sets of convex optimization problems can often be expressed as

$$\arg\min_{x \in \mathcal{X}} f(x) + g(Ax) = \Xi\left(\mathrm{Fix}(T)\right)$$

where

$T : \mathcal{H} \to \mathcal{H}$: a **computable nonexpansive operator** defined on a certain Hilbert space $\mathcal{H}$

$\Xi : \mathcal{H} \to 2^{\mathcal{X}}$: a certain set-valued operator


K-M alg allows us to access only one unspecial $\widehat{x} \in \mathrm{Fix}(T)$
You will be hungry again in one hour.
$\mathrm{Fix}(T)$
The one you love is closer than you think.
Look same, but actually VERY DIFFERENT

Can we choose best one without crunching all cookies ?

This mission
can not be accomplishd by K-M alg
but could be accomplished by
Hybrid steepest descent method !

K-M algorithm

$$x_{n+1} := (1-\alpha_n)x_n + \alpha_n T(x_n) \rightharpoonup \exists \hat{x} \in \mathrm{Fix}(T)$$

Hybrid Steepest Descent Method

$$x_{n+1} := T(x_n) - \lambda_{n+1} \nabla \Psi\left(T(x_n)\right) \to \text{Best Point} \in \mathrm{Fix}(T)$$

A Key for Hierarchical Convex Optimization

Hybrid Steepest Descent Method

[Yamada et al '96, Deutsch-Yamada'98, Yamada'01, Yamada-Ogura'04 etc]

$$x_{n+1} := T(x_n) - \lambda_{n+1} \nabla \Psi \left(T(x_n) \right)$$

can minimize Ψ over

$$\mathrm{Fix}(T) := \{ \boldsymbol{x} \in \mathcal{H} \mid T(\boldsymbol{x}) = \boldsymbol{x} \}$$

where

$\Psi : \mathcal{H} \to \mathbb{R}$, Smooth Convex Function

$\nabla \Psi : \mathcal{H} \to \mathcal{H}$, Lipschitz Continuous

$T : \mathcal{H} \to \mathcal{H}$, Nonexpansive operator

$(\lambda_n)_{n=1}^{\infty} \subset [0, \infty)$: Slowly decreasing

1. This is extension of [Halpern'67/Reich'74/Lions'77/Wittmann'92/...].
2. This can select a very best solution among all fixed points !

Theorem (Convergence of HSDM, see, e.g. [Yamada'01])

$T : \mathcal{H} \to \mathcal{H}$ Nonexpansive with $\mathrm{Fix}(T) \neq \varnothing$

$\Psi : \mathcal{H} \to \mathbb{R}$ Gâteaux differentiable s.t.

$$(\exists \kappa, \eta > 0, \forall x, y \in T(\mathcal{H})) \quad \begin{aligned} &\|\nabla\Psi(x) - \nabla\Psi(y)\| \leq \kappa \|x - y\| \\ &\langle \nabla\Psi(x) - \nabla\Psi(y), x - y \rangle \geq \eta \|x - y\|^2 \end{aligned}$$

$$(\lambda_n)_{n\geq 1} \subset [0, \infty) \text{ satisfies } \begin{cases} \text{(i)} & \lim_{n\to\infty} \lambda_n = 0 \\ \text{(ii)} & \sum_{n\geq 1} \lambda_n = \infty \\ \text{(iii)} & \sum_{n\geq 1} |\lambda_n - \lambda_{n+1}| < \infty \end{cases}$$

$$(\forall x_0 \in \mathcal{H}) \quad x_{n+1} := T(x_n) - \lambda_{n+1} \nabla\Psi(T(x_n))$$

satisfies $\displaystyle \lim_{n\to\infty} \|x_n - x^{\star\star}\| = 0$

where $\displaystyle x^{\star\star} \in \Omega := \arg \min_{x \in \mathrm{Fix}(T)} \Psi(x)$ (Note: $|\Omega| = 1$)

Theorem (nonstrictly convex, $\dim(\mathcal{H}) < \infty$ [Ogura-Yamada'03])

Suppose

$T : \mathcal{H} \to \mathcal{H}$ Nonexpansive with bounded $\mathrm{Fix}(T) \neq \emptyset$

$\Psi : \mathcal{H} \to \mathbb{R}$ Smooth Convex function, s.t.

$$(\exists \kappa > 0, \forall x, y \in T(\mathcal{H})) \quad \|\nabla\Psi(x) - \nabla\Psi(y)\| \leq \kappa\|x - y\|$$

$$(\lambda_n)_{n\geq 0} \in \ell^2_+ \setminus \ell^1_+.$$

Then

$$(\forall x_0 \in \mathcal{H}) \quad x_{n+1} := T(x_n) - \lambda_{n+1}\nabla\Psi(T(x_n))$$

satisfies $$\lim_{n\to\infty} \mathrm{d}(x_n, \Omega) = 0$$

where $$\Omega := \arg \min_{x\in\mathrm{Fix}(T)} \Psi(x) \neq \emptyset.$$

How can we combine Nonexpansive Operators with Hybrid Steepest Descent Method for Hierarchical Convex Optimization ?

We have found many ways ! See for example

[Yamada-Ogura-Shirakawa '02],[Yamada-Yukawa-Yamagishi '11],
[Ono-Yamada '15], [Yamagishi-Yamada '17]

Next we demonstrate a simple strategy in an application to statistical estimation problem !

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A Hierarchical Enhancement of Lasso

Consider the estimation of $\mathbf{b}^{\mathrm{tru}} \in \mathbb{R}^p$ in the standard linear model:

$$\mathbf{z} = \mathbf{X}\mathbf{b}^{\mathrm{tru}} + \sigma\mathbf{e}$$

where

Response vector

$$\mathbf{z} = (z_1, z_2, \ldots, z_n)^t \in \mathbb{R}^n$$

Design matrix

$\mathbf{X} \in \mathbb{R}^{n \times p}$ ($p > n$ for High-dimensional case)

: assumed to have no zero column vector !

Noise vector

$$\mathbf{e} = (\varepsilon_1, \ldots, \varepsilon_n)^t \in \mathbb{R}^n$$

ε_i : realization of normalized random variable with mean 0 and variance 1

Standard Deviation of Entire noise

$$\sigma > 0$$

Lasso [Robert Tibshirani '96]

$$\widehat{\mathbf{b}}_{\text{Lasso}}(\lambda) \in \arg\min_{\mathbf{b}\in\mathbb{R}^p} \left\{ \frac{1}{2n}\|\mathbf{z} - \mathbf{X}\mathbf{b}\|_2^2 + \lambda\|\mathbf{b}\|_1 \right\}.$$

A Prediction Bound for Lasso

[Koltchinskii, Lounici, and Tsybakov'11], [Rigollet and Tsybakov'11]

If $\lambda \geq \dfrac{2\|\mathbf{X}^t(\mathbf{z} - \mathbf{X}\mathbf{b}^{\text{tru}})\|_\infty}{n}$,

it holds $\dfrac{\|\mathbf{X}\widehat{\mathbf{b}}_{\text{Lasso}}(\lambda) - \mathbf{X}\mathbf{b}^{\text{tru}}\|_2^2}{n} \leq 2\lambda\|\mathbf{b}^{\text{tru}}\|_1.$

A powerful enhancement of Lasso

TREX [Lederer, Müller '15]: Nonconvex Optimization

$$\widehat{\mathbf{b}}_{\text{TREX}} \in \arg\min_{\mathbf{b}\in\mathbb{R}^p} \left\{ \frac{\|\mathbf{X}\mathbf{b} - \mathbf{z}\|_2^2}{\|\mathbf{X}^t(\mathbf{X}\mathbf{b} - \mathbf{z})\|_\infty} + \alpha\|\mathbf{b}\|_1 \right\}, \text{default choice } \alpha = \frac{1}{2}$$

TREX: an enhancement of Lasso [Lederer, Müller '15]

Nonconvex

$$\widehat{\mathbf{b}}_{\mathrm{TREX}} \in \arg\min_{\mathbf{b}\in\mathbb{R}^p} \left\{ \frac{\|\mathbf{X}\mathbf{b}-\mathbf{z}\|_2^2}{\alpha\|\mathbf{X}^t(\mathbf{X}\mathbf{b}-\mathbf{z})\|_\infty} + \|\mathbf{b}\|_1 \right\}, \quad \text{default choice } \alpha = \frac{1}{2}$$

where $\|\mathbf{X}^t(\mathbf{X}\mathbf{b}-\mathbf{z})\|_\infty = \max_{1\le j\le p} |\mathbf{X}^t_{:j}(\mathbf{X}\mathbf{b}-\mathbf{z})|$

Great News 1

[Bien, Gaynanova, Lederer, and Müller '16]

Find best one among solutions of $2p$ convex subproblems:

$$\widehat{\mathbf{b}}^{(j)}_{\mathrm{TREX}} \in \operatorname*{argmin}_{\substack{\mathbf{b}\in\mathbb{R}^p \\ \mathbf{x}_j^t(\mathbf{X}\mathbf{b}-\mathbf{z})>0}} \left\{ \frac{\|\mathbf{X}\mathbf{b}-\mathbf{z}\|_2^2}{\alpha\mathbf{x}_j^t(\mathbf{X}\mathbf{b}-\mathbf{z})} + \|\mathbf{b}\|_1 \right\}$$

where $\mathbf{x}_j = \pm\mathbf{X}_{:j} \quad (j = 1, 2, \ldots, p)$

j th convex subproblem of TREX [Bien, Gaynanova, Lederer, and Müller '16]

$$\widehat{\mathbf{b}}^{(j)}_{\mathrm{TREX}} \in \underset{\substack{\mathbf{b} \in \mathbb{R}^p \\ \mathbf{x}_j^t(\mathbf{X}\mathbf{b}-\mathbf{z}) > 0}}{\operatorname{argmin}} \left\{ \frac{\|\mathbf{X}\mathbf{b}-\mathbf{z}\|_2^2}{\alpha \mathbf{x}_j^t(\mathbf{X}\mathbf{b}-\mathbf{z})} + \|\mathbf{b}\|_1 \right\}$$

A Reformulation for Proximal Splitting [Combettes, Müller '17]

$$\widehat{\mathbf{b}}^{(j)}_{\mathrm{pTREX}} \in \mathcal{S}_j := \underset{\mathbf{b}\in\mathbb{R}^p}{\operatorname{argmin}} \left\{ g_j(\mathbf{M}_j \mathbf{b}) + \|\mathbf{b}\|_1 \right\},$$

Great News 2

$$g_j : \mathbb{R} \times \mathbb{R}^n \to (-\infty, \infty] : (\eta, \mathbf{y}) \mapsto \begin{cases} \frac{\|\mathbf{y}-\mathbf{z}\|_2^2}{\alpha(\eta - \mathbf{x}_j^t \mathbf{z})}, & \text{if } \eta > \mathbf{x}_j^t \mathbf{z}; \\ 0, & \text{if } \mathbf{y} = \mathbf{z} \text{ and } \eta = \mathbf{x}_j^t \mathbf{z}; \\ +\infty, & \text{otherwise.} \end{cases}$$

$$\in \Gamma_0(\mathbb{R} \times \mathbb{R}^n) \; : \; \text{proper lower-semicontinuous convex,}$$

$$\operatorname{prox}_{g_j} := (\mathrm{Id} + \partial g_j)^{-1} \; : \; \text{has a closed form expression,}$$

$$\mathbf{M}_j : \mathbb{R}^p \to \mathbb{R} \times \mathbb{R}^n : \mathbf{b} \mapsto (\mathbf{x}_j^t \mathbf{X}\mathbf{b}, \mathbf{X}\mathbf{b}) \; : \; \text{Bounded Linear}$$

Product Space Reform. TREX subproblem [Combettes, Müller '17]

$$\underset{\mathbf{b}\in\mathbb{R}^p}{\text{minimize}} \quad g_j(\mathbf{M}_j\mathbf{b}) + \|\mathbf{b}\|_1 \qquad \text{Convex Optimization over } \mathbb{R}^p$$

$$\underset{\mathbf{x}=(\mathbf{b},\mathbf{c})\in\mathbb{R}^p\times\mathbb{R}^{n+1}}{\text{minimize}} \quad F_j(\mathbf{x}) + G_j(\mathbf{x}) \qquad \text{Convex Optimization over } \mathbb{R}^{p+n+1}$$

$$F_j : (\mathbf{b},\mathbf{c}) \mapsto \|\mathbf{b}\|_1 + g_j(\mathbf{c}), \qquad G_j(\mathbf{b},\mathbf{c}) = \begin{cases} 0 & \text{if } \mathbf{M}_j\mathbf{b} = \mathbf{c}, \\ \infty & \text{otherwise} \end{cases}$$

$$\text{prox}_{F_j}(\mathbf{b},\mathbf{c}) = \left(\text{prox}_{\|\cdot\|_1}(\mathbf{b}), \text{prox}_{g_j}(\mathbf{c})\right) \qquad \text{prox}_{G_j}(\mathbf{b},\mathbf{c}) = \text{projection onto}$$

soft thresholder

$$\mathcal{M}_j := \{(\mathbf{b},\mathbf{c}) \in \mathbb{R}^p \times \mathbb{R}^{n+1} \mid \mathbf{M}_j\mathbf{b} = \mathbf{c}\}$$

A Fixed Point Characterization with Douglas-Rachford Operator

$$\mathbf{x}^\star \text{ minimizes } F_j + G_j \quad \Leftrightarrow \quad \begin{cases} \mathbf{x}^\star = \text{prox}_{\gamma G_j}(\mathbf{y}) = P_{\mathcal{M}_j}(\mathbf{y}) \\ \mathbf{y} \in \text{Fix}(T_j), \end{cases}$$

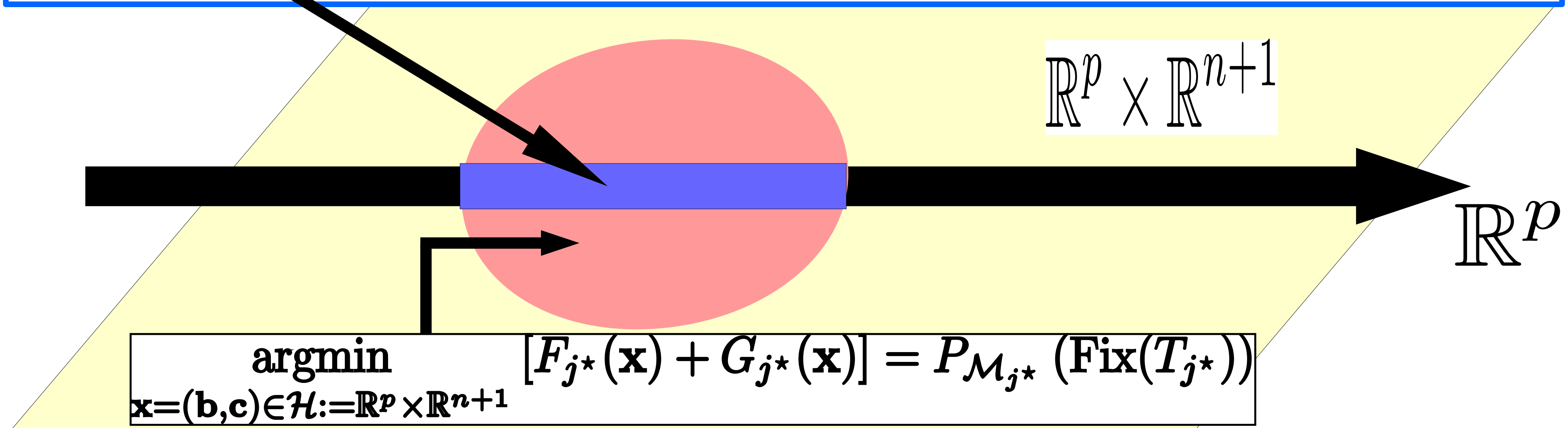
where $T_j := \left(2\text{prox}_{\gamma F_j} - \text{Id}\right) \circ \left(2P_{\mathcal{M}_j} - \text{Id}\right) : \mathbb{R}^{p+n+1} \to \mathbb{R}^{p+n+1}$: Nonexpansive

The solution set $\mathcal{S}$ of TREX can be expressed completely in terms of computable operators !

Solution set of **TREX**

$$\mathcal{S} := \underset{\mathbf{b}\in\mathbb{R}^p}{\operatorname{argmin}} \{g_{j^\star}(\mathbf{M}_{j^\star}\mathbf{b}) + \|\mathbf{b}\|_1\} = \mathcal{Q}_p \circ P_{\mathcal{M}_{j^\star}}(\operatorname{Fix}(T_{j^\star}))$$

where $\mathcal{Q}_p : \mathbb{R}^p \times \mathbb{R}^{n+1} \to \mathbb{R}^p : (\mathbf{b}, \mathbf{c}) \mapsto \mathbf{b}$



We have shown

The nonexpansive operator $T_{j^\star} := \left(2\operatorname{prox}_{\gamma F_{j^\star}} - \operatorname{Id}\right) \circ \left(2P_{\mathcal{M}_{j^\star}} - \operatorname{Id}\right)$ has **BOUNDED** $\operatorname{Fix}(T_{j^\star}) = \{\mathbf{y} \in \mathbb{R}^{p+n+1} \mid T_{j^\star}(\mathbf{y}) = \mathbf{y}\}$!

Consider Hierarchical Enhancement of TREX for

$$\mathbf{z} = \mathbf{X}\mathbf{b}^{\mathrm{tru}} + \sigma \mathbf{e}$$

TREX: Reformulation of convex subproblems [Combettes, Muller '17]

Find vectors in $\mathcal{S}_j := \underset{\mathbf{b} \in \mathbb{R}^p}{\operatorname{argmin}} \{g_j(\mathbf{M}_j \mathbf{b}) + \|\mathbf{b}\|_1\} \quad (j = 1, 2, , 2p),$

where $\quad g_j : \mathbb{R} \times \mathbb{R}^n \to (-\infty, \infty] : (\eta, \mathbf{y}) \mapsto \begin{cases} \frac{\|\mathbf{y}-\mathbf{z}\|_2^2}{\frac{1}{2}(\eta - \mathbf{x}_j^t \mathbf{z})}, & \text{if } \eta > \mathbf{x}_j^t \mathbf{z}; \\ 0, & \text{if } \mathbf{y} = \mathbf{z} \text{ and } \eta = \mathbf{x}_j^t \mathbf{z}; \\ +\infty, & \text{otherwise,} \end{cases}$

Further enhancement of TREX with prior knowledge ?

Example

Suppose that we know the target $\mathbf{b}^{\mathrm{tru}}$ is **fairly flat** !

$$\|D\mathbf{b}^{\mathrm{tru}}\|^2 \text{ is expected to be small,}$$

where $D : \mathbb{R}^p \to \mathbb{R}^{p-1}$ returns differences of all neighboring pairs.

A Hierarchical Convex Optimization for Enhancement of TREX

For $j = 1, 2, \ldots, 2p$,

Problem (Hierarchical enhancement of Lasso for promoting Flatness)

$$\left.\begin{array}{ll} \text{Minimize} & \left\| D\mathbf{b}^\star \right\|^2 \\ \text{subject to} & \mathbf{b}^\star \in \mathcal{S}_j := \underset{\mathbf{b}\in\mathbb{R}^p}{\operatorname{argmin}} \left\{ g_j(\mathbf{M}_j \mathbf{b}) + \|\mathbf{b}\|_1 \right\} \\ & \qquad\;\; = \mathcal{Q}_p \circ P_{\mathcal{M}_j}\left(\operatorname{Fix}(T_j)\right), \end{array}\right\} \quad (\star)$$

where

$$D := \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

(Equivalent Problem for **Hybrid Steepest Descent Method**)

$$\begin{array}{ll} \text{Minimize} & \Psi(\mathbf{x}^\star) := \left\| D \circ \mathcal{Q}_p \circ P_{\mathcal{M}_j}(\mathbf{x}^\star) \right\|^2 \\ \text{subject to} & \mathbf{x}^\star \in \operatorname{Fix}(T_j) \end{array} \quad (\star\star)$$

where $T_j := \left(2\operatorname{prox}_{\gamma F_j} - \mathrm{Id}\right) \circ \left(2P_{\mathcal{M}_j} - \mathrm{Id}\right) : \mathbb{R}^{p+n+1} \to \mathbb{R}^{p+n+1}$: Nonexpansive

Numerical Test (Underdetermined case)

$$\mathbf{z} = \mathbf{X}\mathbf{b}^{\mathrm{tru}} + \sigma \mathbf{e} \quad (n = 20, p = 30)$$

$\mathbf{X} \in \mathbb{R}^{20\times 30}$: generated by zero-mean Gaussian

$\|\mathbf{X}_{:i}\| = \sqrt{20}$ $(i = 1, 2, \ldots, 30)$ and $\mathbf{X}_{:2} = \mathbf{X}_{:3} = \mathbf{X}_{:4}$

$$\mathbf{b}^{\mathrm{tru}} = \frac{1}{\sqrt{30}}(0, 0, 0, 1, 1, 1, 0, 0, \ldots, 0)^t \in \mathbb{R}^{30}$$

$$T_j := \left(2\mathrm{prox}_{F_j} - \mathrm{Id}\right) \circ \left(2P_{\mathcal{M}_j} - \mathrm{Id}\right) : \mathbb{R}^{p+n+1} \to \mathbb{R}^{p+n+1} : \text{ Nonexpansive}$$

K-M algorithm with Douglas-Rachford Operator

$$(\mathbf{b}_{k+1}, \mathbf{c}_{k+1}) := (1 - \alpha_k)(\mathbf{b}_k, \mathbf{c}_k) + \alpha_k T_j(\mathbf{b}_k, \mathbf{c}_k) \quad (\alpha_k = 1.95/2)$$

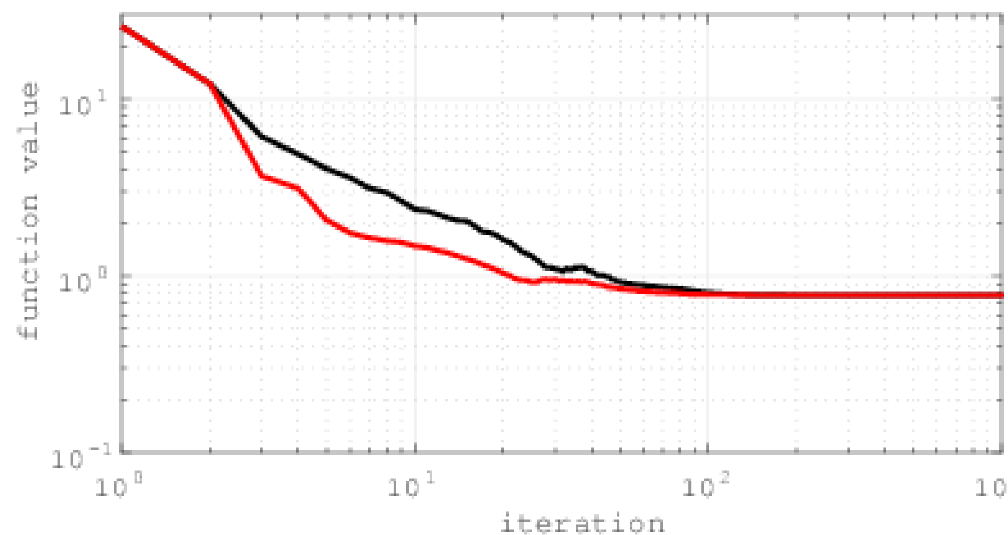
Hierarchical TREX for Promoting Flatness

$$(\mathbf{b}_{k+1}, \mathbf{c}_{k+1}) := T_j(\mathbf{b}_k, \mathbf{c}_k) - \lambda_{k+1} \nabla \Psi\left(T_j(\mathbf{b}_k, \mathbf{c}_k)\right) \quad (\lambda_k = 1/k)$$

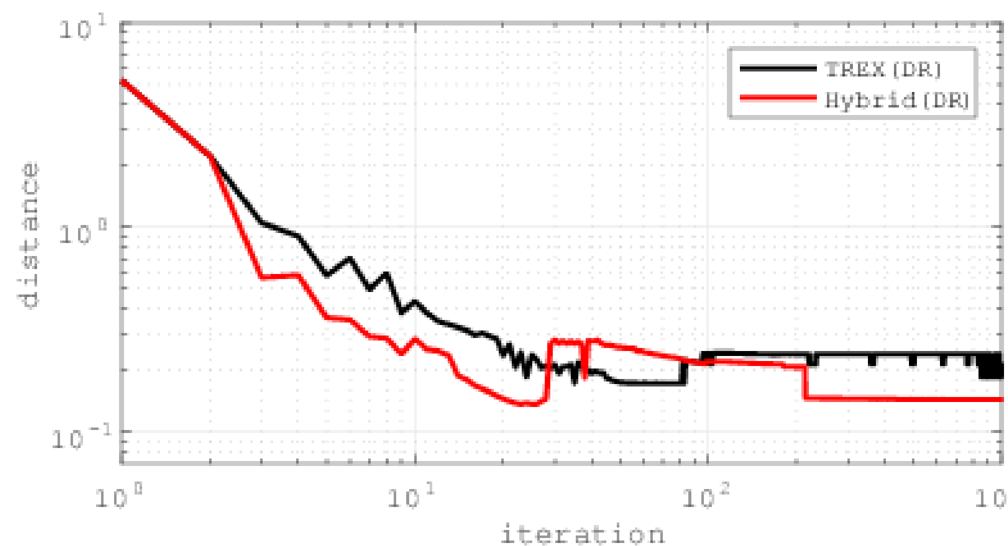
Numerical Performance

$$\mathrm{SNR} = 10\log_{10}\frac{\|X\mathbf{b}^{\mathrm{tru}}\|^2}{\|\sigma\mathbf{e}\|^2} = 20(\mathrm{dB})$$

$$\min_{1\leq j\leq 60}\{g_j(\mathbf{M}_j\mathbf{b}) + \|\mathbf{b}\|_1\}$$



$$\|\mathbf{b} - \mathbf{b}^{\mathrm{tru}}\|$$



Conclusion

1. We introduced a simple strategy for Hierarchical Convex Optimization which can enhance further existing proximal splitting algorithms without losing their optimality.

2. The proposed strategies are based on destined mariage: Proximal splitting operators + Hybrid steepest descent method.

3. We have demonstrated an application to Hierarchical Enhancement of Lasso estimator.

References (Hierarchical convex optimization)

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