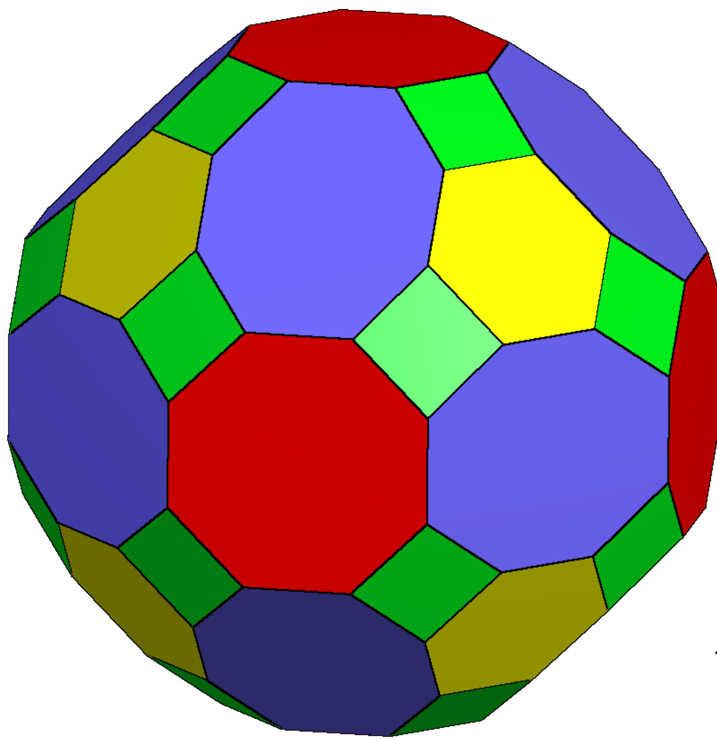
Improved upper bounds on the diameter of lattice polytopes



Antoine Deza, McMaster

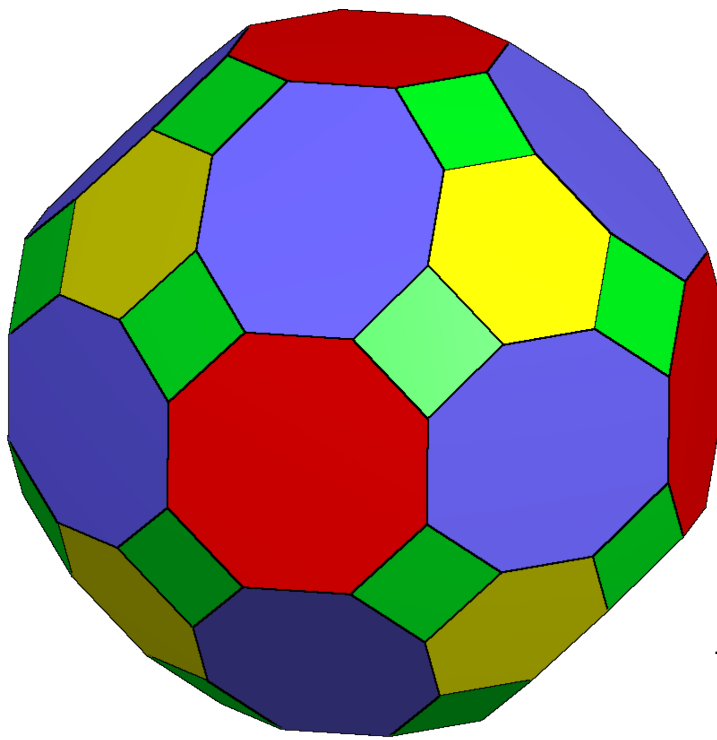
based on a joint work with

Lionel Pournin, Paris XIII

McMaster University

Advanced Optimization Laboratory

Primitive lattice polytopes and convex matroid optimization



Antoine Deza, McMaster

based on joint works with

George Manoussakis, Paris XI

Lionel Pournin, Paris XIII

Shmuel Onn, Technion

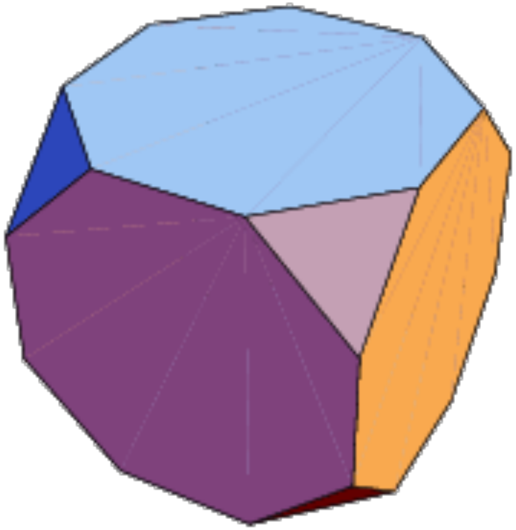
lattice polytopes with large diameter

lattice (d,k)-polytope : convex hull of points drawn from $\{0,1,\ldots,k\}^d$

diameter $\delta(P)$ of polytope P : smallest number such that **any two vertices** of P can be connected by a **path with at most** $\delta(P)$ **edges**

$\delta(d,k)$: largest diameter over all **lattice** (d,k)-polytopes

ex. $\delta(3,3) = 6$ and is achieved by a ***truncated cube***



lattice polytopes with large diameter

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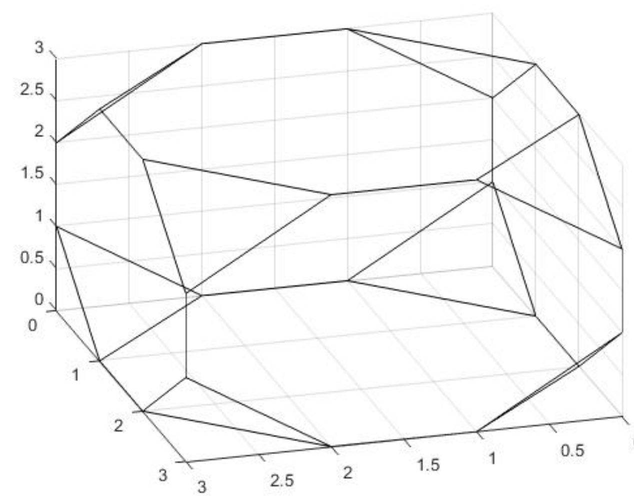


lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

upper bounds :

$\delta(d,1) \leq d$		[Naddef 1989]
$\delta(2,k) = O(k^{2/3})$		[Balog-Bárány 1991]
$\delta(2,k) = 6(k/2\pi)^{2/3} + O(k^{1/3} \log k)$		[Thiele 1991] [Acketa-Žunić 1995]
$\delta(d,k) \leq kd$		[Kleinschmid-Onn 1992]
$\delta(d,k) \leq kd - \lceil d/2 \rceil$	for $k \geq 2$	[Del Pia-Michini 2016]
$\delta(d,k) \leq kd - \lceil 2d/3 \rceil$	for $k \geq 3$	[Deza-Pournin 2016]
$\delta(d,k) \leq kd - \lceil 2d/3 \rceil - (k-2)$	for $k \geq 4$	[Deza-Pournin 2016]

lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

lower bounds :

$\delta(d,1) \geq d$ [Naddef 1989]

$\delta(d,2) \geq \lfloor 3d/2 \rfloor$ [Del Pia-Michini 2016]

$\delta(d,k) = \Omega(k^{2/3} d)$ [Del Pia-Michini 2016]

$\delta(d,k) \geq \lfloor (k+1)d/2 \rfloor$ for $k < 2d$ [Deza-Manoussakis-Onn 2016]

lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	7+	9+	?	?	?	?
	4	4	6	8	10+	12+	14+	16+	?	?
	5	5	7	10+	12+	15+	17+	20+	22+	25+

$\delta(d,1) = d$ [Naddef 1989]

$\delta(2,k) =$ close form [Thiele 1991] [Acketa-Žunić 1995]

$\delta(d,2) = \lfloor 3d/2 \rfloor$ [Del Pia-Michini 2016]

$\delta(4,3) = 8$ [Deza-Pournin 2016]

lattice polytopes with large diameter

δ(*d*,*k*)		*k*								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	7+	9+	?	?	?	?
	4	4	6	8	10+	12+	14+	16+	?	?
	5	5	7	10+	12+	15+	17+	20+	22+	25+

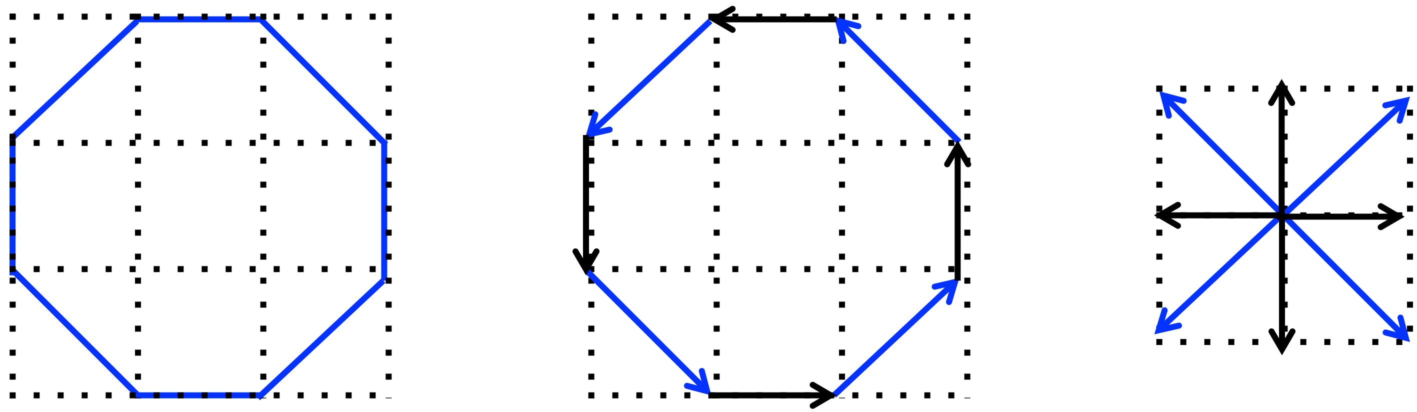
All known entries of δ(*d*,*k*) are achieved, up to translation, by a *Minkowski sum of primitive lattice vectors* (some uniquely)

Conjecture: $\delta(d,k) \le \lfloor (k+1)d/2 \rfloor$ [Deza-Manoussakis-Onn 2016]

lattice polygons with many vertices

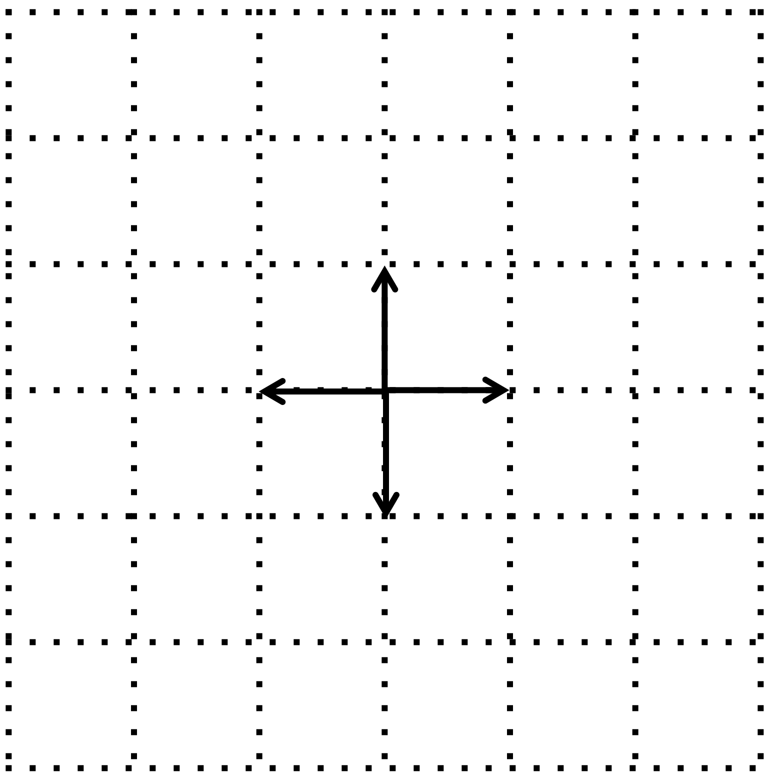
Q. What is δ(2,***k***) : largest diameter of a polygon which vertices are drawn form the ***k*** x ***k*** grid?

A polygon can be associated to a set of vectors (*edges*) *summing up to zero*, and *without a pair of positively multiple vectors*



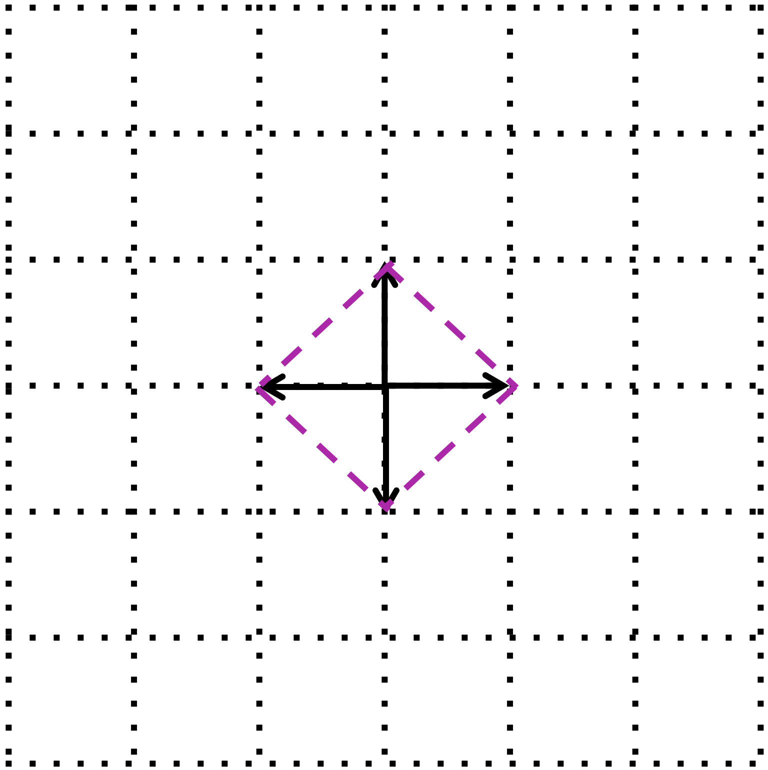
δ(2,3) = 4 is achieved by the 8 vectors : (±1,0), (0,±1), (±1,±1)

lattice polygons with many vertices



$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

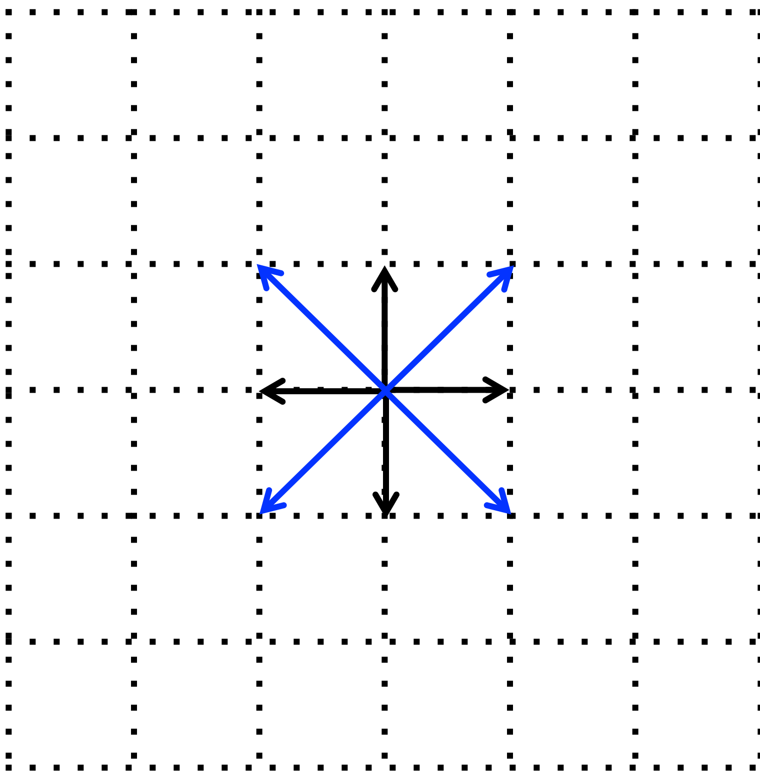
lattice polygons with many vertices



$||x||_1 \leq 1$

$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

lattice polygons with many vertices



δ(2,2) = 2 ; vectors : (±1,0), (0,±1)

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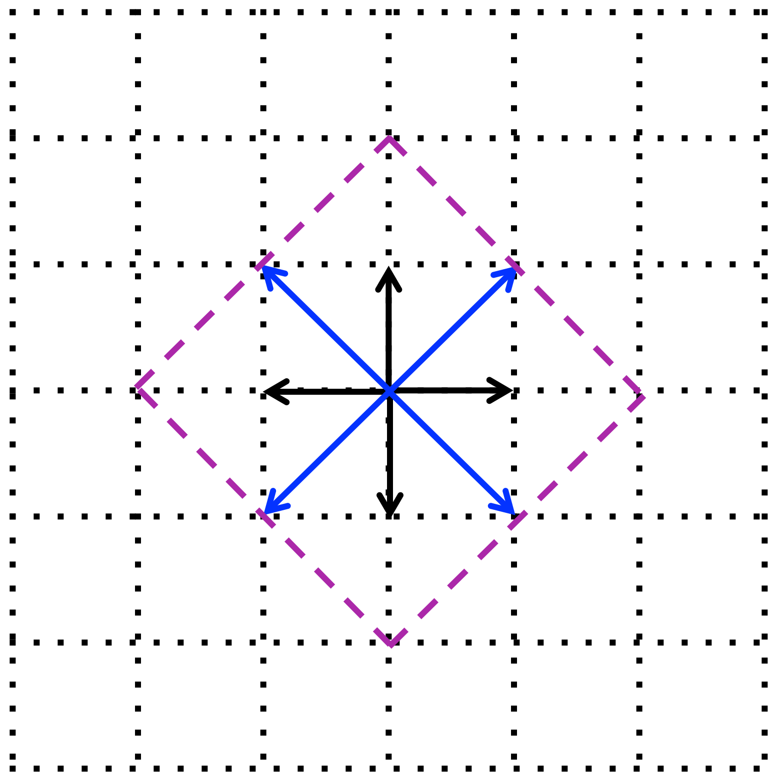
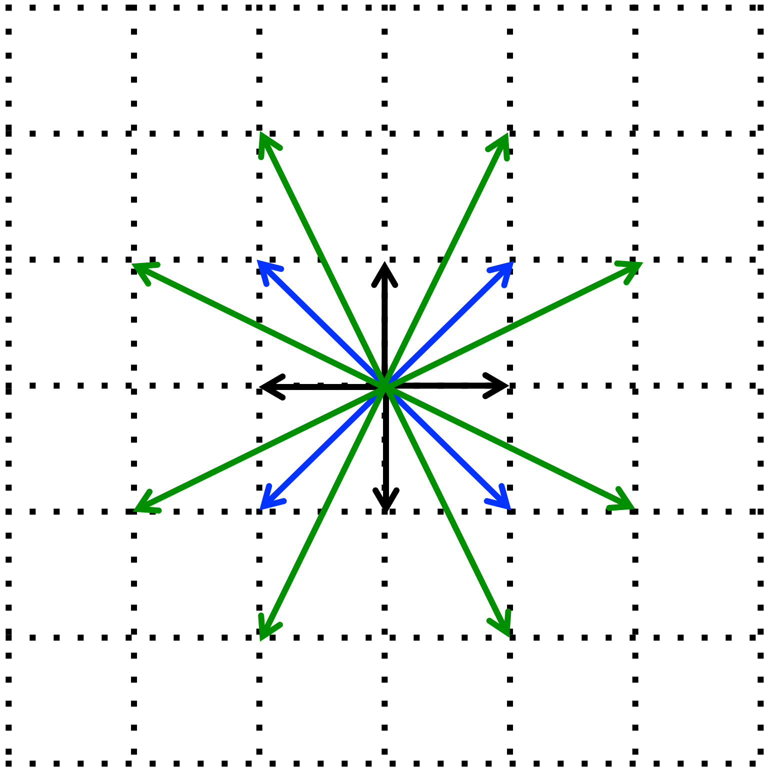
lattice polygons with many vertices



$||x||_1 \leq$ **2**

$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$
$\delta(2,3) = 4$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$

lattice polygons with many vertices

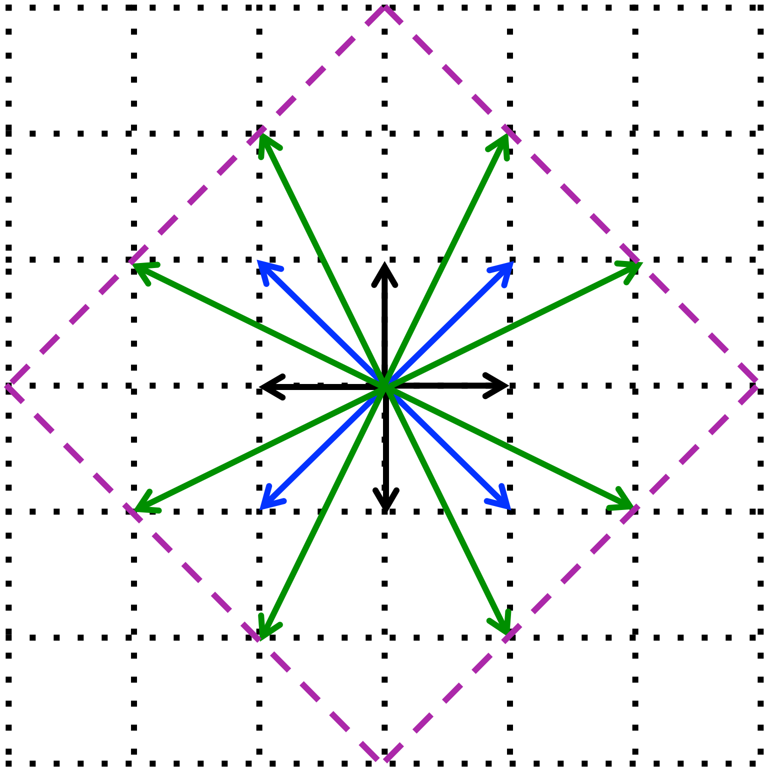


$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

$\delta(2,3) = 4$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$

$\delta(2,9) = 8$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$, $(\pm1,\pm2)$, $(\pm2,\pm1)$

lattice polygons with many vertices



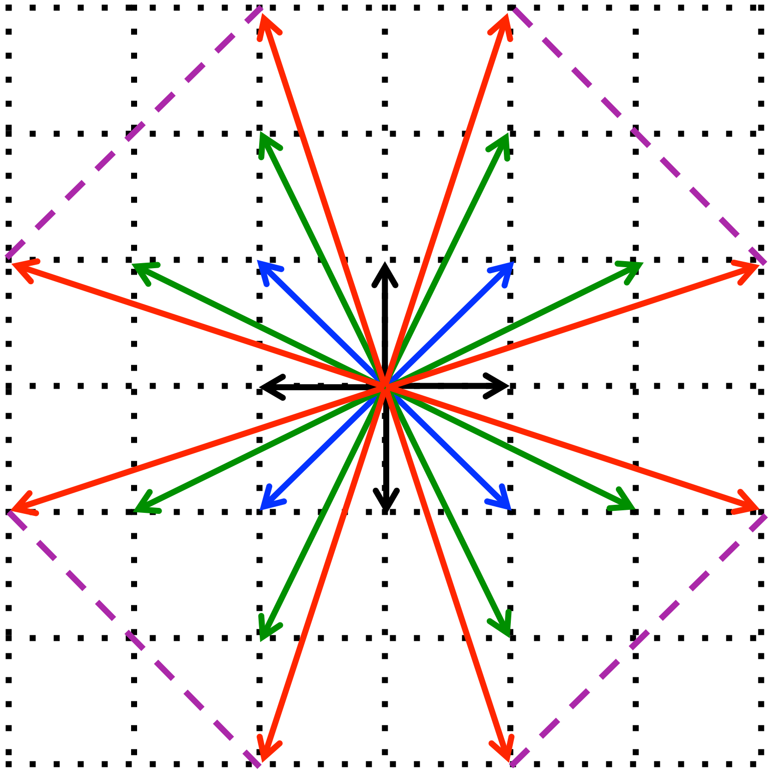
$||x||_1 \leq$ **3**

δ(2,2) = 2 ; vectors : (±1,0), (0,±1)

δ(2,3) = 4 ; vectors : (±1,0), (0,±1), (±1,±1)

δ(2,9) = 8 ; vectors : (±1,0), (0,±1), (±1,±1), (±1,±2), (±2,±1)

lattice polygons with many vertices



$||x||_1 \leq 4$

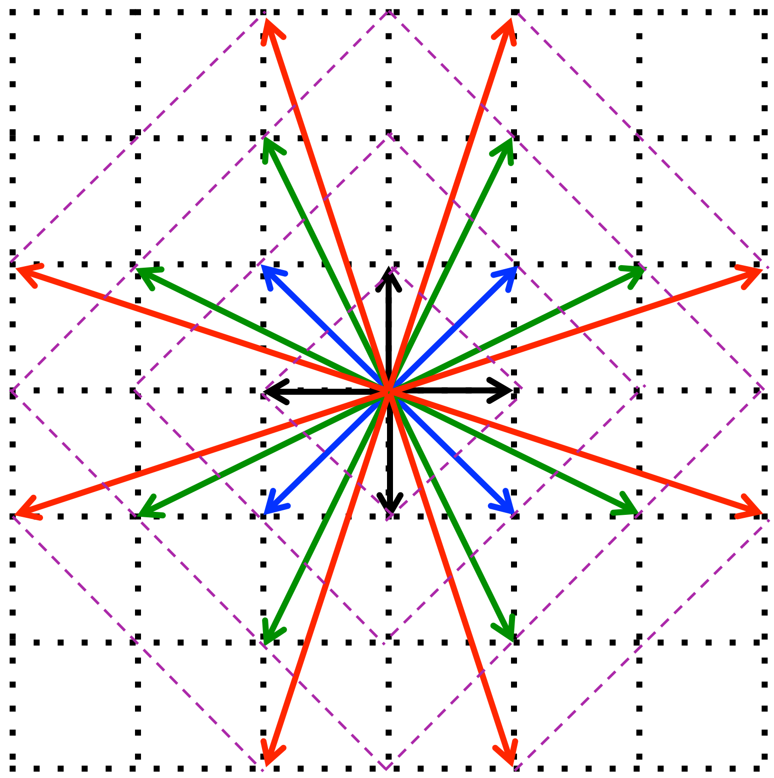
$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

$\delta(2,3) = 4$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$

$\delta(2,9) = 8$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$, $(\pm1,\pm2)$, $(\pm2,\pm1)$

$\delta(2,17) = 12$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$, $(\pm1,\pm2)$, $(\pm2,\pm1)$, $(\pm1,\pm3)$, $(\pm3,\pm1)$

lattice polygons with many vertices



$||x||_1 \leq p$

$$\delta(2,k) = 2\sum_{i=1}^{p}\varphi(i) \text{ for } k = \sum_{i=1}^{p} i\varphi(i)$$

φ(*p*) : ***Euler totient function*** counting positive integers less or equal to *p* relatively prime with *p*

φ(1) = φ(2) = 1, φ(3) = φ(4) = 2,…

lattice polygons with many vertices

δ(2,*k*)		*k*								
		1	2	3	4	5	6	7	8	9
	p	1		2						3
	v	4	6	8	8	10	12	12	14	16
	δ	2	3	4	4	5	6	6	7	8

$\delta(2,k) = 2\sum_{i=1}^{p}\varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

φ(*p*) : ***Euler totient function*** counting positive integers less or equal to *p* relatively prime with *p*
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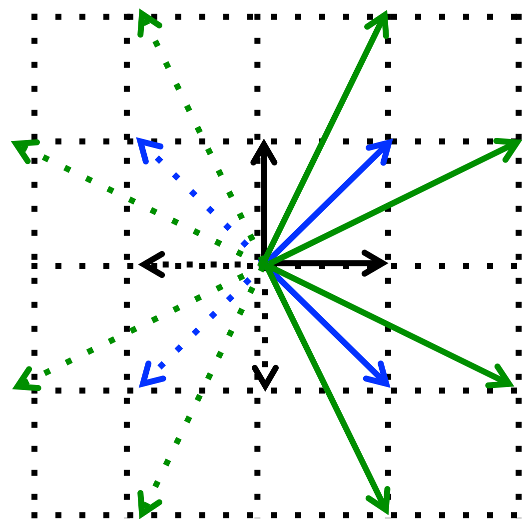
lattice polygons with many vertices



$||x||_1 \leq p$

$H_1(2,p)$: Minkowski sum generated by $\{x \in \mathbb{Z}^2 : ||x||_1 \leq p, \gcd(x)=1, x \succeq 0\}$

$H_1(2,p)$ has diameter $\delta(2,k) = 2\sum_{i=1}^{p} \varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

Ex. $H_1(2,2)$ generated by (1,0), (0,1), (1,1), (1,-1) (fits, *up to translation*, in 3x3 grid)

$x \succeq 0$: first nonzero coordinate of x is nonnegative

primitive lattice polytopes

as generalization of the permutahedron of type B_d

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p)$: Zonotope $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

Given a set G of m vectors (generators)

Minkowski (G) : convex hull of the 2^m sums of the m vectors in G
Zonotope (G) : convex hull of the 2^m ***signed*** sums of the m vectors in G

up to translation $Z(G)$ is the image of $H(G)$ by an homothety of factor 2

- ***Primitive lattice polytopes*****:** Minkowski sum generated by ***short integer*** vectors which are ***pairwise linearly independent***

primitive lattice polytopes

as generalization of the permutahedron of type B_d

$$H_q(d,p) : \text{Minkowski } (x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \succeq 0)$$

$$Z_q(d,p) : \text{Zonotope } (x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \succeq 0)$$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $Z_q(d,p)$: invariant under symmetries induced by coordinate permutations and reflections induced by sign flips
- Coordinates of the vertices of $Z_q(d,p)$ are odd, thus the number of vertices of $Z_q(d,p)$ is a multiple of 2^d
- $H_q(d,p)$ is, up to translation, a lattice (d,k)-polytope where k is the sum of the first coordinates of all generators of $Z_q(d,p)$
- diameter of $Z_q(d,p)$ is equal to the number of its generators

primitive lattice polytopes

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- $H_q(d, 1)$: $[0, 1]^d$ cube for finite q

primitive lattice polytopes

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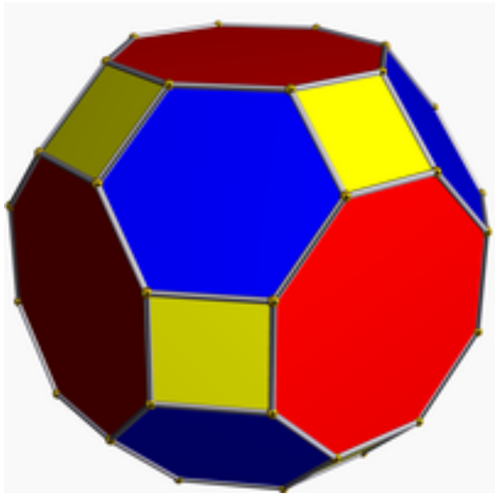
$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $H_1(3,2)$: truncated cuboctahedron
  (great rhombicuboctahedron)



primitive lattice polytopes

as generalization of the permutahedron of type B_d

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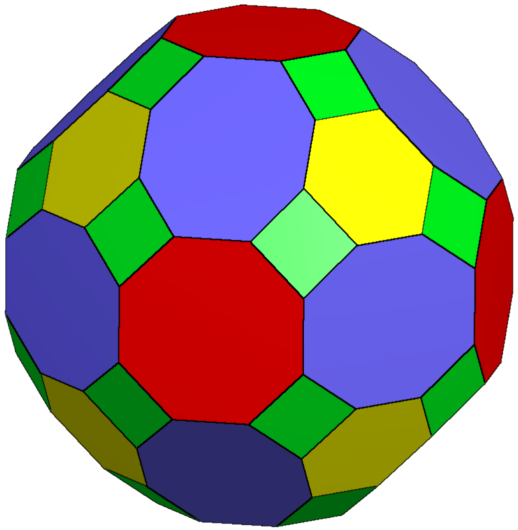
$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $H_\infty(3,1)$: truncated small rhombicuboctahedron



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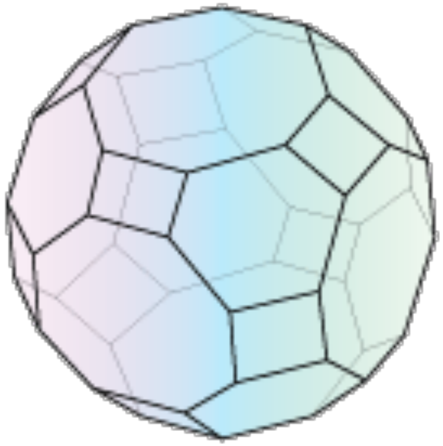
$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $Z_1(d,2)$: permutahedron of type B_d



primitive lattice polytopes

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$x \succeq 0$: first nonzero coordinate of x is nonnegative
H^+ / Z^+: ***positive*** primitive lattice polytope $x \in \mathbb{Z}^d_+$

- $H_1(d,2)^+$: Minkowski sum of the permutahedron with the $\{0,1\}^d$

primitive lattice polytopes

as generalization of the permutahedron of type B_d

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$x \succeq 0$: first nonzero coordinate of x is nonnegative

H^+ / Z^+: ***positive*** primitive lattice polytope $x \in \mathbb{Z}^d_+$

- $H_1(d,2)^+$: Minkowski sum of the permutahedron with the $\{0,1\}^d$, i.e., graphical zonotope obtained by the ***d***-clique with a loop at each node

graphical zonotope Z_G: Minkowski sum of segments $[e_i,e_j]$ for all *edges* $\{i,j\}$ of a given graph G

primitive lattice polygons

*as lattice (2,**k**)-polygons with large diameter*

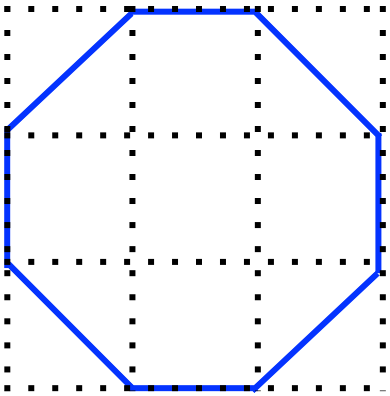
Q. (revisit) What is δ(2,***k***) : largest diameter of a polygon which vertices are drawn form the ***k*** x ***k*** grid?

For any ***k***, there exists ***p*** so that δ(2,***k***) is achieved, up to translation, by the Minkowski sum of a subset of the generators of $H_1(2,p)$.

Moreover, for any ***p***, and for ***k*** $= \sum_{i=1}^{p} i\varphi(i)$, δ(2,***k***) is uniquely achieved, up to translation, by $H_1(2,p)$ (φ : Euler's totient function)

Ex. ***p*** =2

$H_1(2,2)$: lattice (2,3)-polygon with diameter 4



primitive lattice polytopes

as lattice (d,k)-polytopes with large diameter

For $k < 2d$, Minkowski sum of a subset of the generators of $H_1(d,2$ is, up to translation, a lattice (d,k)-polytope with diameter $\lfloor (k+1)d/2 \rfloor$

Proof sketch. Assume d even (odd case similar).
$H_1(d,2)$: lattice $(d,2d-1)$-polytope with diameter d^2 (permutahedron of type B_d)

removing the $d/2$ generators (0,…,0,1,0,...,0,-1,0,...0) forming one of the d-1 *perfect matchings of the d-clique* [Berge 1983] yields a lattice $(d,k-1)$-polytope with diameter decreasing by $d/2$. After d removal, one obtains $H_1(d,2)^+$ a lattice (d,d)-polytope with diameter $d(d+1)/2$

primitive lattice polytopes

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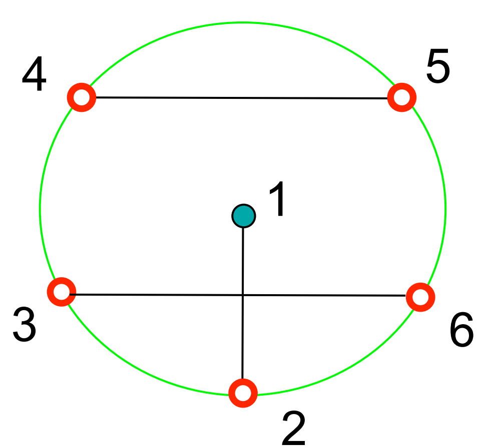
(1,-1,0,0,0,0), (0,0,1,0,0,-1), (0,0,0,1,-1,0)



primitive lattice polytopes

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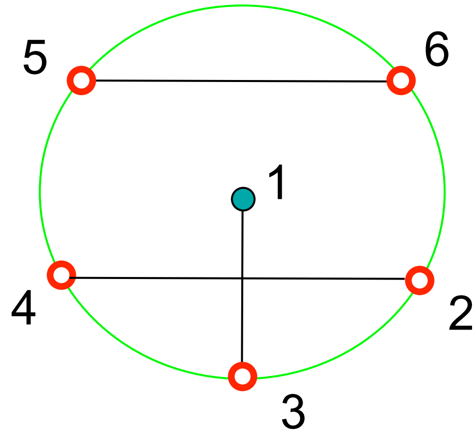

primitive lattice polytopes

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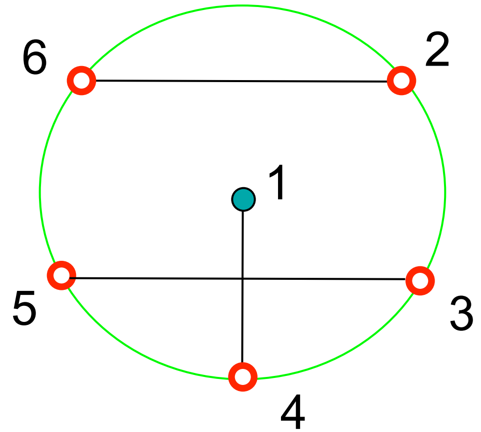
(1,0,-1,0,0,0), (0,1,0,-1,0,0), (0,0,0,0,1,-1)

(1,0,0,-1,0,0), (0,0,1,0,-1,0), (0,1,0,0,0,-1)

….



primitive lattice polytopes

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$H_1(d,2)$: lattice $(d,2d-1)$-polytope with diameter d^2 (permutahedron of type B_d)

removing the $d/2$ generators (0,…,0,1,0,...,0,-1,0,...0) forming one of the d-1 *perfect matchings of the d-clique* [Berge 1983] yields a lattice $(d,k-1)$-polytope with diameter decreasing by $d/2$. After d removal, one obtains $H_1(d,2)^+$ a lattice (d,d)-polytope with diameter $d(d+1)/2$

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lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

upper bounds :

$\delta(d,1) \leq d$		[Naddef 1989]
$\delta(d,k) \leq kd$		[Kleinschmid-Onn 1992]
$\delta(d,k) \leq kd - \lceil d/2 \rceil$	for $k \geq 2$	[Del Pia-Michini 2016]
$\delta(d,k) \leq kd - \lceil 2d/3 \rceil$	for $k \geq 3$	[Deza-Pournin 2016]

lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

Lemma. (Del Pia-Michini 2016) Consider lattice (d,k)-polytope P, u vertex of P, and vector $c \in R^d$ with integer coordinates, then $d(u,F) \leq c\cdot u - \beta$ where $\beta = \min\{ c\cdot x : x \in P \}$ and $F = \{ x \in P : c\cdot x = \beta \}$

Lemma. Consider lattice (d,k)-polytope P, $I \subseteq \{1,\ldots,d\}$ such that $l_i \leq x_i \leq h_i$ for $x \in P$ and $i \in I$, then : $\delta(P) \leq \delta(d-|I|,k) + \mathrm{sum}_{i\in I}(h_i - l_i)$

Lemma. Consider lattice (d,k)-polytope P, u, v vertices of P, $I \subseteq \{1,\ldots,d\}$ with $|I| \leq 3$ such that $u_i+v_i \leq k$ when $i \in I$, then

$$d(u,v) \leq \delta(d-|I|,k) + \mathrm{sum}_{i\in I}(u_i+v_i)$$

$|I| = 1$: $\delta(d,k) \leq kd$ [Kleinschmid-Onn 1992]

$|I| = 2$: $\delta(d,k) \leq kd - \lceil d/2 \rceil$ for $k \geq 2$ [Del Pia-Michini 2016]

$|I| = 3$: $\delta(d,k) \leq kd - \lceil 2d/3 \rceil$ for $k \geq 3$ [Deza-Pournin 2016]

lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

Consider lattice (d,k)-polytope P with $d \geq 3$, $k \geq 3$, u, v vertices of P, then one of the following inequalities holds:

(i) $d(u,v) \leq \delta(d-1,k) + k - 1$

(ii) $d(u,v) \leq \delta(d-2,k) + 2k - 2$

(iii) $d(u,v) \leq \delta(d-3,k) + 3k - 2$

$\Rightarrow \quad \delta(d,k) \leq kd - \lceil 2d/3 \rceil$ for $k \geq 3$

lattice polytopes with large diameter

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$\Rightarrow$ $\delta(d,k) \leq kd - \lceil 2d/3 \rceil$ for $k \geq 3$

$\delta(d,k) \leq kd - \lceil 2d/3 \rceil - (k - 2)$ for $k \geq 4$

primitive lattice polytopes

related questions

[Soprunov-Soprunova 2016] ***Minkowski length*** $L(\boldsymbol{P})$ of a lattice polytope $\boldsymbol{P}$: largest number of lattice segments which Minkowski sum is contained in $\boldsymbol{P}$

denote $L(\{0,1,\dots,\boldsymbol{k}\}^{\boldsymbol{d}})$ by $L(\boldsymbol{d},\boldsymbol{k})$ (Minkowski length of a box)

$L(2,\boldsymbol{k}) = \delta(2,\boldsymbol{k})$ achieved by a Minkowski sum of a proper subset of generators of $H_1(2,\boldsymbol{p})$ for some $\boldsymbol{p}$

$L(\boldsymbol{d},\boldsymbol{k}) = \lfloor(\boldsymbol{k}+1)\boldsymbol{d}/2\rfloor$ for $\boldsymbol{k} < 2\boldsymbol{d}$ achieved by a Minkowski sum of a proper subset of generators of $H_1(\boldsymbol{d},2)$

Sloane OEI sequences

$H_\infty(\boldsymbol{d},1)^+$ vertices : A034997 = number of generalized retarded functions in quantum Field theory (determined till $\boldsymbol{d}$ =8)

$H_\infty(\boldsymbol{d},1)$ vertices : A009997 = number of regions of hyperplane arrangements with {-1,0,1}-valued normals in dimension $\boldsymbol{d}$ (determined till $\boldsymbol{d}$ =7)

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A034997 Number of Generalized Retarded Functions in Quantum Field Theory. 1

2, 6, 32, 370, 11292, 1066044, 347326352, 419172756930 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

COMMENTS a(d) is the number of parts into which d-dimensional space (x_1,...,x_d) is split by a set of (2^d - 1) hyperplanes c_1 x_1 + c_2 x_2 + ...+ c_d x_d =0 where c_j are 0 or +1 and we exclude the case with all c=0.

Also, a(d) is the number of independent real-time Green functions of Quantum Field Theory produced when analytically continuing from euclidean time/energy (d+1 = number of energy/time variables). These are also known as Generalized Retarded Functions.

The numbers up to d=6 were first produced by T. S. Evans using a Pascal program, strictly as upper bounds only. M. van Eijck wrote a C program using a direct enumeration of hyperplanes which confirmed these and produced the value for d=7. Kamiya et al. showed how to find these numbers and some associated polynomials using more sophisticated methods, giving results up to d=7. T. S. Evans added the last number on Aug 01 2011 using an updated version of van Eijck's program, which took 7 days on a standard desktop computer.

REFERENCES

Björner, Anders. "Positive Sum Systems", in Bruno Benedetti, Emanuele Delucchi, and Luca Moci, editors, Combinatorial Methods in Topology and

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T. S. Evans, N-point finite temperature expectation values at real times, Nuclear Physics B 374 (1992) 340-370.

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Table of n, a(n) for n=1..8.

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convex matroid optimization

Melamed-Onn 2014:

The optimal solution of max { $\mathbf{f}(\mathbf{W}x)$: $x \in \mathbf{S}$} is attained at a vertex of the projection integer polytope in $\mathbb{R}^d$: conv(**WS**) = **W**conv(**S**)

S : set of feasible point in $\mathbb{Z}^n$ (in the talk $\mathbf{S} \in \{0,1\}^n$)
W : integer ***d*** x ***n*** matrix (**W** is mostly {0,1,…, ***p***}-valued)
f : convex function from $\mathbb{R}^d$ to $\mathbb{R}$

Q. What is the maximum number **v**(***d***,***n***) of vertices of conv(**WS**) when $\mathbf{S} \in \{0,1\}^n$ and **W** is a {0,1}-valued ***d*** x ***n*** matrix ?

Obviously v(***d***,***n***) ≤ |**WS**| = $O(n^d)$
In particular v(2,***n***) = $O(n^2)$, and v(2,***n***) = $\Omega(n^{0.5})$

convex matroid optimization

Melamed-Onn 2014

Given matroid **S** of order ***n***, {0,1,…,***p***}-valued ***d*** x ***n*** matrix **W**, maximum number **m**(***d***,***p***) of vertices of conv(**WS**) is independent of ***n*** and **S**

convex matroid optimization

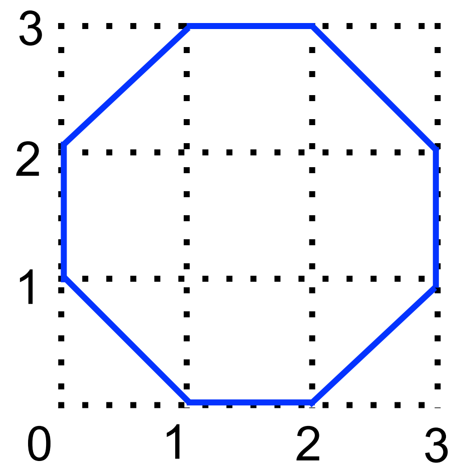
Melamed-Onn 2014

Given matroid **S** of order ***n***, {0,1}-valued ***d*** x ***n*** matrix **W**, maximum number **m**(***d***,1) of vertices of conv(**WS**) is independent of ***n*** and **S**

Ex: maximum number **m**(2,1) of vertices of a planar projection conv(**WS**) of matroid **S** by a binary matrix **W** is attained by the following matrix and uniform matroid of rank 3 and order 8:

$$\mathbf{W} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{S} = U(3,8) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$



conv(**WS**)

convex matroid optimization

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convex matroid optimization

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$$d\,2^d \le \mathbf{m}(d,1) \le 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i}$$

m(2,1) = 8
24 ≤ **m**(3,1) ≤ 158
64 ≤ **m**(4,1) ≤ 19840

convex matroid optimization

Melamed-Onn 2014 Deza-Manoussakis-Onn 2016

Given matroid **S** of order ***n***, {0,1}-valued ***d*** x ***n*** matrix **W**, maximum number **m**(***d***,1) of vertices of conv(**WS**) is independent of ***n*** and **S**

for ***d*** ≥ *3*

$d\,2^{d} \leq \mathbf{m}(d,1) \leq 2\sum_{i=0}^{d-1}\binom{(3^{d}-3)/2}{i}$

$2+2d! \leq \mathbf{m}(d,1) \leq 2\sum_{i=0}^{d-1}\binom{(3^{d}-3)/2}{i} - f(d)$

m(2,1) = 8
24 ≤ **m**(3,1) ≤ 158
64 ≤ **m**(4,1) ≤ 19840

m(2,1) = 8
48 ≤ **m**(3,1) ≤ **96**
370 ≤ **m**(4,1) ≤ **5376**
11292 ≤ **m**(5,1) ≤ **1 981 440**

primitive lattice polytopes

as lower and upper bound for convex matroid optimization parameter

Given matroid **S** of order ***n***, {0,1,…,***p***}-valued ***d*** x ***n*** matrix **W**, maximum number **m**(***d***,***p***) of vertices of conv(**WS**) is independent of ***n*** and **S**

$$| H_\infty(d,p)^+ | \le \mathbf{m}(d,p) \le | H_\infty(d,p) |$$

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$H_\infty(d,1)^{+}$ vertices : A034997 = number of generalized retarded functions in quantum Field theory

$H_\infty(d,1)$ vertices : A009997 = number of regions of hyperplane arrangements with {-1,0,1}-valued normals in dimension ***d***

❖ | *P* | : number of vertices of *P*

primitive lattice polytopes

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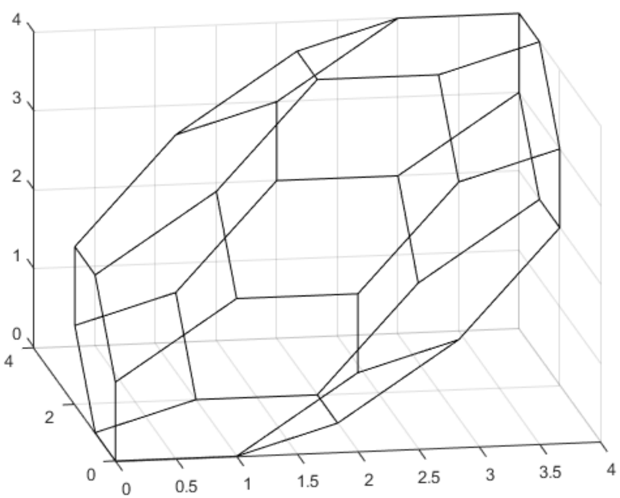
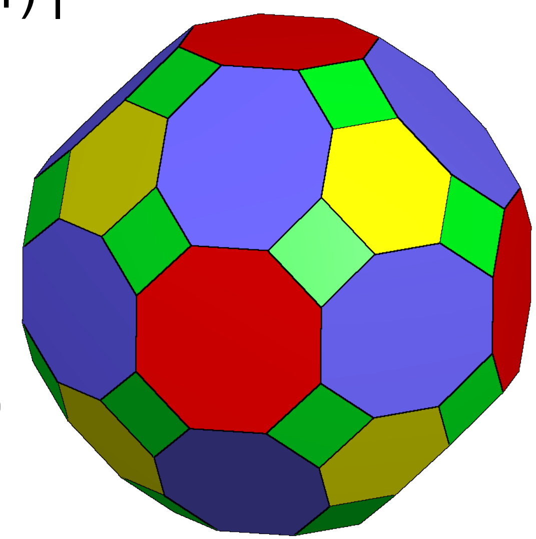


$H_\infty(3,1)^+$

$32 \leq \mathbf{m}(3,1) \leq 96$

$370 \leq \mathbf{m}(4,1) \leq 5\,376$

$1\,292 \leq \mathbf{m}(5,1) \leq 1\,981\,440$



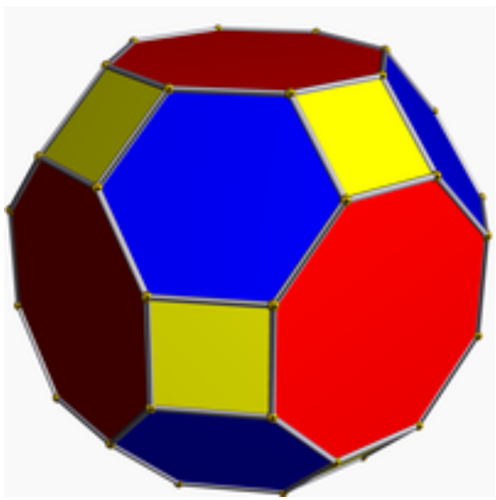
$H_\infty(3,1)$: truncated small rhombicuboctahedron

primitive lattice polytopes

as lower and upper bound for convex matroid optimization parameter

Given matroid **S** of order ***n***, {0,1}-valued ***d*** x ***n*** matrix **W**, maximum number **m**(***d***,1) of vertices of conv(**WS**) is independent of ***n*** and **S**

$$|\, H_\infty(d,1)^+ \,| \le \mathbf{m}(d,1) \le |\, H_\infty(d,1) \,|$$



$$48 \le \mathbf{m}(3,1) \le 96$$

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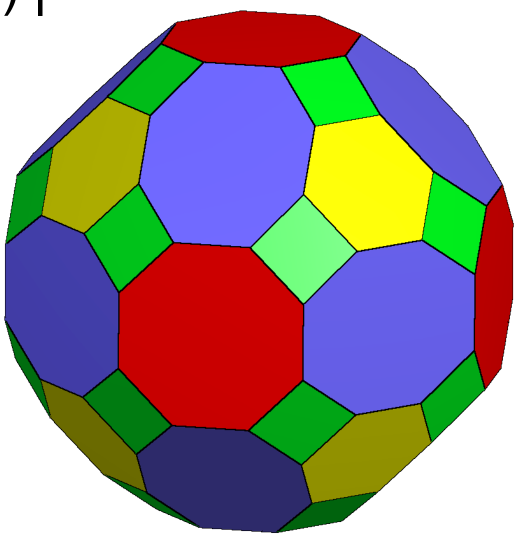

truncated cuboctahedron (great rhombicuboctahedron)

$H_\infty(3,1)$: truncated small rhombicuboctahedron

- lower bound can be further strengthened using computer search for conv(**WS**)

primitive lattice polytopes

complexity questions

For ***fixed p*** and ***q***, linear optimization over $Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$ is polynomial-time solvable, even in ***variable*** dimension ***d*** (polynomial number of generators)

⇒ for ***fixed*** positive ***integers p*** and ***q***, the following problems are polynomial time solvable:

- ***extremality***: given $x \in \mathbb{Z}^{d}$, decide if x is a vertex of $Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$
- ***adjacency***: given $x_1, x_2 \in \mathbb{Z}^{d}$, decide if $[x_1, x_2]$ is an edge of $Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$
- ***separation***: given rational $y \in \mathbb{R}^{d}$, either assert $y \in Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$, or find $h \in \mathbb{Z}^{d}$ separating y from $Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$ i.e, satisfying $h^{T}y > h^{T}x$ for all $x \in Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$

primitive lattice polytopes

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Q. Existence of a *direct* algorithm for fixed ***p*** and ***q***

Existence of an algorithms for fixed ***p*** and $q = \infty$

Existence of *hole* : $x \in Z_q(d,p)^+ \cap \mathbb{Z}^d$ which can not be written as a sum of a subset of generators of $Z_q(d,p)^+$

primitive lattice polytopes diameter and convex matroid optimization bounds

$\delta(d,k)$: largest diameter over all lattice (d,k)-polytopes

- **Conjecture** (holds for all known $\delta(d,k)$: $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$ and $\delta(d,k)$ is achieved, up to translation, by a Minkowski sum of primitive lattice vectors

  $\Rightarrow \delta(d,k) = L(d,k)$ (Minkowski length of cube $\{0,\ldots,k\}^d$)

  $\Rightarrow \delta(d,k) = \lfloor (k+1)d/2 \rfloor$ for $k < 2d$

- $|H_\infty(d,1)^+| \leq \mathbf{m}(d,1) \leq |H_\infty(d,1)|$
  e.g. determination of $\mathbf{m}(3,1)$? ($48 \leq \mathbf{m}(3,1) \leq 96$)

- determination of $\delta(3,k)$ and of $\delta(d,3)$? ($\delta(d,3) = 2d$?)

- Complexity issues, e.g. decide whether a given point is a vertex of $Z_\infty(d,1)$

primitive lattice polytopes diameter and convex matroid optimization bounds

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- Complexity issues, e.g. decide whether a given point is a vertex of $Z_\infty(d,1)$

✓ ***thank you***